KONINKLIJKE AKADEMIE
VAN WETENSCHAPPEN
-:- TE AMSTERDAM -:-

PROCEEDINGS OF THE SECTION OF SCIENCES

VOLUME XXVI

— (Nos. 1—10) —

PUBLISHED BY
"KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN", AMSTERDAM
1923

(Translated from: „Verslag van de Gewone Vergaderingen der Wis- en Natuurkundige Afdeeling" Vols. XXXI and XXXII).

CONTENTS.

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PROCEEDINGS

VOLUME XXVI

Nos. 1 and 2.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vols. XXXI and XXXII).

CONTENTS.

Mathematics. — *"Ueber das Komitantensystem zweier und dreier ternärer quadratischer Formen"*. By B. L. VAN DER WAERDEN. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of February 24, 1923).

Ein volles Komitantensystem für zwei ternäre quadratische Formen ("Kegelschnitte") ist aufgestellt worden von GORDAN, und findet sich bei CLEBSCH [1]). Ein solches für 3 Kegelschnitte ist unabhängig voneinander von CIAMBERLINI [2]), von BAKER [3]) und von FISCHER und MUMMELTER [4]) aufgestellt worden. Das CIAMBERLINI'sche System besteht, wenn man die "identische" Komitante u_x hinzurechnet, aus 128 Formen, das BAKER'sche aus 148, das FISCHER—MUMMELTER'sche aus 185 Formen. In der Tat sind 20 von den BAKER'schen Formen mittels der CIAMBERLINI'schen Formen reduzibel (siehe § 2), während SEELIG [5]) gezeigt hat, wie sich die FISCHER—MUMMELTER'schen Formen auf die CIAMBERLINI'schen reduzieren. Schliesslich rührt ein vollständiges Typensystem für eine unbeschränkte Anzahl Kegelschnitte (oder, was auf dasselbe hinauskommt, für 5 Kegelschnitte) her von TURNBULL [6]), der daraus ein vollständiges Formensystem für 4 Kegelschnitte ableitet, bestehend aus 784 Komitanten.

Mein *Zweck* ist, zu zeigen:

in § 1, dass die 21 Formen von GORDAN *ir*reduzibel sind,

in § 2, dass von den 128 CIAMBERLINI'schen Formen 6 reduzibel sind,

in § 3, dass die übrigen 122 irreduzibel sind.

Die *Methode* der Irreduzibilitätsbeweise beruht auf dem folgenden evidenten Prinzip: Soll eine Reduktionsformel für eine Komitante gelten, so muss sie auch dann noch gelten, wenn man die Urformen spezialisiert, z.B. sie miteinander identifiziert, oder auch statt der symbolischen Quadrate a_x^2 wirkliche Quadrate v_x^2 einführt. Ich werde dementsprechend in den §§ 1 und 3 alle apriori möglichen homogenen Reduktionsformeln für die betreffenden Komi-

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[1]) CLEBSCH-LINDEMANN, Vorlesungen I, Abt. III § VIII, p. 291 (Leipzig 1876)

[2]) Giornale di Battaglini 24 (1886) p 141.

[3]) Trans. Camb. Phil. Soc., Vol. 15, Part I (1894) p. 62.

[4]) Monatshefte für Mathematik und Physik 8 (1897), p. 97.

[5]) " " " " " 29 (1918), p. 225.

[6]) Proc. London Math. Soc. (2) 9 (1910) p. 81.

tanten (mit unbestimmten Koëffizienten) aufstellen und sodann, durch verschiedene Spezialisierungen und nachfolgende geometrische Betrachtungen, deren Unmöglichkeit zeigen.

Die ersten Ansätze zu Irreduzibilitätsbeweisen finden sich bei TURNBULL[1]) er zeigt auf Grund der Identifizierung, dass gewisse Komitanten für 4 Kegelschnitte irreduzibel sind, vorausgesetzt dass gewisse Komitanten für 3 Kegelschnitte es sind. TURNBULL fügt hinzu, er sehe noch nicht ein, wie man sonst noch Irreduzibilitätsbeweise geben könnte.

Die *Bezeichnungen* schliessen sich an CLEBSCH und CIAMBERLINI an, obzwar später bessere Methoden eingeführt worden sind. Die Urformen heissen

$$f_1 = a_x^3 = b_x^3 = \ldots .$$
$$f_2 = a'_x{}^3 = b'_x{}^3 = \ldots .$$
$$f_3 = a''_x{}^3 = b''_x{}^3 = \ldots .$$

Die Kontravarianten der einzelnen Urformen werden bezeichnet als

$$F_{11} = (a\ b\ u)^2 = u_\alpha{}^2 = u_\beta{}^2 = \ldots .$$
$$F_{22} = \text{u.s.w.}$$

Von BAKER übernehme ich noch die folgenden Abkürzungen:

$$u = \overline{xy} \text{ bedeutet } u_1 = x_2 y_3 - x_3 y_2, \text{ u.s.w.}$$
$$(vw\,.\,xy) = (\overline{vw}\ x\ y) = (v\ w\ \overline{xy}) = v_x w_y - v_y w_x.$$

$A \overset{r}{=} 0$ bedeutet: A ist reduzibel zu einfacheren Formen (d. h Formen deren Gesamtgrad in allen Koeffizienten und Variabeln niedriger ist).

$A \overset{r}{=} B$ bedeutet: $A = B +$ reduzibele Glieder (bei BAKER $\equiv$).

$=$ oder $\equiv$ bedeutet: identisch gleich für alle Werte der u, x, a_{ik}, a'_{ik}, a''_{ik}.

Ich werde die folgenden *Reduktions-Identitäten* verwenden[2]):

(a) $a_\alpha a_y v_\alpha = \frac{1}{3} a_\alpha{}^2 \,.\, v_y \overset{r}{=} 0$

(b) $(abv)\, a_y b_z = \frac{1}{2} v_\alpha (ayz) \overset{r}{=} \frac{1}{2} v_\alpha (ayz)$

dual: (c) $(\alpha\beta y)\, v_\alpha w_\beta = \frac{2}{3} a_\alpha{}^2 \,.\, b_y (bvw) \overset{r}{=} 0$

(d) $a_z b_\varphi a_y b_z = a_z{}^2 \,.\, b_y b_z - \frac{1}{2}(a\varphi y)(a\varphi z) \overset{r}{=} -\frac{1}{2}(a\varphi y)(a\varphi z)$

dual: (e) $g_\alpha g_\beta v_\alpha w_\beta = g_\alpha{}^2 \,.\, v_\beta w_\beta - \frac{2}{3} a_\alpha{}^2 \,.\, (bgv)(bgw) \overset{r}{=} 0$

[1]) Proc. London Math. Soc. (2) 9 (1910), p. 120.

[2]) CLEBSCH—LINDEMANN, I, III, § VIII. Am übersichtlichsten findet man die Identitäten, sowie die Ableitung des Formensystems zweier Kegelschnitte, bei GRACE and YOUNG, Algebra of Invariants, § 228.

(f) $a_p b_q a_r b_s = a_p b_q a_r b_s + \frac{1}{2}(apq)(ars) \overset{r}{=} a_p b_q a_r b_s + \frac{1}{2}(apq)(ars)$

dual: (g) $p_\alpha q_\alpha r_\beta s_\beta = q_\alpha p_\beta r_\alpha s_\beta + \frac{2}{3} a_\alpha^2 . (apq)(ars) \overset{r}{=} q_\alpha p_\beta r_\alpha s_\beta$

wo $\varphi, g, y, z, v, w, p, q, r, s$ beliebige Symbole sind.

Dazu kommen die fundamentalen Identitäten des ternären Gebietes.

Bemerkung. Man kann von einer jeden Identität zu der dualistisch entsprechenden übergehen, indem man jedes a durch α, jedes a' durch α', usw., jedes x durch u, und umgekehrt, ersetzt, und sodann, wo nötig, durch Hinzufügung von Faktoren $\frac{4}{3} a_\alpha^3$, $\frac{4}{3} a'^2_{\alpha'}$, usw., die erhaltene Formel homogen macht. Denn wenn man a durch α ersetzt, so müsste man eigentlich α ersetzen durch $\bar{a}$, definiert durch $\bar{a}_x^2 = (\alpha \beta x)^2$; es ist aber $(\alpha \beta x)^3 = \frac{4}{3} a_\alpha^2 . a_x^3$.

§ 1. *Irreduzibilität des Systems für zwei Kegelschnitte.*

Ich werde das Gordansche System hinschreiben, dabei aber von je zwei Formen, die durch Vertauschung der beiden Kegelschnitte in einander übergehen, nur eine behalten. In Klammern füge ich hinzu die 4 Grade der Komitanten in $a_{ik}, a'_{ik}; u, x$. Eine daneben-stehende Zahl bezeichnet die Anzahl der analogen Formen. Dualistisch gegenüberstehende Formen sind mit den entsprechenden Griechischen und Lateinischen Buchstaben benannt, oder auch durch obere Querstriche unterschieden.

	u_x	(00.11)	1	$N_{12} = (a\,a'\,u)\,a_x\,a'_x$	(11.12)	1
f_1	$= a_x$	(10.02)	2	$C_{1,2} = (a\,a'\,u)\,a'_\alpha\,a_\lambda\,u_\alpha$	(31.21)	2
F_{11}	$= u_\alpha^2$	(20.20)	2	$\bar{N}_{12} = (\alpha\,\alpha'\,x)\,u_\alpha\,u_{\alpha'}$	(22.21)	1
F_{12}	$= (a\,a'\,u)^2$	(11.20)	1	$\Gamma_{1,2} = (\alpha\,\alpha'\,x)\,a_{\alpha'}\,u_\alpha\,a_x$	(32.12)	2
A_{111}	$= a_\alpha^2$	(30.00)	2	$D_{12} = (a\,a'\,u)\,a_{\alpha'}\,a'_\alpha\,u_\alpha\,u_{\alpha'}$	(33.30)	1
A_{112}	$= a'^2_\alpha$	(21.00)	2	$\Delta_{12} = (\alpha\,\alpha'\,x)\,a_{\alpha'}\,a'_\alpha\,a_x\,a'_x$	(33.03)	1
$B_{1,2}$	$= a'_\alpha\,a'_x\,u_\alpha$	(21.11)	2			
Φ_{12}	$= (\alpha\,\alpha'\,x)^2$	(22.02)	1			

Die apriori möglichen homogenen Reduktionsformeln sind:

(1)	$F_{11} = 0$	(7)	$N_{12} = 0$
(2)	$F_{12} = 0$	(8)	$C_{1,2} = 0$
(3)	$A_{111} = 0$	(9)	$\bar{N}_{12} = 0$
(4)	$A_{112} = 0$	(10)	$\Gamma_{1,2} = \lambda . A_{112} . N_{12}$
(5)	$B_{1,2} = \lambda . A_{112} . u_x$	(11)	$_{12} = 0$
(6)	$\Phi_{12} = \lambda A_{112} f_2 + \mu A_{122} f_1$	(12)	$\Delta_{12} = 0$

Jetzt gehe ich daran, die Unmöglichkeit jeder dieser Formeln zu beweisen:

(1) Aus der Geometrie des Kegelschnittes weiss man, dass (1)
(3) und (3) nicht gelten.
(2) (1) und (3) sind aber Spezialisierungen von (2) en (4). Daher
(4) können auch diese nicht gelten.
(5) $B_{1,2} = 0$ stellt, bei variabelem x, die Gleichung der Polare ° des Pols ′ von u dar (Pol ° bedeutet: Pol bezüglich f_1, Pol ′ bedeutet: Pol bezüglich f_2). $B_{1,2}$ ist somit nicht identisch Null. Auch fällt diese Polare nicht für jedes u mit u selbst zusammen, es sei denn, dass die beiden Polarsystemen identisch seien; $B_{1,2}$ enthält also nicht allgemein den Faktor u_x: (5) gilt nicht.
(6) In (6) spezialisiere man $a_x^2 = v_x^2$. Jede Form, welche ein Symbol a enthält, verschwindet dann. Das ergibt $\Phi_{1,2} = 0$, $A_{11,2} = 0$, $A_{12,2} \neq 0$, $f_1 \neq 0$, und daher $\mu = 0$. Ebenso beweist man $\lambda = 0$. Aus (6) wird $\Phi_{1,2} = 0$. Die dazu duale Formel (2) gilt aber nicht, daher kann auch (6) nicht gelten.
(7) Die beiden Polaren des Punktes x seien:

$$v = a \,.\, a_x \quad ; \quad w = a' \,.\, a'_x \,.$$

Sie sind im Allgemeinen weder unbestimmt, noch miteinander identisch. Daher ist $N = (v\, w\, u) \not\equiv 0$, oder (7) gilt nicht.
(9) Die dualistische Betrachtung gilt für (9).
(8) Die Polare ′ des Pols ° von u sei

$$v = a' \,.\, a'_x \, u_x$$

Weiter sei $\overline{uv} = y$. Da v nicht mit u zusammenfällt (siehe unter (5)), so ist y nicht unbestimmt. Nun ist $C_{1,2} = a_x \, a_y \not\equiv 0$, oder (8) gilt nicht. Die dualistische Betrachtung ergibt, dass in (10) $\lambda \neq 0$ sein muss.
(10) In (10) setze man $a'^2_x = v_x^2$. Jede Form, welche ein Symbol a' enthält, verschwindet dann, und es wird $\Gamma_{1,2} = 0$, $N_{1,2} \neq 0$, $A_{11,2} \neq 0$, und daher $\lambda = 0$, in Widerspruch mit dem Vorhergehenden.
(11) $D_{1,2}$ stellt das Produkt der linken Seiten der Gleichungen der drei Seiten des den beiden Kegelschnitten gemeinsamen Polardreiecks dar, und kann somit nicht identisch verschwinden
(12) Der dualistische Beweis gilt für $\Delta_{1,2}$.

§ 2. *Reduktion der Formen $\overline{M}_{ik}$ und $\overline{T}_i$ von* Ciamberlini.

Es sei

$$\overline{M}'_{2,3} = (a\, a''\, x)\, a_{x'}\, a_{x''}\, u_{a'}\, u_x \quad ; \quad \overline{M}'_{3,2} = (a\, a'\, x)\, a_{x''}\, a_{x'}\, u_x\, u_{x'} \,.$$

Ciamberlini hat bewiesen [1])

[1]) Giornale di Battaglini, 24, p. 150, a.

$$\overline{M}'_{2,3} - \overline{M}'_{3,2} \overset{r}{=} 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Um $\overline{M}'_{2,3}$ zu reduzieren, multipliziere man die Identität

$$(a\,a''\,x)\,a_{x'} = (a'\,a''\,x)\,a_x + (a\,a'\,x)\,a_{x''} + (a a''\,a')\,a_x$$

mit $a_{x''} u_{x'} u_x$. Das erste Glied rechts is reduzibel nach (a), weil es den Faktor a_x enthält; das zweite Glied enthält den wirklichen Faktor $a_{x''}{}^2$.

Also:

$$\overline{M}'_{2,3} \overset{r}{=} (a\,a''\,a')\,a_x\,a_{x''}\,u_{x'}\,u_x \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Ebenso

$$\overline{M}'_{3,2} \overset{r}{=} (a\,a'\,a'')\,a_x\,a_{x'}\,u_{x''}\,u_x \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Man multipliziere weiter die Identität

$$\begin{vmatrix} a_\alpha & a_{\alpha'} & a_x \\ a''_\alpha & a''_{\alpha'} & a''_x \\ u_\alpha & u_{\alpha'} & u_x \end{vmatrix} = (a\,a'\,x)\,(a\,a''\,u)$$

mit $(a''\,a\,u)\,b''_\alpha\,b''_{\alpha'}$. Die rechte Seite spaltet sich in zwei wirkliche Faktoren $(a''\,a\,u)^2$ und $(a\,a'\,x)\,b''_\alpha\,b''_{\alpha'}$, ist also $\overset{r}{=} 0$. Bei der Entwicklung der Determinante auf der linken Seite kann man die Glieder die den wirklichen Faktor u_x enthalten, vernachlässigen; ebenso nach (a) die Glieder mit a_α. Es bleibt

$$-(a''a\,u)b''_\alpha b''_{\alpha'}a''_{\alpha'}a_x u_\alpha + (a''a\,u)b''_\alpha b''_{\alpha'}a''_\alpha a_x u_{\alpha'} + (a''a\,u)b''_\alpha b''_{\alpha'}a_{\alpha'}a''_x u_\alpha \overset{r}{=} 0 \quad (4)$$

Das erste Glied wird umgeformt mittels (d):

$$-(a''a\,u)\,b''_\alpha\,b''_{\alpha'}a''_{\alpha'}a_x u_\alpha \overset{r}{=} \tfrac{1}{2}\,(a''a'\,.\,a\,u)\,(a''a'\,a)\,a_x u_\alpha$$
$$= \tfrac{1}{2}\,a_{x''}\,u_{x'}\,(a''\,a'\,a)\,a_x\,u_\alpha - \tfrac{1}{2}\,a_{x'}\,u_{x''}\,(a''\,a'\,a)\,a_x\,u_\alpha$$

Das zweite Glied von (4) gibt ebenso

$$(a''\,a\,u)\,b''_\alpha\,b''_{\alpha'}a''_\alpha\,a_x\,u_{\alpha'} \overset{r}{=} -\tfrac{1}{2}\,(a''a\,.\,a\,u)\,(a''a\,a')\,a_x u_{x'}$$
$$= -\tfrac{1}{2}\,a_{x''}\,u_\alpha\,(a''\,a\,a')\,a_x\,u_{x'} + \tfrac{1}{2}\,a_x\,u_{\alpha''}\,(a''\,a\,a')\,a_x\,u_\alpha$$

oder, da das zuletzt angeschriebene Glied den Reduzenten a_α enthält,

$$\overset{r}{=} -\tfrac{1}{2}\,a_{x''}\,u_\alpha\,(a''\,a\,a')\,a_x\,u_{x'} + 0.$$

Das dritte Glied von (4) wird reduziert mittels (f):

$$(a''\,a\,u)\,b''_\alpha b''_{\alpha'}a_{x'}a''_x\,u_\alpha = (b''a\,u)\,a''_\alpha b''_{\alpha'}a_{\alpha'}a''_x\,u_\alpha + \tfrac{1}{2}\,(a''a\,.\,a\,u)\,(a''a'\,x)\,a_{x'}u_\alpha$$
$$= (b''a\,u)\,b''_{\alpha'}a_{\alpha'}\,.\,a''_\alpha a''_x\,u_\alpha + \tfrac{1}{2}\,a_{x''}\,(a''a'\,x)\,a_{x'}\,.\,u_\alpha{}^2 -$$
$$- \tfrac{1}{2}\,a_\alpha\,u_{x''}\,(a''\,a'\,x)\,a_{x'}\,u_\alpha \overset{r}{=} 0 + 0 + 0$$

Damit wird die Gleichung (4) zu:

$$\tfrac{1}{2}\, a_{\alpha''}\, u_{\alpha'}\, (a''a'a)\, a_x\, u_\alpha - \tfrac{1}{2}\, a_{\alpha'}\, u_{\alpha''}\, (a''a'a)\, a_x\, u_\alpha - \tfrac{1}{2}\, a_{\alpha''}\, u_\alpha\, (a''a\,a')\, a_x\, u_{\alpha'} \overset{r}{=} 0$$

oder, da das erste und dritte Glied einander gleich sind,

$$(a\,a''\,a')\, a_x\, a_{\alpha''}\, u_{\alpha'}\, u_\alpha + \tfrac{1}{2}\, (a\,a'\,a'')\, a_x\, a_{\alpha'}\, u_{\alpha''}\, u_\alpha \overset{r}{=} 0 \quad . \quad . \quad . \quad (5)$$

Aus (2), (3), (5) folgt:

$$\overline{M}'_{2,3} + \tfrac{1}{2}\, \overline{M}'_{3,2} \overset{r}{=} 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Endlich folgt aus (1) und (6):

$$\overline{M}'_{2,3} \overset{r}{=} \quad ; \quad \overline{M}'_{3,2} \overset{r}{=} 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

und somit: die drei Ciamberlini'schen Formen

$$\overline{M}_{23} = \overline{M}'_{2,3} + \overline{M}'_{3,2} \quad ; \quad \overline{M}_{31} = \overline{M}'_{3,1} + \overline{M}'_{1,3} \quad ; \quad \overline{M}_{12} = \overline{M}'_{1,2} + \overline{M}'_{2,1}$$

sind reduzibel.

Der andere reduzibele Typus des Ciamberlini'schen Systems ist:

$$\overline{T}_1 = (a\,a'a'')\, (a\,a''x)\, (\beta\, a'x)\, u_\beta\,.$$

Nach (*e*) ist in einer Komitante jedes a mit jedem β vertauschbar, daher:

$$(a'a''\,u)\, a'_\beta\, a''_\alpha\, b'_\beta\, b''_\alpha\, b'_x\, b''_x \overset{r}{=} (a'\,a''\,u)\, a'_\alpha\, a''_\alpha\,.\, b'_\beta\, b''_\beta\, b'_x\, b''_x \overset{r}{=} 0$$

Auf der linken Seite wenden wir (*d*) an auf die Faktoren $a'_\beta\, b'_\beta$

$$(a'\,\beta\,.\,a''\,u)\, (a'\,\beta\,x)\, a''_\alpha\, b''_\alpha\, b''_x \overset{r}{=} 0$$

$$a''_{\alpha'}\, u_\beta\, (a'\,\beta\,x)\, a''_\alpha\, b''_\alpha\, b''_x - a''_\beta\, u_{\alpha'}\, (a'\,\beta\,x)\, a''_\alpha\, b''_\alpha\, b''_x \overset{r}{=} 0.$$

Im zweiten Gliede dieses Ausdruckes ergibt abermalige Vertauschung eines α mit einem β einen wirklichen Faktor a''^2_x. Folglich ist das zweite Glied zu vernachlässigen. Auf das erste Glied wenden wir wiederum (*d*) an, jetzt auf die Faktoren $a''_\alpha\, b''_\alpha$ und finden

$$(a''\,a\,a')\, (a''\,a\,x)\, u_\beta\, (a'\,\beta\,x) \overset{r}{=} 0$$

oder

$$\overline{T}_1 \overset{r}{=} 0.$$

Ebenso

$$\overline{T}_2 \overset{r}{=} 0 \quad ; \quad \overline{T}_3 \overset{r}{=} 0.$$

§ 3. *Irreduzibilität des Systems für drei Kegelschnitte.*

Mit Weglassung der Formen die nur von zwei der drei Kegelschnitte abhängen, und der reduzibelen Formen $\overline{M}$ und $\overline{T}$, besteht das Ciamberlini'sche volle System für drei Kegelschnitte aus den folgenden Formen:

L	$= (a\,a'\,a'')^2$	(111.00)	1
V_1	$= (a\,a'\,a'')\,(a'\,a''\,u)\,a_x$	(111.11)	2 [1]
$S_{2,2}$	$= a'_\alpha\,a''_\alpha\,a'_x\,a''_x$	(211.02)	3
$\Sigma_{2,2}$	$= a_{\alpha'}\,a_{\alpha''}\,u_{\alpha'}\,u_{\alpha''}$	(122.20)	3
A	$= (a\,a'\,a'')^3$	(222.00)	1
$\overline{V}_1$	$= (a\,a'\,a'')\,(a'\,a''\,x)\,u_x$	(222.11)	2 [2]
$P_{2,3}$	$= a_{\alpha''}\,a_{\alpha'}\,a''_{\alpha'}\,a''_x\,u_{\alpha''}$	(123.11)	6 [3]
H	$= (a'\,a''\,u)\,(a''\,a\,u)\,(a\,a'\,u)$	(111.30)	1
I	$= (a\,a'\,a'')\,a_x\,a'_x\,a''_x$	(111.03)	1
O_1	$= (a'\,a''\,u)\,a'_\alpha\,a''_\alpha$	(211.10)	3
$E_{2,3}$	$= (a'\,a''\,u)\,a''_\alpha\,u_\alpha\,a'_x$	(211.21)	6 [4]
T_1	$= (a\,a'\,a'')\,(a\,a'\,u)\,(b\,a'\,u)\,b_x$	(211.21)	3
X_1	$= (a\,a'\,a'')\,a_x\,a'_\alpha\,a''_\alpha$	(311.01)	3
Ω_1	$= (a'\,a''\,x)\,a_{\alpha'}\,a_{\alpha''}$	(122.01)	3
$M_{2,3}$	$= a'_\alpha\,a''_\alpha\,a_x\,[a\,a''\,u)\,a'_x + (a\,a'\,u)\,a''_x]$	(311.12)	3
$\overline{E}_{2,3}$	$= (a'\,a''\,x)\,a_{\alpha''}\,a_x\,u_{\alpha'}$	(122.12)	6
Υ	$= (a\,a'\,a'')\,u_\alpha\,u_{\alpha'}\,u_{\alpha''}$	(222.30)	1
$U_{2,3}$	$= (a'\,a''\,u)\,a'_{\alpha''}\,a''_\alpha\,u_\alpha\,u_{\alpha''}$	(213.30)	6 [5]
$\overline{H}$	$= (a'\,a''\,x)\,(a''\,a\,x)\,(a\,a'\,x)$	(222.03)	1
$Y_{2,3}$	$= (a'\,a''\,x)\,a''_{\alpha'}\,a_{\alpha''}\,a''_x\,a_x$	(123.03)	6
Ξ_1	$= (a\,a'\,a'')\,u_\alpha\,a_{\alpha'}\,a_{\alpha''}$	(322 10)	3
G_1	$= (a'a''u)\,a'_{\alpha''}\,a_{\alpha''}\,a_{\alpha'}\,a''_{\alpha'}$	(133.10)	3 [6]
$\overline{G}_1$	$= (a'a''x)\,a''_{\alpha'}\,a''_\alpha\,a'_\alpha\,a'_{\alpha''}$	(233.01)	3 [7]

Die Methode der Irreduzibilitätsbeweise ist dieselbe wie in § 1. Die Formen L, V, Ω, X, G, U, Y sind irreduzibel, denn wären sie reduzibel, so wären auch A_{11}, B, $\overline{N}$, C, D, D, Δ (s. § 1), die aus

[1]) Die Summe der drei V ist, wie man sogleich sieht, $= L.u_x$.

[2]) Die Summe der drei $\overline{V}$ ist reduzibel.

[3]) Bei CIAMBERLINI heissen diese 6 Formen $P_1\,P_2\,P_3\,\Pi_1\,\Pi_2\,\Pi_3$.

[4]) Bei CIAMBERLINI heissen diese 6 Formen E_{23}, E_{31}, E_{12}, E'_{23}, E'_{31}, E'_{12}.

[5]) Bei CIAMBERLINI heissen diese 6 Formen $U_{23}\,U_{31}\,U_{12}\,U'_{23}\,U'_{31}\,U'_{12}$.

[6]) Die Formen G und $\overline{G}$ finden sich nicht in der Ciamberlinischen Tafel (a. a. O. (p. 153). Das ist aber offenbar ein Schreib- oder Druckfehler, denn auf S 145 ist die Form G genannt unter den "Forme con un determinante fattore"; bei den reduzibelen Formen p. 148 wird G nicht genannt (d. h. sie wird zu den "forme fundamentale" gerechnet); in der Tafel der "forme fundamentale" S. 153 wird sie nicht genannt, wohl aber mitgezählt, und in den geometrischen Anwendungen S. 157 taucht sie wieder auf. Vgl. SEELIG, Monatshefte f. Math. u. Phys. **29**, p. 265, Fussnote 21.

[7]) Bei BAKER finden sich ausserdem noch die Formen $(810)_2$, (911), (1010), die reduzibel sind nach CIAMBERLINI (p. 151 *g*, p. 149 *c*, p. 151 *g*).

den erstgenannten durch Identifizierung von 2 der 3 Kegelschnitte entstehen, reduzibel.

Für die beiden Formen $E_{2,3}$ und $E_{3,2}$ findet man durch Specialisierung:

$[E_{2,3}]_{1=2} = C_{2,3} =$ irreduzibel; $[E_{2,3}]_{1=3} =$ reduzibel;

$[E_{3,2}]_{1=2} =$ reduzibel; $[E_{3,2}]_{1=3} = C_{3,2} =$ irreduzibel;

Daraus folgt: Es kann weder $E_{2,3}$, noch $E_{3,2}$, noch auch eine lineare Kombination der beiden, reduzibel sein, denn sonst wäre auch eine der Formen C reduzibel.

Die dualistische Betrachtung gilt für $\overline{E}_{2,3}$ und $\overline{E}_{3,2}$.

Für die übrigen Formen werde ich alle apriori möglichen homogenen Reduktionsformeln aufstellen. Dabei ist folgendes zu beachten. Wenn eine Komitante K_{23} symmetrisch ist bezüglich der Formen f_2 und f_3, so kann man in einer Reduktionsformel für diese Komitante rechts die Indizes 2 und 3 überall vertauschen, ohne dass die Formel ihre Geltung verliert. Bildet man dann die halbe Summe der beiden Ausdrücke, so fallen alle alternierenden Glieder heraus, die symmetrischen bleiben stehen, und die anderen Glieder bilden Gruppen von je zwei ähnlichen mit gleichen Koeffizienten. Ist hingegen K_{23} alternierend bez. 2 und 3, so kehren sich die Verhältnisse gerade um: man bildet die halbe Differenz, die symmetrischen Glieder heben sich weg, die alternierenden bleiben. usw. Diese beiden Fälle werden mit s (symmetrisch) und a (alternierend) bezeichnet. In den jetzt folgenden Formeln sind diese beiden Operationen bereits ausgeführt; beispielsweise sind in der ersten Formel die letzten beiden Glieder mit gleichen Koefficienten versehen.

(1)s $S_{33} = \lambda f_1 L + \mu (f_2 A_{113} + f_3 A_{112})$

(2)s $\Sigma_{33} = \lambda (F_{13} A_{233} + F_{12} A_{322}) + \mu (F_{22} A_{133} + F_{23} A_{122})$

(3)s $A = \lambda L^2 + \mu (A_{112} A_{233} + A_{113} A_{322} + A_{231} A_{123})$

(4)s $\overline{V}_1 = \lambda L^2 u_x + \mu (A_{112} A_{233} + A_{113} A_{322}) u_x + \nu A_{122} A_{133} u_x + \varrho A u_x + \sigma L V_1$ [1])

(5) $P_{2,3} = \lambda L A_{233} u_x + \mu A_{122} A_{233} u_x + \nu A_{133} A_{223} u_x + \varrho A_{233} V_1 + \sigma A_{223} V_2$

(6) $H = 0$

(7) $I = 0$

(8) $O_1 = 0$

(9)s $T_1 = \lambda (E_{2,3} + E_{3,2}) + \mu O_1 u_x$

(10)s $M_{23} = \lambda (A_{112} N_{13} - A_{113} N_{12})$

(11)a $\Upsilon = \lambda (O_1 F_{23} + O_2 F_{31} + O_3 F_{12}) + \mu L H$

(12)a $\overline{H} = \lambda (f_1 \Omega_1 + f_2 \Omega_2 + f_3 \Omega_3) + \mu L H$

---

[1]) Das Glied $\tau L (V_2 + V_3)$, das noch möglich wäre, ist gleich $\tau L u_x - \tau L V_1$, und somit in anderen Gliedern der Gleichung aufzunemen.

(13)a $\Xi_1 = \lambda L O_1 + \mu (A_{112} O_3 + A_{113} O_2)$

(14)a $G_1 = \lambda L \Omega_1 + \mu (\Omega_2 A_{223} + \Omega_3 A_{332}) + \nu (A_{122} X_3 + A_{133} X_2)$.

(6), (7). (6) und (7) gelten nicht, denn H und I stellen JACOBIANA und CAYLEYANA des Bündels $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$ dar [1]).

(1). In (1) setze man $a_x^2 = v_x^2$ (kurz: $a = v$). Es verschwinden dann alle Ausdrücke die ein Symbol a enthalten. Daher $S_{23} = 0$, $A_{113} = 0$, $A_{112} = 0$. Weiter ist dann $f_1 \not\equiv 0$, $L \not\equiv 0$. Daraus folgt $\lambda = 0$. Zweitens wähle man x in einem der 4 Schnittpunkte von f_2 und f_3. (1) wird dann $S_{23} = 0$. Geometrisch würde das bedeuten, dass die Tangenten in x zu f_2 and f_3 konjugiert sind bezüglich f_1, was nicht immer der Fall ist, weil f_1 ganz beliebig.

(2). In (2) setze man $a' = v$. Dann wird $\Sigma_{23} = 0$, $F_{22} = 0$, $A_{122} = 0$, $A_{322} = 0$, $F_{13} \not\equiv 0$, $A_{233} \not\equiv 0$. Das ergibt $\lambda = 0$. Weiter verläuft der Beweis dualistisch entsprehend zu (1).

(3). In (3) setze man $a' = v$, $a'' = w$. Das ergibt in derselben Weise wie bei den früheren Beweisen $\lambda = 0$. Setzt man nur $a' = v$, so findet man $u = 0$. (3) wird damit $A = 0$. Dualistisch müsste dann auch $L = 0$ sein, was falsch ist.

(4). In (4) setze man zuerst $a' = v$, $a' = w$, $a'' = s$. Dann findet man

$$0 = \lambda (v\, w\, s)^4\, u_x + \sigma (v\, w\, s)^3 (w\, s\, u)\, v_x .$$

Da aber die Linien u und v unabhängig sind, müssen die Koeffizienten von u_x und v_x einzeln verschwinden, somit $\lambda = 0$, $\sigma = 0$. Oder: V_1 enthält den Faktor u_x. Daraus folgt aber dualistisch, dass auch V_1 den Faktor u_x enthalten müsste, was nicht der Fall ist.

(5). In (5) setze man $a' = v$. $P_{2,3}$, A_{122} und A_{233} verschwinden dann. Nach Division durch A_{233} erhält man:

$$0 = \lambda (a\, a''\, v)^3\, u_x + \varrho (a\, v\, a'')(v\, a''\, u)\, a_x + \sigma (a\, v\, a'')(a''\, a\, u)\, v_x .$$

Setzt man hier $a = s$, $a' = w$, so findet man eine lineare Abhängigkeit der drei Linien u.s.w. welche aber ganz beliebig sind. Das ist nur dann möglich, wenn alle Koeffizienten Null sind, also wenn $\lambda = 0$, $\varrho = 0$, $\sigma = 0$. Daraus folgt dass P den Faktor u_x enthält. Setzt man in P aber $a = v$, so zerfällt P in zwei nichtverschwindende Faktoren, der eine linear in u, der andere in x. Diese beiden Tatsachen sind unvereinbar.

(8). Die zu (8) duale Formel $\Omega = 0$ gilt nicht, daher kann (8) auch nicht gelten.

[1]) Siehe CLEBSCH-LINDEMANN, a.a.O., oder besser BAKER, a a.O., wo man die geometrischen Untersuchungen von CLEBSCH, ROSANES, usw. über die Figur dreier Kegelschnitte zusammengestellt findet.

(9). In (9) setze man $a = v$. Die rechte Seite verschwindet, und man erhält

$$(v\, a'\, a'')\,(a''\, u\, v)\,(a'\, u\, v)\,.\,v_x = 0$$

oder

$$(v\, a'\, a'')\,(a''\, u\, v)\,(a'\, u\, v) = 0$$

und somit: die beiden Polaren des Punktes $\overline{uv}$ bezüglich f_2 und f_3 schneiden sich auf v. Das ist aber offenbar nicht immer der Fall, da diese beiden Polaren nach Wahl der Linien u und v noch beliebig gewählt werden können.

(10) In (10) setze man $a' = v$, $a'' = w$, und erhält

$$v_\alpha w_\alpha\,.\,\{a_x\,(a\,w\,u)\,.\,v_x + a_x\,(a\,v\,u)\,.\,w_x\} = \\ = \lambda\,\{v_\alpha^2\,.\,(a\,w\,u)\,a_x\,.\,w_x + w_\alpha^2\,.\,(a\,v\,u)\,a_x\,.\,v_x\}.$$

Da diese Gleichung für jedes u gelten muss, so müssen die Koeffizienten von $(avu)\ a_x$ und $(awu)\ a_x$ jeder für sich Null sein. Folglich wäre

$$v_\alpha w_\alpha\,.\,w_x - \lambda\,.\,w_\alpha^2\,.\,v_x = 0$$

was, wegen der Unabhängigkeit der Linien u und v unmöglich ist.

(11) In (11) setze man $a = v$, $a' = w$, $a'' = s$, und erhält $\mu = 0$. Setzt man nur $a = v$, $a' = w$, so findet man $\lambda = 0$. (11) wird damit $\Upsilon = 0$; die dualistische Formel (7) gilt aber nicht, daher kann auch (11) nicht gelten.

(12) In (12) setze man $a' = v$, $a' = w$, $a'' = s$, und erhält $\mu = 0$. Setzt man nur $a = v$, so findet man $\lambda = 0$. (12) wird damit $\overline{H} = 0$. Die duale Formel gilt aber nicht, daher kann auch (12) nicht gelten.

(13). In (13) setze man $a' = v$, $a'' = w$, und findet $\lambda = 0$. Setzt man nur $a' = v$, so findet man $\mu = 0$. (13) wird damit $\Xi = 0$. Die dualistische Formel gilt aber nicht, daher kann auch (13) nicht gelten.

(14). In (14) setze man $a = v$, und findet

$$\lambda\,(v\,a'a'')^2\,.\,(a'a''x)\,v_{\alpha'}\,v_{\alpha'} + \mathrm{r}\,\{v_{\alpha''}^2\,(v\,a'a'')\,a''_{\alpha'}\,v_{\alpha'}\,a'_x + v_{\alpha''}^2\,(v\,a'a'')\,a'_{\alpha''}\,v_{\alpha''}\,a''_x\} = 0.$$

Jetzt wähle man für v eine der gemeinsamen Tangenten von f_2 und f_3. Dann ist $v_{\alpha'}^2 = 0$, $v_{\alpha''}^2 = 0$, $(v\,a'\,a'')^2 \not\equiv 0$ (denn $(v\,a'\,a'')^2$ ist nur dann Null, wenn v die beiden Kegelschnitte in harmonischen Punktpaaren schneidet), $(a'\,a''_x)\,v_{\alpha'}\,v_{\alpha''} \not\equiv 0$ (denn diese Form ist nur dann identisch Null in x, wenn die Verbindungslinie der beiden Pole, oder Berührungspunkte von v, unbestimmt wird). Also $\lambda = 0$. Nimmt man sodann für v eine beliebige Tangente von f_3, die nicht zugleicherzeit Tangente von f_2 ist, so wird $v_{\alpha'}^2 = 0$, $v_{\alpha''}^2 \neq 0$, während man geometrisch leicht einsieht, dass $(v\,a'\,a'')\ a''_{\alpha'}\,v_{\alpha'}\,a'_x \not\equiv 0$. Also $\nu = 0$. Jetzt ist die Formel (14) homogen in a und a; sie kann daher dualisiert werden ohne dass Faktoren $\frac{4}{3}\ a_x^2$ hinzutreten. Aus der Irreduzibilität von G^J folgt dann ihre Unmöglichkeit.

Astronomy. — "*On the magnitude equation of* OSTHOFF's *estimates of star-colours*". By EJNAR HERTZSPRUNG.

(Communicated at the meeting of February 24, 1923).

In Annalen van de Sterrewacht te Leiden Vol. XIV, Part 1, p. 14; 1922 I have noticed an unexplained magnitude equation for the derived c_2/T values of stars of the spectral classes AO, A2, A3 and A5. Now the c_2/T values used l.c. depend for about 58 percent of the total weight on direct colourestimates. A redetermination of the magnitude equation of those estimates is therefore very desirable. The opportunity for this is given by the new catalogue of OSTHOFF (Specola Astronomica Vaticana Vol. VIII; 1916) extending his estimates with the 4 inch refractor one magnitude farther viz. to about 6^m. A card catalogue was made containing the hour of R. A., the degrees of declination, the spectrum of the new Draper Catalogue H.D. (taken from the Index Catalogue, Spec. Astr. Vat. IX; 1917), the magnitude to one tenth and the estimated colour. The cards were divided into groups according to spectrum. After some trial the subdivisions of spectral class were combined in the way as shown in Table 1. For each of the 6 combined groups corresponding values of mean magnitude and mean estimated colour are given. On the accompanying diagram the figures of Table 1 are represented graphically.

The most striking fact is, that the estimated colour does not, as hitherto adopted [1]), increase continuously with decreasing apparent brightness but shows a maximum in the neighbourhood of 4^m or 5^m. Especially for the white stars the decrease in estimated colour between 5^m and 6^m is very marked. This is nothing more, than should be expected from the known peculiarities in colourconception by the human eye. If the spectrum of the sun is made to increase in intensity starting just below the limit of visibility, the blue and green portion will appear first, but without showing any colour, until by still greater intensity the colours green and blue are

[1]) A. Pannekoek, Koninklijke Akademie van Wetenschappen te Amsterdam, Proceedings of the Meeting of Saturday October 27, 1906, and E. HERTZSPRUNG, Zeitschr. für wiss. Photographie Bd. 5, 100; 1907.

seen. On the other hand the red end of the spectrum will appear red, as soon as it is perceived. By very great intensities the colours will again loose in saturation [1]. The magnitude equation found for OSTHOFF's colours is in accordance with these facts.

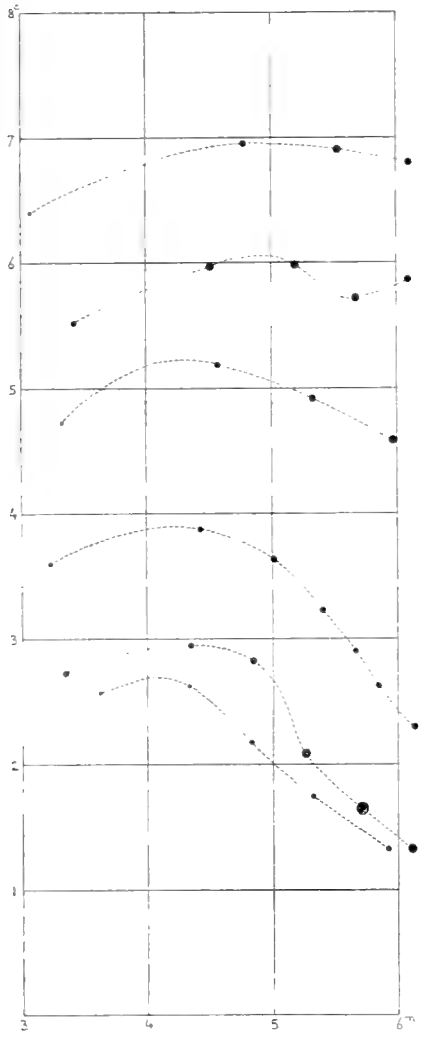

Figure 1.

TABLE I

B3, B5			B8, B9, A0, A2, A3			A5, F0, F2, F5			F8, G0, G5			K0			K2, K5, M_a, M		
n	m	c	n	m	c	n	m	c	n	m	c	n	m	c	n	m	c
18	3.63	2.57	65	3.35	2.73	30	3.24	3.60	23	3.32	4.72	51	3.42	5.52	22	3.07	6.40
23	4.34	2.63	59	4.35	2.95	51	4.43	3.87	61	4.58	5.19	106	4.51	5.96	71	4.78	6.95
38	4.83	2.18	84	4.85	2.83	73	5.01	3.64	77	5.33	4.92	105	5.19	5.98	99	5.53	6.90
48	5.32	1.75	145	5.26	2.09	63	5.40	3.24	108	5.97	4.59	107	5.66	5.72	87	6.09	6.80
48	5.02	1.32	104	5.55	1.75	52	5.66	2.90				105	6.09	5.86			
			173	5.80	1.59	59	5.85	2.63									
			132	6.11	1.33	70	6.14	2.30									

[1] E.g. the wire of the electric lamp behind the darkroom glass, only letting red light through, appears yellow.

The results obtained are able to clear up the discrepancies cited above from Leiden Ann. XIV. At the same time they form an instructive example of the unsafety of extrapolation, as just at about 5^m, which was the limit of brightness of the stars concerned in Leiden Ann. XIV, the magnitude equation of the estimated colour changes its character.

The above considerations rest on the assumption, that stars of the same spectrum do not show any systematic change of effective temperature with apparent magnitude. As long as we have no other reliable colourequivalents of these stars, this seems to be the most plausible supposition, which can be used.

Anatomy. — *"On the development of the shoulder-girdle and episternum in Reptiles"*. By CHR. VAN GELDEREN. (Communicated by Prof. L. BOLK).

(Communicated at the meeting of December 30, 1922).

In comparative anatomy we distinguish in the primary shoulder-girdle of most Sauria the scapula and suprascapula, coracoid, epicoracoid and procoracoid. Procoracoid and coracoid are usually homologised with the similarly-named portions of the shoulder-girdle of the Urodela and Anura. This was long ago contested by GÖTTE [1]). According to him there is nothing in the ontogenesis to justify such an independence being attributed to the so-called procoracoid of the Sauria, for he holds that all the parts of the ventral portion of the primary shoulder-girdle (in Chemidophorus) develop from one massive formation. The procoracoid of the Sauria, therefore, he says, will not originate as a free cranio-ventral process of the coracoid, to unite with it ventro-medially into a ring, as GÖTTE found it in Anura, and as in comperative anatomy it is frequently termed with respect to the Sauria. Although such a development was observed at a later date by WIEDERSHEIM [2]), BROOM [3]) and BOGOLJUBSKI [4]) in a few other Sauria also, no hand-or text-books (with the exception of that by WIEDERSHEIM) make any reference to this. It is this which has led to the present article, treating of the development of the primary as well as the secondary shoulder-girdle (including episternum).

In intimate connection with the question as to the ontogenesis of the episternum is another, namely, that of the development of the clavicula. And attention will also be devoted to this in the following lines.

The episternum, for the first data of the development of which we have to thank RATHKE [5]), was seen by the latter to originate

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[1]) A. GÖTTE, Archiv. f. mikrosk. Anat. Bd. XIV, 1877.

[2]) R. WIEDERSHEIM, Das Gliedmaszenskelett der Wirbelthiere. Jena, 1892.

[3]) R. BROOM, Trans. South. Afric. Philos. Soc. Vol. XVI, Pt. 4, 1906.

[4]) S. BOGOLJUBSKI, Zeitschr. f. Wissensch. Zool. Bd. 110, 1914.

[5]) H. RATHKE, Ueber den Bau und die Entwickl. des Brustbeines der Saurier. Königsberg 1853.

unpaired between the medial ends of the claviculae. GÖTTE holds the opinion that, also on account of its paired formation, the episternum develops from a part of the clavicular formation which is bent caudally. Moreover, in his opinion, the clavicle originates as a blastemic process of the primary shoulder-girdle. GEGENBAUER [1]), on the contrary holds that the connection of the clavicle and shoulder-girdle is a secondary one. HOFFMAN [2]) observed the paired development of the episternum in the crocodile, and also on the basis of GÖTTE's researches, he speaks of a clavicular sternum. WIEDERSHEIM was not able to find any real genetic connection of the episternum with the clavicula either in Lacerta or in Crocodilus, although he succeeded in recognizing the clavicula, the embryonal existence of which GÖTTE had already surmised in a rudimentary form. As regards the relation between the clavicula and the scapulo-coracoideum, WIEDERSHEIM shares GÖTTE's opinion. SCHAUINSLAND [3]) did not find in Sphenodon any primary connection of episternum and clavicula in stadia where the medial portion of the latter contained no bone as yet. Besides a primary connection of clavicula and scapulo-coracoideum BOGOLJUBSKI mentions a paired formation of the episternum, in which the ossification takes place from paired centra. Of the genetic relations of clavicula and episternum he gives no details.

None of the researchers ever found any cartilage in episternum and clavicula. GÖTTE and WIEDERSHEIM, however, describe a form of ossification which is strongly suggestive of the formation of perichondral bone round about a nucleus of cartilage. The bony clavicle, they say, first canaliculate and afterwards cylindrical, enclosing a soft medullar cord, just like a cartilaginous process. SCHAUINSLAND and BOGOLJUBSKI specially mention to have found no trace of such a peculiar ossification process. According to these writers the medullar cavity is produced by osteoklastic action.

I had for my investigations seventeen embryos of the common lizard, lacerta agilis, all of which I prepared in cross-sections. (Section thickness 10 μ). Further, the collection belonging to the Anatomic Laboratory contained a dozen series of Gongylus ocellatus and two of Ptychozoon homalocephalum. The direction in which sections were made in the thorax-region depended intimately upon the age of the embryos, namely, they were all made frontal on the jaw. This, with the slight curve in the region of the neck in the older

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[1]) C. GEGENBAUR, Untersuch. z. Vergleich. Anat. der Wirbelthiere. 2 Teil. Schultergürtel. Leipzig 1865.

[2]) C. K. HOFFMANN, Niederl. Archiv. f. Zoologie, Bd. V, 1879.

[3]) H. SCHAUINSLAND, Archiv. f. Mikrosk. Anat. u. Entw.gesch. Bd. LVI, 1900.

embryos, was practically identical with frontal on the thorax. According as the neck-curve was more pronounced in the younger embryos, the sections were made more transverse on the thorax. In the account of my observations I shall commence with Lacerta, as my material of this was the most complete.

Fig. 1*a* shows the shoulder-girdle of Lacerta agilis spread out in one flat plane, whereby the sternum and episternum have been left in position in order to show the relative positions. Fig. 1*b* shows only the primary shoulder-girdle.

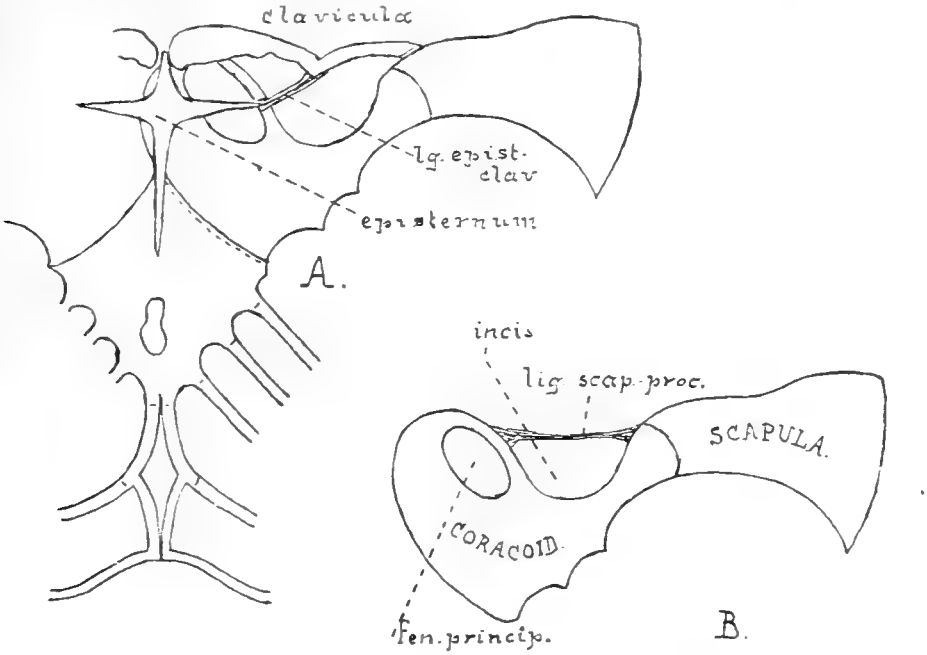

Fig. 1. Sternum and clavicula shoulder-girdle of Lacerta agilis.

The primary shoulder-girdle, i. e. the cartilaginous preformation, consists of a dorsal portion: the scapula and the non-ossifying large supra-scapula, and a ventral portion, viz. the coracoid, in which we usually distinguish three parts: coracoideum, s. str., procoracoid and epicoracoid. They surround an oval opening, the fenestra coracoidea principalis (FÜRBRINGER)[1]. Besides this cranial to the fossa glenoidalis humeralis, there is generally another fine canal, through which the n. musc. sopracoracoidei runs. This canal will henceforth never be counted among the coracoidal fenestrae 'many reptiles possess more than one fenestra'. The cranial border of the primary shoulder-girdle exhibits a deep incisura scapulo-procoracoidea which is bridged by a strand of connective tissue, lig. scapulo-procoracoideum. The cora-

[1]) M. FÜRBRINGER, Jenaïsche Zeitschr. Bd. 34, 1900.

coideum is received diarthrotically in the sulcus articularis coracoidalis sterni. The clavicula is connected syndesmotically with the suprascapula. Between the medial extremities of the claviculae the cranial point of the dagger-like episternum interposes. The latter lies mainly cranial to the sternum; a small part, however, lies ventral to the sternum and is quite separated from it by connective tissue.

The youngest embryo that I was enabled to examine, Lacerta ag. D. (N. T.)[1]) about 22), possessed no shoulder-girdle yet. Only in the inarticulated epiphysal limbbud was a central blastema. Besides the primary shoulder-girdle, also the clavicle was lacking.

In the embryo Lacerta ag. S. (N. T. about 24) the central blastema of the superior limb has extended proximally as a blastematic formation of the shoulder-girdle as yet very difficult to define. There is no trace yet of the clavicula.

The embryos Lacerta ag. E. and F. (N. T., about 26) contain a well defined shoulder-girdle which still consists entirely of dense mesenchyme.

Any clavicular formation is still lacking. In the humeral formation there is already praechondrium. The line of demarcation of the coracoid with respect to the growing sternal formation is much more obscure than it was in embryo S. Specially noteworthy is the fact that the formation of the shoulder-girdle, apart from the nerve-canal, one solid whole.

Embryo Lacerta ag. I. (N.T. about 28) is clearly in a more advanced stage of development. Cartilage is found in the humerus, which passes over proximally into praechondrium of which also a part of the primary shoulder-girdle consists. The latter still forms one continuous whole with the humerus. In this embryo the boundary of coracoid and sternal formations has almost disappeared, a transition stadium which will speedily be followed by the formation of the definite articular cavity. In the process of the primary girdle the praechondrium occupies the caudal region; the rest is still compact mesenchymatous, but quite homogeneous.

From the cranial border of the scapulo-coracoideum, dorsal from the humerus-formation a blastematic spur proceeds. There is also a very small fragment of bone tissue to be seen, quite dorsally close the point of attachment to the primary shoulder-girdle.

Lacerta ag. K. (N. T. about 29). The line of demarcation between coracoid and sternum is indicated (now permanently) by a loose mesenchymatous layer. There is an increase of praechondrium in

[1]) K. Peter, Normentafel Lacerta agilis.

the scapulo-coracoideum. That portion, however, which is still blastematic, has lost its homogeneity. (Compare the schemata of fig. 5.) A more compact cranial border can be plainly distinguished; the ventro-medial border is also more compact than the rest of the blastema. Further, a likewise denser strip of blastema connects the cranial border with the praechondral caudal portion. In these compacter regions there is no praechondrium however. The blastematic spur proceeding from the cranial scapular border has become slightly longer, as also the fragment of bone lying in it. It is from this process that the clavicle develops; we shall therefore henceforth term it the clavicular process. The connection of the scapulo-coracoid and the clavicular process will be evident from the two consecutive sections illustrated in fig. 2. The left section lies cranial to the right one. In the former the ventral outgrowth of the clavicular process can be seen; in the latter the connection with the scapulo-coracoideum.

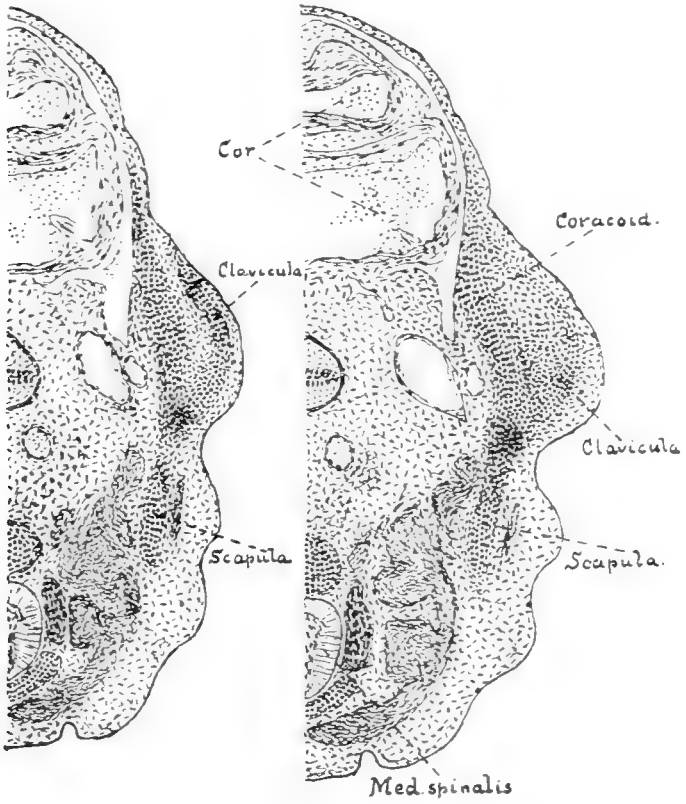

Fig. 2. Lacerta agilis K. Cross-section.

The dark spot at the place where the clavicular-process goes out

from the scapula shows the thickening of the cells against the bone fragment present in the following (not drawn) section.

Embryo lacerta ag. G. (N. T. about 29) is distinguished from the former one by a piece of bone which has grown larger in a ventro-medial direction in the blastematic process of the clavicula, which has grown out in the same direction. The blastematic clavicular process of the scapulo-coracoid still reaches much further ventrally than the bone fragment.

Embryo Lacerta ag. H. (N. T. about 30). Both the form and the histological differentiation of the formation of the parts of the skeleton have undergone marked changes. The calcified cartilaginous diaphysis is surrounded by a covering of perichondral bone. The articulatio humeri is indicated by a layer of thick mesenchyme which lies between the cartilaginous proximal humerus extremity and the shoulder-girdle. Cartilage is found in the scapulo-coracoid in accordance with the position of the scapula and of the later coracoideum s. str., i. e. in the dorsal and ventro-caudal parts. The ventro-cranial half consists of praechondrium and blastema, except that where, in the adult lizard, the epicoracoid, procoracoid and lig. scapulo procoracoideum are found, we find in this embryo praechondrium, and that at the place of the future fenestra principalis and of the membrana scapulo-procoracoidea, only thickened mesenchyme blastema is found. For the rest the scapulo has grown out in a dorsal direction as well as the coracoid has done in wider sense in a ventral direction. The clavicular-process has grown longer ventro-medially, especially the bony nucleus lying in it. Moreover it is also striking that caudal to the bony clavicle a compact blastema-mass has developed. The significance of this will become plain later.

Embryo Lacerta ag. J. (N.T. about 31). The organs, the development of which are examined here, show no striking differences from embryo H. More cartilage is present in the scapulo-coracoid than before. The outline of the praechondrium against the blastema is more easily definable. The blastema, of which the bony clavicle occupies the cranial border, has increased in size but is still perfectly homogeneous. It is distinctly loose of the coracoid lying underneath it (properly dorsal to it). The schemata of fig. 6 may now be compared.

Embryo Lacerta ag. L. (N.T. about 31). In this specimen many of the parts still praechondral in the former embryo have become cartilaginuos. The blastema, of which the bony clavicle occupies the cranial border, has increased in size, but is still homogeneous. The bony clavicle is now nearly as long as the clavicular blastema.

In the ventral medial line the clavicular-blastema of both sides are still distinctly separated.

Those parts of the coracoid which were still blastematic in the former embryos have decreased in density and have become somewhat lighter. (fenestra principalis and membr. scapulo-procoracoidea). The later lig. scapulo-procoracoideum has remained praechondral. Pro- and epicoracoid now consist of cartilage. Fig. 3 shows four sections taken from this series (not consecutive). Section *a* contains the procoracoid, the lig. scapulo-procoracoideum and the scapula; section *b* has already passed (more caudally) through the later membrana scapulo-procoracoidea; section *c* contains also the thinner blastema which corresponds to the later fenestra principalis; section *d* finally contains only coracoideum s. str. (and scapula).

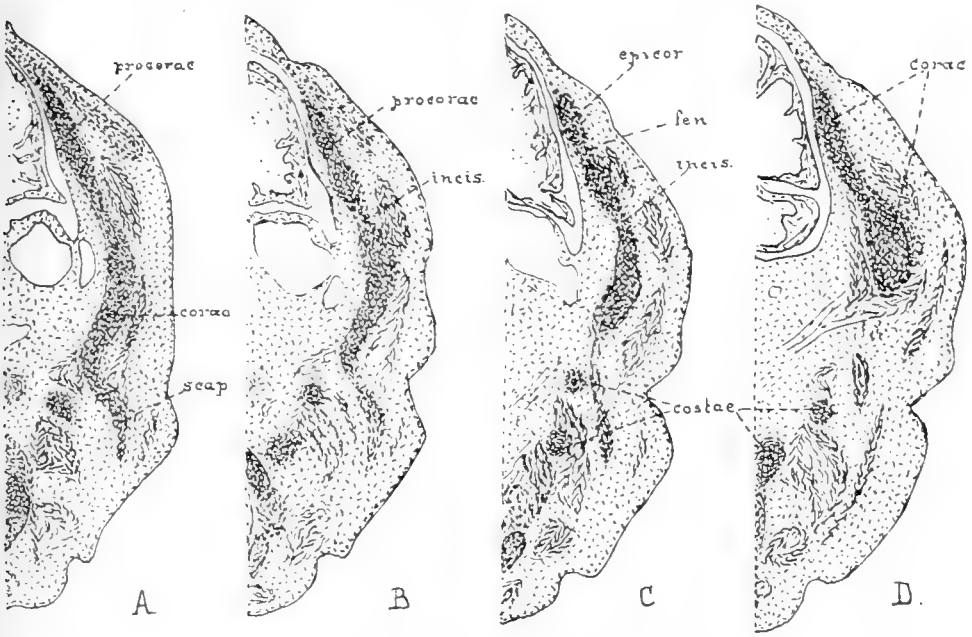

Fig. 3. Lacerta agilis L. Cross-section.

Embryo Lacerta ag. N. (N.T. about 32). The entire definite coracoid in a wider sense is now present in cartilage. In the bridging over of the incisura scapulo-procoracoidea, the praechondrium has diminished while the blastema has increased. The thinning of the blastema corresponding to the membrana scapulo-procoracoidea and to the fenestra principalis, already seen in the preceding embryo, is continued here. The definite form of incisura and fenestra is clearly recognisable.

In the blastema which joins the clavicula caudally a still slight central thinning can be seen. Simultaneously, in the unthinned,

medial and caudal boundaries of the thinned centrum, a small trace of bone-tissue is seen, at a place almost corresponding to the crossing-point of the definite episternum. The blastematic medial ends of the claviculae are no longer sharply defined; and there is no connection yet between the two by way of the medial line.

Embryo Lacerta ag. O. (N. T. about 32). The changes in the primary girdle are confined here to the non-cartilaginous portions. At the place of the incisura scapulo-procoracoidea and of the fenestra principalis the thinning of the tissue is fairly complete; the blastema once present has become nearly a membrane of connective tissue. In the cranial border of the membrana scapulo-procoracoidea a thicker strand is distinguishable, in which a few praechondrium insulae are lying, as the remains of a entirely praechondral bridging. In the retro-clavicular blastema the central thinning has proceeded further. Fig. 4 shows five partial illustrations of sections from this series. Each has been drawn exactly to the medial plane. There was still loose mesenchyme between the right and left claviculae. In Fig. *a* the bony clavicula has been taken for the greater part lengthwise. In Fig. *b* only the thickened medial end of the clavicula is to be seen. Lateral to it comes a thinner blastematic region (thinned centre), still more laterally the cut caudal border. In Fig. *c* only the thinned centre with the caudal surrounding border medial and lateral to it has been reproduced. Of the clavicular bone no more traces are to be seen. Figs. *d* and *e* have been chosen caudal to the thinned centre. Fig. *e*, the most caudal, shows the last vestige of the retroclavicular blastema.

Embryo Lacerta ag. M. (N.T. about 33). In this one the thinning into a connective tissue membrane has been fully accomplished at the place of the incisura scapulo-procoracoidea and of the fenestra principalis. Apart from a praechondral insula, the incisura bridging consists of a strand connective tissue, ligament. The central thinning of the blastema lying caudal to the bony clavicula has here, too, practically led to the formation of a connective-tissuemembrane. The bony episternum has grown in size. There is thus now one connected complex present, consisting of a thin bony episternal transverse bar (situated in the caudal boundary), from which a thin blastema bundle can be traced to a point ventral of the equal-sided sternal band and in which a still much thinner fragment of bone, (even broken perhaps locally) is found. Thus, for the first time, in this embryo a small piece of the episternum is met with ventral from the sternal formation. From the transverse bar a blastema bundle (likewise caudal border) runs in a lateral and cranial direction and

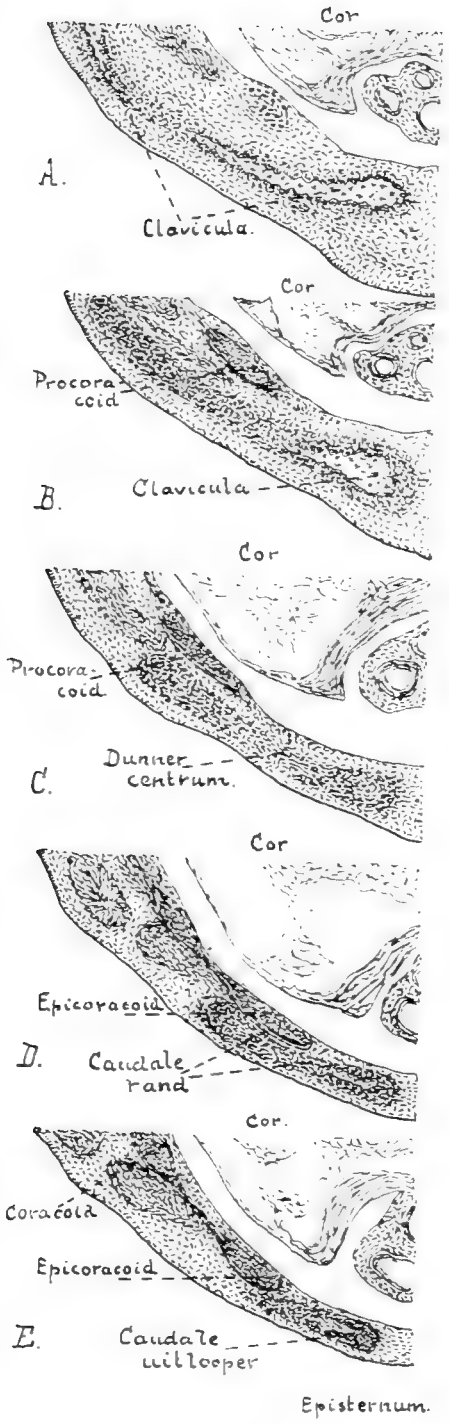

Fig. 4. Lacerta agilis cross-section.
Dunner centrum = Thinner centre.
Caudale rand = Caudal margin.
Caudale uitlooper = Caudal process.

comes to insertion at the clavicula. Finally, the medial boundary forms a blastema strip, in which the half of the little cranial point of the episternum will develop later. The complex of clavicular and episternal formations is connected with that of the other half of the body at the level of clavicle and episternal transverse bar; the caudal processes are still separated.

Embryo Lacerta ag. P. (N. T. about 33). In this embryo the sternal borders are already blended cranially. Apart from a commencement of calcifying cartilage, there is nothing to remark at the primary shoulder-girdle except the occurrence of a cartilaginous insula in the lig. scapulo-procoracoideum. The coracoidea have passed the medial line, and are thus partly overlapping each other. The central thinning in the episterno-clavicular blastema has completely given place to the membrana-clavicularis. In the caudal border the episternal transverse bar has elongated, and its lateral extremity is attached by ligament to the clavicula. The episternum, now grown unpaired, has also acquired a cranial point which interposes itself between the two clavicles. In cross-sections it shows traces of paired formation, (deep medial groove on the dorsal side); the paired bony formation I have not seen however. On the medial half of the clavicula a thick cranial border and a thinner caudal bone-plate can be distinguished, the latter being evidently an ossified portion of the claviculo-episternal membrane.

In embryo Lacerta ag. Q. (N.T. about 33—34) the cranio-caudal measurement of the episternum has attained its definite relative size. Several cross-sections show a paired cranial episternal point. This duality is merely local however. Nevertheless I take it as a proof that also this part of the episternum is formed pairedly, in which case the whole bony episternum was originally paired. In the primary shoulder-girdle the calcification has extended.

Regarding embryo Lacerta ag. R. (N.T. about 34—35), in which the portions of the bony skeleton discussed here have all attained their definite form, although on a small scale, there is nothing of note except that in the lig. scapulo-procoracoideum vestiges of cartilage are still to be found.

When the scapulo-coracoideum passes into praechondrium and later into cartilage, a narrow strip of tissue remains between it and the bony clavicula at the blastema stage. We can now for the first time speak of a syndesmosis scapulo-clavicularis, although the connection between scapula and clavicula was already long present. Only with the histological differentiation of the scapulo-coracoideum is it possible to indicate the boundary region as syndesmosis.

Of Gongylus ocellatus I had eleven series at my disposal, but without any older stadia, such as P. Q. and R. of Lacerta. As I had neither a full-grown specimen nor a good illustration of the shoulder-girdle of Gongylus, I am forced to describe the full-grown shoulder-girdle from data taken from the literature on the subject. Of the primary shoulder-girdle the coracoid only contains a fenestra principalis (apart from the canalis nervi supracoracoidei), just as in Lacerta; and further the cranial border of the scapulo-coracoideum shows a deep hollow. The clavicula, viz. the thin medio-caudal portion, according to SIEBENROCK, has an extremely irregular border.

Embryo Gongylus oc. T. possesses a blastematic shoulder-girdle continuous with the humerus. The diaphysis humeri already contains praechondrium. No trace of the clavicle is to be seen yet. The vaguely defined scapulo-coracoideum consists everywhere of blastema of equal denseness. The sternum lies at some distance from the coracoideum. Thus this embryo, as also Gongylus oc. G. which shows the same degree of development, corresponds to Lacerta ag. S.

The embryos Gongylus oc. A and B are of very nearly the same age. I shall base my description on embryo A on account of its better preserved colouring. The humerus diaphysis contains cartilage, which passes proximally over into praechondrium. This continues into the scapulo-coracoideum, but is there limited to the region bordering on the humerus. For the rest the primary girdle is blastematic, only more sharply defined than in embryo T. At this stage the sternal formation (temporary) has practically become one blastematic continuum with the coracoid. From the cranial border of the scapula a blastematic clavicular process goes out in a ventral direction. In the dorsal portion of it I found already a small fragment of bone tissue. The scapulo-coracoideum still forms one compact whole. This embryo thus agrees with Lacerta ag. I.

Embryo Gongylus oc. D. (embryo C. represents the same stadium). As in Lacerta ag. embryo J., the scapulo-coracoideum is here largely cartilaginous (scapulo, coracoideum s. str.). Epicoracoid and procoracoid are still praechondral. Two thinned blastemic parts have appeared; they correspond to the fenestra principalis and to the incisura scapulo-procoracoidea. The latter is closed by a ligament containing praechondrium. The clavicular blastema, as also the bony clavicula lying in it, have become longer (in a ventro-medial direction). Between clavicula and the praechondral-cartilaginous scapula is a strip which is still blastematic, representing the syndesmosis claviculo-scapularis.

In the next embryo of Gongylus oc. E. the thinner blastematic

parts in the primary girdle have given place to thin membranes of connective tissue, or, in other words, the fenestration is complete. The lig. scapulo-procoracoideum contains praechondrium which is connected with the cartilage of the girdle only by ligament. The bony tissue in the clavicular process has increased in extent.

In the remaining older embryos there is but little that is new to be remarked about the scapulo-coracoideum, (increase in size and commencement of calcification). The further development of the clavicular formation could not be traced. In the older embryos the latter appears in the ventral body-wall, and as it is but thinly covered with the skin, it is hardly possible in the frontally cut series to define the cell-thickening under the almost tangentially cut breast-skin from the blastemic clavicular formation. For the same reason the development of the episternum could not be traced in detail. In the oldest series a paired bony episternal formation was present. (Gongylus oc. E. and L.). The episternal formation of one half of the body has been demonstrated by me elsewhere. (Fig. 6)[1]).

Ptychozoon homalocephalum. Embryo A is still very young, the diaphysis humeri contains no cartilage as yet. The shoulder-girdle formation is continuous with the humerus formation. The blastematic scapulo-coracoideum is still rather vaguely outlined. The mesenchyme thickening, of which it is formed, is quite homogeneous. Nothing is to be seen yet of the fenestra principalis which occurs in the adult scapulo-coracoid; nor of the incisura scapulo procoracoidea. The bony clavicle, or even the blastematic formation of it, is still lacking.

Ptychozoon embryo B. Round the diaphysis-humeri lies a covering of perichondral bone. The primary shoulder-girdle shows cartilage. The more cranial portions are still praechondral (epicoracoid!). The fenestration of the first homogeneous compact coracoideum is already fairly complete. So the conditions correspond completely to those found in Lacerta J. and Gongylus D—E.

From the cranial border of the scapulo-coracoideum the bony clavicula proceeds, connected with the scapula by syndesmosis. Joined to the clavicula, just as in Lacerta, is a retroclavicular blastema. Of the episternum no traces of bone are to be found yet.

The examination of the embryos of Gongylus and Ptychozoon has thus led to the confirmation of most of the facts observed in Lacerta, namely the origin of the fenestra principalis and of the incisura scapulo-coracoidea by reduction of parts of an originally compact primary shoulder-girdle and also the primary connection of the

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[1]) Ch. van Gelderen, Proceedings. Kon. Acad. v Wetensch. Vol. XXIV, 1922.

blastematic clavicular formation with the scapula-coracoid. Others again of the results found in Lacerta could not be further verified, namely, the formation of each episternum half connected with the formation of the homolateral clavicula.

After the casuistic description in the above lines, I shall now with the help of figs. 5 and 6 summarize the development of the skeleton parts.

In Lacerta, as well as in Gongylus and Ptychozoon the cora-

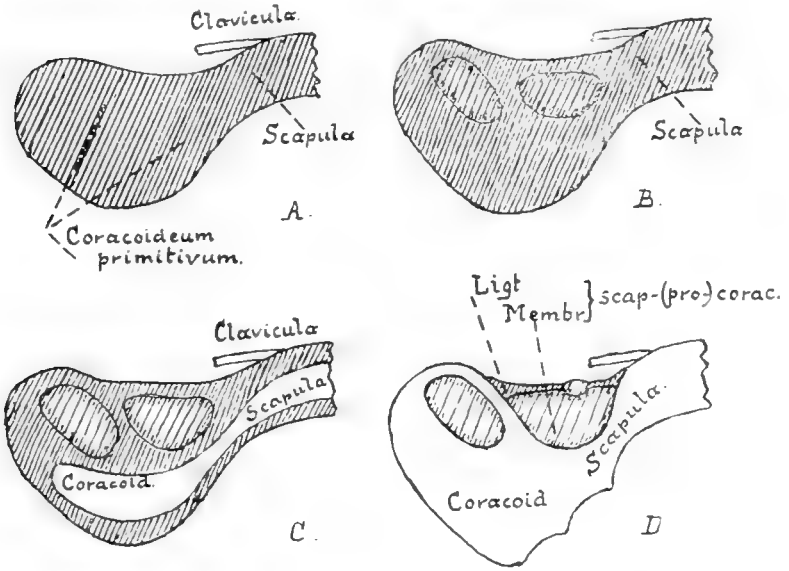

Fig. 5. Lacerta agilis. Schemata of the development of the coracoid.

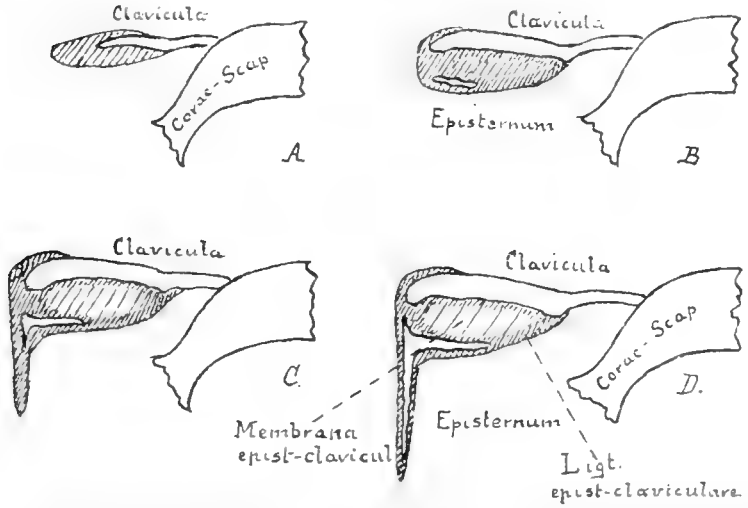

Fig. 6. Lacerta agilis. Schemata of the development of the secondary shoulder-girdle.

coideum in youthful stadia does not show a single trace of the fenestra principalis, and the region of the later incisura scapulo-procoracoidea still forms part of the homogeneous compact formation of the scapulo-coracoideum. When later a progressive histological differentiation occurs at the place where scapula, coracoideum s. str., epicoracoid and procoracoid will originate (formation of cartilage) this is accompanied by regressive changes at the place of the fenestra principalis and of the incisura scapulo-coracoidea, viz. a thinning of the blastema and finally reduction to a thin membrane of connective tissue. The cranial closing of the incisura does not occupy such a prominent place in these regressive changes. In the lig. scapulo-procoracoideum there are still cartilaginous insulae in the oldest embryos, which prove that this ligament is a reduced portion of the coracoid (in a wider sense). In fig. 5 the four schemata show the process of development of the coracoideum. The scapular end of the clavicle has not been hatched in each of the figures, and has been indicated in the same form. In the primary girdle hatching indicates blastema, praechondrium or connective tissue according as the hatching is more or less close. Entire absence of hatching indicates cartilage. The ab origine present nerve-canal has been omitted. The figures require no further explanation. Thus genetically both the fenestra principalis and the incisura scapulo-coracoidea, i.e. the membranes which enclose them, are parts of the shoulder-girdle. The lig. scapulo-coracoideum is, as it were, a reduced procoracoid.

As regards the episternum, in the youngest embryo in which a blastematic clavicular process was found, it was continuous with the primary shoulder-girdle. From which I deduce a genetic connection, in a sense that the clavicular blastema originates as a process of the scapulo-coracoid. It might still be opposed that the stadium in which this connection did not yet exist has not come into my hands, to which I might return that the bone in the blastematic clavicula first occurs dorsally and enlarges in a ventral direction, a symptom which, in my opinion, is strongly in favour of the genesis of the clavicula as a process of the scapulo-coracoideum.

The further development of the clavicular blastema I shall describe shortly with the help of fig. 6. In illustration *a*, already a fairly large bony clavicula is seen to be present in the blastemic clavicular process. In illustration *b* this is not more than a strip of bone lying in the cranial border of a large, for the rest homogeneous, blastema. In illustration *c* a further differentiation in the said blastema has commenced. It consists now of a centrum poorer in cells and a denser mesenchymatous border. In the latter, which represents a

portion of the episternum, a commencement of bone appears. Moreover, a thin caudally-directed blastematic process has also appeared. Finally illustration *d* shows the state of the episternum just before the right and left parts blend to one unpaired episternum. One blastema thus gives rise to one clavicula + the half of the episternum, augmented by the membrana episterno-clavicularis lying between them, which is nothing else than the reduced centrum of the original homogeneous blastema and by the lig. episterno-claviculare, that lies in the lateral border of the membrane of the same name.

If we now consider that of this joint process only that portion exists first from which the clavicula develops, I believe I may conclude that the episternum is pairedly formed from the clavicular processes. This manner of growth would imply that without the clavicle there would be no episternum, a state of matters as is seen in Rhiptoglossa. The conditions as found in adult crocodiles (an episternum but no clavicula) is explained by WIEDERSHEIM's discovery, namely that embryos of crocodilus contain a rudimentary clavicula. Of the peculiar manner of ossification of the clavicula, as described by GÖTTE and others, I could not find any trace.

We have still to see what comparative anatomical conclusions may be drawn from the above.

In the large comprehensive works upon comparative anatomy the opinion formulated by GEGENBAUR is expressed i.e. a great independence is ascribed to the cranial boundary of the fenestra principalis. This boundary, the procoracoid, is said to be the homologue of the similarly-named shoulder-girdle part of the Anura, Urodela and Chelonia. The procoracoid would thus occur in two main types, viz. as cranio-ventral process of the coracoideum in Urodela and Chelonia, and as cranial border of a fenestra in Anura and Sauria. Sphenodon has no procoracoid. The publications of GÖTTE, WIEDERSHEIM, BROOM and BOGOLJUBSKI have not been able to bring about any change in this theory. Now the coracoideum (in a wider sense) of the Sauria occurs in very different forms viz. 1st entirely without fenestrae in Sphenodon and Chamaeleo; 2nd with one fenestra, which has been named fenestra principalis on account of its frequent occurrence (FÜRBRINGER; "Hauptfenster" GEGENBAUR); 3rd with, besides the fenestra principalis, one or two more "Nebenfenster". The latter are said to have no morphological value, whereas the „Hauptfenster" has. Now we know from GÖTTE that in Chemidophorus spec. (3rd group: one principal and two minor fenestrae) all the fenestrae develop secondarily by regression of parts of the shoulder-girdle, or in other words, that the early-embryonal Saurian-

coracoid has the same form as that of Sphenodon. And, moreover, it follows from the above description of the development of the coracoid of the lizard, that the incisura scapulo-coracoidea has the same genesis as the dorsal fenestra of Chemidophorus, save that in Lacerta the cranial border also is practically entirely reduced (except for the remains of cartilage). Thus also the coracoideum of Lacerta with one fenestra contains, although it seems somewhat paradoxial, a second, dorsal fenestra. Consequently the so-called procoracoid of the lizard is the sum of what is in multiple fenestrated coracoidea termed the procoracoid and mesocoracoid (mesocoracoid lies between fenestra principalis and dorsal "Nebenfenster"). By the procoracoid in the order of Sauria are thus understood different parts of the girdle.

This fact, as well as the development of the coracoid (taken in a wider sense), induce me to side with GÖTTE; the whole ventral portion of the primary shoulder-girdle of the Sauria, with or without fenestrae, corresponds merely to the coracoideum of the Urodela and Anura. Respecting the latter GÖTTE has already demonstrated that their shoulder-girdle (with one fenestra) does not acquire its definite form by fenestration, but that it passes through an Urodelan stage (Rana esculenta). The fact that the adult shoulder-girdle of Lacerta corresponds to that of e.g. Rana thus depends upon caeno-genesis. The different parts of the two shoulder-girdles are not homologous.

The crocodilia, in which a procoracoid is lacking, will thus, like Sphenodon and Chamaeleo, possess a coracoid homologous with the whole pars coracoidea of the primary girdle of Lacerta. In short, as far as our knowledge extends at present (regarding Chelonia there are no genetic data) we are not obliged in the case of any reptile to assume a procoracoid that is homologous with the procoracoid of the Amphibia.

GEGENBAUER postulated the homology of the episternum of the reptiles and mammals; the difference in the histological structure (reptilia : bone; mammals : cartilage or bone), and in the histogeny reptilia : desmal, and in mammals chondral ossification) was evidently no objection, although he did consider as an objection the fact that the episternum of the Sauria lies ventral and that of the mammalia cranial from the sternum. Another weak point in the theory of this homology is that the episternum of the mammals is generally held to be a clavicular sternum, i.e. that we see in this episternum a product of the claviculae, whereas most of the researchers who studied the episternum of the Reptilia did not succeed in establishing a genetic connection between the clavicle and episternum.

Only GÖTTE saw (in Cnemidophorus) the episternum commence as caudal process of the clavicle. Well then, the foregoing casuistic demonstration shows the genetic connection of the clavicula and episternum, even though this is not so simple in Lacerta as was described by GÖTTE for Cnemidophorus. Herewith a new point of agreement with the episternum of the mammals has been found. Further, we have seen that the episternal halves lie at first quite cranially from the halves of the breast-bone. Only later, with the commencement of the longitudinal bar does a small portion of the episternum of the Sauria come to lie ventrally from the sternum. The different position of the episternum thus seems to exist but partially, and it occurs secondarily. The only difficulty in homologizing the episterna of Reptilia and Mammalia is thus the histogenetic difference. And GAUPP[1]) has demonstrated that too much importance must not be attached to this in general.

As for the clavicle, respecting the development of which my researches confirmed its primary connection with the scapulo-coracoideum, I do not deem it advisable for the present to enter into the discussion which is being carried on as to its homology, although in the theory developed bij GÖTHE (homology of the clavicle with the "Procoracoid" of the Amphibia, which he terms the clavicula) and of which WIEDERSHEIM,[2]) on the basis of his own investigations, proves himself an advocate (in the last edition of his "Vergleichende Anatomie" WIEDERSHEIM has changed his opinion, for what reason I do not know) there is undoubtedly a certain attraction.

SUMMARY.

1. Fenestrae in the shoulder-girdle of the Sauria develop secondarilly, in a girdle of the type of Sphenodon.

2. The incisura scapulo-(pro)coracoidea is likewise a fenestra of which the cranial border, except for some cartilaginous remains, is reduced to a ligament.

3. The clavicle originates as a blastematic process from the scapulo-coracoideum.

4. The episternum proceeds from a paired formation. This formation is the product of the homolateral clavicular process.

5. As long as there are no data of the development of the girdle of Chelonia, there is nothing which obliges us to assume a procoracoid in any reptile, homologous with that of the Amphibia.

[1]) E. GAUPP, Kopfskelett in HERTWIG's Handbuch. Jena, 1905.

[2]) R. WIEDERSHEIM, Grundrisz d. Vergl. Anat. d. Wirbelthiere. 4e Aufl. Jena 1898.

Chemistry. — "*Provisional Communication on Boric Acid Compounds of some Organic Substances containing more than one Hydroxyl-Group. Boron as a Pentavalent Element.*" By P. H. HERMANS. (Communicated by Prof. J. BÖESEKEN).

(Communicated at the meeting of December 30, 1922).

The behaviour of boric acid towards hydroxyl-containing organic substances is striking in many respects. The extra-ordinary ease and rapidity with which it forms esters of the type of $B(OR)_3$ with the ordinary saturated alcohols, also when a catalyst is absent, is a totally unexpected property for a weak, and for the rest mono-basic acid such as boric acid, and in this respect it is unequalled.

Still more interesting is the action of boric acid on the aqueous solutions of multi-valent alcohols and other substances rich in hydroxyl, such as some sugars. It has been known for a long time that these mixed solutions sometimes present a much greater hydrogen ion concentration than a solution of boric acid only. The alkaline reaction of a borax solution can even become an acid one by addition of substances such as mannite[1]). Also the influence of boric acid or borates on the optical rotatory power of such substances rich in hydroxyl, was early observed. Undoubtedly these phenomena point to compounds which boric acid forms with the substances mentioned above. Several investigators have expressed their opinion about the nature of these compounds[1]). Mostly it is assumed that acid boric acid esters are formed which possess a higher degree of acidity than free boric acid. Systematic attempts to find out more about these compounds through their isolation, have seldom been made, at least they have not been very successful.

In 1869 DUVE[2]) described a series of salts of different boro-tartaric acids, which however present the apparance of glassy, non-crystallizing masses or amorphous precipitates, the individuality of which is open to doubt. The same principle applies to most of the boro-citric acid

[1]) We will postpone the older and more recent literature on this subject to a following publication.

[2]) Vierteljahrsschr. pr. Pharm. XVIII, 321.

salts described by SCHEIBE[1]) in 1879 and 1880, with the exception however of a potassium salt, which was considered to have the formula $C_{12}H_{13}K(BO)_2O_{14} . 2H_2O$, crystallizes beautifully, and to which we refer below[2]).

Also among the salts of boro-salicylic acid described for the first time by JAHNS in 1878[3]) there are some well-crystallised compounds.

The first who inquired more systematically into the influence on the acidity of boric acid by hydroxyl-containing substances, was MAGNANINI, who published a series of papers on the influence which these substances have on the conductivity (and some other physical constants) of boric acid solutions.[4]) The number of compounds examined by him is very large, and he pointed out the influence of the constitution in connection with the occurrence or non occurrence of an increase of conductivity. He found a.o. that this was only observed in *α*-oxy acids, and *not* when the OH-group is somewhere else, it *was* found in aromatic *o*-oxy carbonic acids, *not* in the *m*- and *p*-isomers, it *was* found in *o*-diphenols, not in *m*- and *p*-diphenols.

These researches have been continued and extended by BÖESEKEN (and collaborators)[5]), who assumed, discovered, and worked out an influence of the *steric* configuration by the side of the constitional influence.

In his hands the Magnanini "boric acid method" became an important instrument, not only for the determination of the constitution and configuration, but also for our stereo-chemical views in general. These results reached their acme in the application of the method to the sugars and their derivatives, the isomeric tartaric acids, and the saturated cyclic vic. diols.

In his "Lagerung der Atome im Raume" VAN 'T HOFF already expressed his opinion that in the compounds which are responsible for the phenomena in question, the boron atom might be part of a ring-system, and that this ring could close only when certain condi-

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[1]) Russ. Zeitschr. f. Pharm. **18**, 257, 289, 321; **19**, 513. Pharm. Journ. and Trans. (3) **11**, 389.

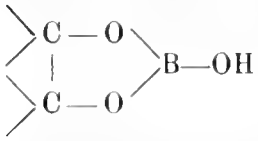
[2]) We have not yet been able to test entirely the records given by KLEIN in 1878 on mannite-boric acid salts of rather complicated constitution. Probably we have to do with not accurately defined substances also here.

[3]) Arch. der Pharm. (3) **12**, 212.

[4]) Z. phys. Ch. **6**, 58. Gazz. chim. Ital. **20**, 441, 448, 453; **21**, 134, 215; **22**, 541; **23**, 197. Acad. dei Lincei Rend. (4) **6a**, 411, 457.

[5]) E.g. These Proc. Vol. XV, p. 216 (1912); Vol. XVIII, p. 1647, 1654 (1915); Vol. XXI, p. 80 (1918); Vol. XXIII, p. 69 (1920); Verslag van de gewone vergaderingen K. Akad. v. Wet. Amsterdam Dl. XXIX, p. 368, 924 (1921). Chem. Weekbl. **19**, 207. Recueil **40**, 354, 558.

tions were fulfilled. The accompanying formula was the simplest, and was used by different investigators (see e.g. ABEGG's Handboek III pg. 43), and also served BÖESEKEN as a working-scheme. According to this assumption one molecule of di-oxy-compound, therefore, combines with one molecule of boric acid with loss of water to a cyclic compound, which would have stronger acid properties.



Different investigators have supposed other relations as to the number of molecules combining than these to account for the physico-chemical behaviour of the mixed boric acid-polyoxyderivative-solutions, but the grounds on which these suppositions were based, are generally uncertain, and often conflicting. Up till now nothing could be said with certainty about the general type on which the acid complexes are based. The observations agree with each other only in so far that these compounds are almost completely dissociated in aqueous solution, and that their formation is favoured by increase of the concentration of the components. It further appeared from different investigations that in some cases (a.o. with substances like mannite and dulcite); several compounds of different composition must be present in the solution, of which however, it can, not be ascertained which are the typically acid ones.

In 1911 FOX and GAUGE[1]) described the first compound of boric acid with a multi-valent alcohol which is well crystallized. From an alcoholic solution they obtained a mannito boric acid $C_6H_{15}O_8B$. They do not however, say, anything about the constitution. A second compound was described by DERX[2]), viz. the crystallized cis-cyclo heptane diol boric acid. He determined the boron content by titration and gave the following formula based on that C_7H_{14} :  BOH,H_2O.

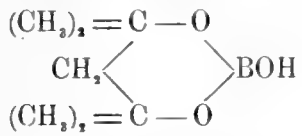
We might consider the mannito boric acid in an analogous way as $C_6H_{13}O_4$:  BOH, H_2O. Both alcohols increase the conductivity of the boric acid in a high degree. There were, as we see, reasons to look upon these compounds as being the strong complex acids that bring about these phenomena. The impetus for making a new attempt to ascertain the nature of the strongly acid boric acid compounds was given by the following accidental discovery. The 2.4. dimethyl pentane 2.4. diol[3]) when reacting on boric acid (even in very diluted

[1]) Soc. **99**, 1075.

[2]) Recueil **41**, 340 (1922).

[3]) Obtained from CH_3MgJ and diacetone alcohol. Mr. LANGEDIJK drew my attention to a new method of preparation, which renders this latter substance very easily accessible. (ADAMS, Organic Syntheses 1921, p. 45).

aqueous solutions) gives a beautifully crystallizing and only slightly soluble compound, to which the following formula must be assigned as a result of analysis and examination of properties:



	C	H	B
Calculated	53,16	9,17	6,90
Found	53,10	9,07	6,80
	53,27	9,12	6,95

The compound has an exceedingly great crystallizing power, melts not entirely sharply at 100—102°, is somewhat volatile, and has a pleasant odour strongly reminiscent of saffron. It is, however, by no means a strong acid, and like boric acid it hardly colours blue litmuspaper red, and in accordance with this the said diol (which is readily soluble in water) does not bring about any increase of conductivity of the boric acid, as was to be expected from such a 1.3 diol according to the data collected by BÖESEKEN up till now.

As a result of this I believed that the really stronger acid boro-complexes must possess another structure than had been assumed so far, and this was soon confirmed. I succeeded, though only after some more difficulties, in isolating analogous and likewise only exceedingly weakly acid boron-compounds of 2.4.dimethyl hexane 2.4.diol, 2.4.pentane diol, and pinacone, likewise diols which do not appreciably influence the conductivity of boric acid. In the case of tri-methylene glycol, ordinary glycol, and cis-1.2.cyclo hexane diol the existence of compounds could be shown, but attempts to isolate them in a pure state failed [1]). It is probable that all the 1.2 and 1.3 diols are able to form such compounds with boric acid, that in many cases, however, they can be separated only with great difficulty, if at all, in consequence of unfavourable solubility conditions and similar difficulties. In aqueous solution they are partially split up into their components. The compound described first, at 0° in 0,1 normal solution for 75 %. The readiness with which this compound is obtained, is owing to its slight solubility in water (4,46 g. in 100 cc. solution at 25°), which is still smaller than that of boric acid. It dissolves in diluted alkalies, probably accompanied by formation of a potassium salt, which is, however, also dissociated, as pure diol is withdrawn from the solution by ether. The liquid free diol is salted out by strong potassium hydroxide; the potassium salt itself could not be isolated as yet. In the cis. 1.2 cyclo hexane diol, however, the corresponding potassium salt is slightly soluble in an excess of strong potassium hydroxide, and crystallizes out,

[1]) I hope to discuss the details of the preparation in my doctor's dissertation.

before the limit of solubility of the free diol is reached. It can be obtained pure by sucking off and washing with alcohol of 96 %. It is a compound that was discovered already before by BÖESEKEN and VAN GIFFEN [1]) but which was not isolated and more closely examined then.

The aqueous solution of this potassium salt has an alkaline reaction, and the potassium can be determined quantitatively by titration with methyl orange, which proves anew that the corresponding complex boric acid is an exceedingly weak acid. Besides this compound $C_6H_{12}O_3BK$, a dipotassium compound $C_6H_{13}O_4BK$ was obtained from very strong potassium hydroxide.

Under the same circumstances crystalline compounds can also be obtained from cis-1.2 hydridene diol, cis 1.2 cyclopentane diol, and cis-1.2 and cis 2.3 tetrahydronaphtalene diols. These compounds consist of potassium salts of either of the two types or of both, some of which could, however, not yet be obtained pure and free from potassium hydroxide being sometimes (o.a. in the case of hydridene diol) too easily soluble, both in water and in alcohol, so that they cannot be washed with one of these solvents.

It is remarkable that in the di-potassium compound of cis-cyclohexane diol boric acid both potassium atoms can be titrated with methyl orange and HCl, but that in the corresponding compound of cis-cyclo pentane diol only one of the two potassium atoms can thus be titrated. We reserve a further discussion of this point till some future occasion.

We may point out here that these compounds supply us with a method to separate isomer cis- and trans- cyclic 1.2 diols from each other, the latter not giving such compounds. For in many cases the beautiful method of separation with the aid of acetone compounds found by VAN LOON [2]) is attended with important drawbacks, as has appeared from another investigation (to which I hope to refer later on. [3])

The fact that the formation of these cyclic and only exceedingly weakly acid compounds seems to be a general property of the 1.2 and 1.3 diols, leads to the conclusion that the more strongly acid boro-complexes, which arise in appreciable quantities only in poly

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[1]) Recueil **39**, 183 (1920).

[2]) These Proc. Vol. XXIII p. 60 (1920), and proefschrift Delft p. 59; of BÖESEKEN and DERX, Recueil **40**, 519.

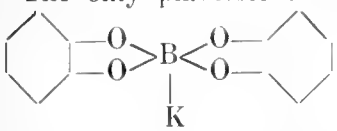
[3]) The new method of separation has already been sucessfully applied by Mr. MAAN to the methyl 1. cyclohexane 1.2 diols. The cis-diol was obtained in a much purer state than by the acetone method, as the action of acids is now fully eliminated.

oxy derivatives with "favourably" orientated OH-groups, belong to another group. In 1917 BÖESEKEN (in collaboration with OBREEN and Miss VAN HAEFTEN, Recueil **37**, 184) described several salts of pyro catechin boric acids of pretty complicated constitution. As the former had already for some time considered the analysis values found to be uncertain, and as the boric acid compounds described above appeared to be by no means particularly complex, there was sufficient occasion to subject the beautifully crystallized salts of pyro catechin boric acid to a renewed investigation, the more so because pyro-catechin greatly increases the conductivity of boric acid, so that accordingly the other type of compounds might be expected here. This expectation was confirmed: the potassium salt appeared to possess the forumla $C_{12}H_8O_4BK$ [1]).

The carbon was determined by the wet way according to the method of MEISENHEIMER, the hydrogen according to a simplified method worked out by myself, about which more will be given later. The following values were found:

	C 54,28; 54,26	H 3,03	B 4,3	K 14,7
Calculated	54,14	3,00	4,1	14,7

The only plausible structure that answers to this is:



(potassium dipyrocatechinborate)

The boron is here pentavalent, or has (according to WERNER's nomenclature) the coordination value four, just as in the well-known compounds KBF_4 and $Na[B(OC_2H_5)_4]$. The latter was obtained from $NaOC_2H_5$ and boric acid triethyl ester [2]).

The free dipyro catechin boric acid can be obtained by heating the anilin salt in a vacuum of some mm. Hg. at 100—120°, in which the aniline escapes quantitatively. Mr. MEULENHOFF (who has undertaken a closer investigation of these derivatives at Prof. BÖESEKEN's request), found that the acid obtained in this way can be prepared in perfectly pure condition by sublimation in vacuum at about 200°.

The potassium salt described is very little soluble in cold water, and this solution gives an alkaline reaction, probably in consequence of the fact that a dissociation in pyrocatechin and potassiumborate (possibly first in pyro catechin and mono-pyro catechin borate) sets in.

[1]) The erroneous constitution, given in the last-mentioned paper is owing, partly to an error of calculation that has crept in, partly to the fact that substances containing boron and being rich in C, are not easily combustible.

[2]) COPEAU C.r. **127**, 721 (1898) e.g. LIVIO CAMBI, Acad. dei Lincei Rend (5), **23a**, p. 244.

The potassium can, however, *not* be determined quantitatively by titration, from which it appears that we have to do here with an acid that is stronger than the mono diol boric acids described. Unchanged pyro catechin can again be withdrawn with ether from the aqueous solution.

The other complexes stronger than boric acid are probably also built up according to the type of di-pyro catechin boric acid.

So far, however, the separation of a derivative that probably belongs to this type, has succeeded only in one diol of abiphatic character, i. e. in the cis-cyclo heptane 1.2 diol. This diol was first prepared by DERX from suberic acid; he ascertained that it increases the conductivity of boric acid in a great degree, and states that he has succeeded in separating a solid boric acid compound, the B-content of which agrees with the formula $C_7H_{14}:\begin{matrix}O\\O\end{matrix}>BOH.H_2O$[1]). As only 0.2 gramme of this diol were available (prepared by DERX), I have carried out the following experiments on micro-chemical scale under the microscope.

With an almost saturated boric acid solution the diol gives rise to the formation of an oil which is only soluble in much water.[2]) This oil is probably the liquid diciscycloheptanediolboric acid, from which more or less accidentally DERX obtained the mono ciscyclo heptanediolboric acid as a solid substance. On addition of a little strong potassium hydroxide an aqueous suspension of this oil gives crystals of a potassium salt, while also a drop of aniline is dissolved with separation of beautiful crystal needles, which are, however, pretty readily soluble in water and other solvents.

In connection with the small quantity of material available it was better to abandon the idea of an examination of the liquid compound itself, and to try and separate one of the salts. For this purpose I chose the aniline salt to avoid the possibility that with KOH, as with the other cyclic diols, a compound of the monotype would again crystallize out. It might, however, be expected of aniline that it would give a crystallized salt only with a stronger acid.

Only a few tenths of milligrammes of the aniline salt were obtained in a sufficiently pure condition with a melting-point of about 50°. Mr. H. GRAVESTEIN was so kind as to take the execution

[1]) Proefschrift Delft and Recueil **43**, 340 (1922).

[2]) As Mr. DERX communicated to me in a conversation, this oil was also observed by him, but considered as an impurity. He has obtained the solid boric acid compound described by him in a small quantity from a pretty large quantity of this oil and through rather complicated manipulations.

of a micro-elementary analysis upon him. The combustion of this boron-containing compound requires, however, special preliminary experiments, and has not yet been accomplished; the results will be published later. A determination of the boron-content yielded the following results: 9,76 mgr. were dissolved with 1 gr. of pure mannite in 10 cc. of water, and titrated with 0,0097 N barite water (under similar circumstances tested by pure boric acid) and phenolftalein as indicator. Consumed 2,60 cc. Calculated for $C_{19}H_{26}O_4NB$ ... 2,99 % B; found 2,8 % B.

To all probability we have here actually to do with diciscycloheptane diol boric acid aniline.

In this compound the aniline is bound still more loosely than in aniline dipyro catechin boric acid. In vacuum at room temperature it already escapes, the remaining part becoming liquid. The liquid residue becomes solid again by the addition of aniline. Also on evaporation of the aqueous solution over concentrated sulphuric acid an oil remains behind, which becomes solid again by the addition of aniline. Beside a dish with pumice saturated with aniline the salt can, however, be regained unchanged by evaporation of the aqueous solution in vacuum.

Di-ciscycloheptanediol-boric acid is, therefore probably a much weaker acid than dipyrocatechin-boric acid, and the great increase of conductivity of boric acid by pyro-catechin must, therefore, be put to the account not only of the favourable orientation of the OH-groups, but also partially to the account of the acidifying influence of pyrocatechin as such. This admonishes to caution in making comparisons with regard to the orientation of the HO-groups between diols that are not very much a like in structure, exclusively on the ground of measurements of the conductivity. This point was, indeed, already foreseen by BÖESEKEN, and was a.o. mentioned by VAN LOON [1]) and LIEMPT [2]).

That also the increase of conductivity caused by the α-oxy acids in the boric acid is probably to be attributed to the formation of complex acids built in an analogous way, we have been able to make plausible by showing that the analyses of the SCHEIBE's boro dicitric acid potassium [3]) and of the zincous salt of JAHNS' boro disalicylic acid are in agreement with the formulae:

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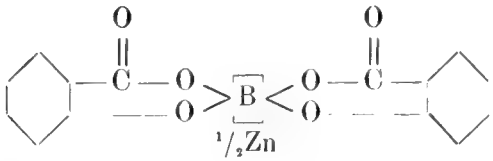
[1]) Proefschrift Delft, p. 56.

[2]) Recueil **39**, 359.

[3]) Also the free acid has been separated crystalline by SCHEIBE and by me. It is, however, difficult to purify and dry. SCHEIBE's analysis, which I have not yet checked, is in harmony with my view.

and



In the cis 1.2 tetrahydro naphthalene diol and the cis 1.2 hydrindene diol (both increasers of conductivity) the formation of an oil can also be observed in supersaturated solution by addition of boric acid. Aniline dissolves in these solutions, but a salt does not crystallize out.

I will state here that a further proof of the constitution of these boron compounds can be furnished, if experiments to split one of the asymmetric derivatives e.g. boron dicitric acid or nitro pyro catechin derivatives into optical antipodes, should be successful.

It is, therefore, probable, that we shall have to see the derivates of an unknown acid $\begin{bmatrix} HO. & \\ HO. & B \end{bmatrix} \begin{matrix} .OH \\ .OH \end{matrix}$ H in the more strongly acid boron complexes. The material described here may possibly be able to throw some light on the so far obscure constitution of the boron acids. As a working hypothesis we will now assume what follows:

1. Maintaining the coordination value four for boron, the formula for meta-boric acid becomes:

$$[O : B : O] H.$$

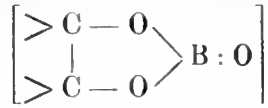
2. The mono-basic ortho boric acid is considered as meta-boric acid being hydrated one-sidedly:

$$\begin{bmatrix} HO \\ HO \end{bmatrix} > B : O \Big] H$$

which can, however, pass into (is in equilibrium with) the genuine trihydroxyl boron $B(OH)_3$, from which the well-known esters $B(OR)_3$ have been derived. The first form is present to a certain percentage particularly in aqueous solution, the second form especially in organic solvents such as alcohol. The volatility of boric acid might be ascribed to the presence of $B(OH)_3$.

3. We start from the principle that a hydroxyl group bound to boron forms exceedingly easily an esterlike compound with alcohols.

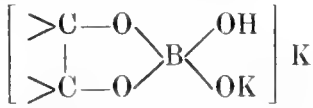
This enables the $[(HO)_2B:O]H$ present in water to form compounds with a number of glycols and α-oxy acids, of the following type:



[1])

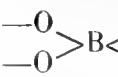
Like boric acid these acids are very weak.

4. On the other side of the boron atom a compound can now be formed with a second molecule of diol or oxy acid with loss of water. Whether then a molecule of water is first admitted, may be left undecided for the present. The existence of dipotassium salts, to which we can assign the structure:



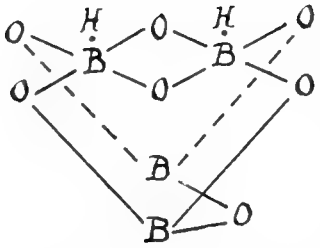
may possibly plead in favour of this, like the presence of an extra molecule of water in DERX's solid mono cyclo heptane diol boric acid and FOX and GAUGE's mono mannite boric acid.

A second molecule of dioxy compound is, however, received in diluted aqueous solution in appreciable quantities only when certain favourable conditions are realized, i.e. with a favourable steric situation of the hydroxyl groups in the diol or oxy acid. The tendency to the formation of a di-compound is, accordingly, smaller than that to the formation of the mono-derivatives, and the former seems, therefore, to be very sensitive to the value of the ring-tension in the ring to be formed. This fact constitutes the hypothetical foundation of BÖESEKEN's boric acid method.

5. It is known that the poly-boric acids whose presence must be assumed in alkaline solutions, are stronger acids than ortho-boric acid. Plausible structure formulae could not be drawn up for this large series of acids as yet on the basis of trivalent boron. Possibly they too possess the grouping  . Maintaining the assumption that to each H-atom that can be replaced by metals belongs one pentavalent B-atom, the other B-atoms being trivalent, a structure schema may be constructed for a great number of poly boric acids. Tetra borix acid, which forms the foundation of borax, possesses e.g. the scheme: [2])

[1]) In this connection it will be of importance to examine whether in the saponification of $B(OR)_3$ by water the presence of a relatively stable intermediate product $(RO)_2BOH$ can be shown.

[2]) This representation does not lay claim, of course, to be anything more than a scheme.



In conclusion it may still be mentioned that the question what place three remarkable "acid boric acid esters" described by WOHL and NEUBERG [1]) and also the boric acid complexes [2]) found by GRÜN and NOSSOWITCH, occupy in this respect must still be made a subject of investigation.

I may still be allowed to express my great indebtedness to Prof. BÖESEKEN for the kind interest which he evinced in this investigation carried out in his laboratory.

Delft, December, 1922. *Organic Lab. of the Technical Univ.*

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[1]) Ber **32**, 3488 (1899).

[2]) Sitz. Ber der Akad. der Wiss. Wien M. N. Cl. **125**, 2B. 171 (1916).

Chemistry. — "*The Electro-viscous Effect in Rubbersol.*" By Prof. H. R. KRUYT and W. A. N. EGGINK.

(Communicated at the meeting of January 27, 1922).

1. Researches on agarsol [1] have taught that the relation [2]) between the charge of dispersed particles and the viscosity of the dispersed system manifests itself clearly in those sols in which the charge can be considerably modified without the colloid system as such being annihilated, i.e. in those systems of which the stability does not only depend on their charge, but in which also hydration (more general: solvation) protects the system. The conceptions about the stability of the lyophile sol may be applied throughout the territory of the emulsoids [3]), at least when water is taken as the substance in which the dispersion takes place. Our attention was, however, drawn by a remark on p. 570 of O. DE VRIES' Estate Rubber [4]), where it is stated that increase resp. decrease of the viscosity of a benzene rubber solution is brought about by shaking it with a few drops of a solution of alkali resp. of acid or salt.

As it seems as if this is a question of an electro-viscous effect, we have examined what influence electrolytes have on the viscosity of solutions of rubber in benzene.

2. Sols were used prepared in the following way: 1 gr. of a certain crêpe-rubber was added to 300 cm. of benzene, after 24 hours it was carefully shaken, and the sol was poured through a folded paper filter. Then benzene solutions of the electrolytes were made; the liquids which were to be examined viscosimetrically, were prepared by mixing a volume of sol with a volume of the solution of the electrolyte (resp. a volume of benzene, for the zero-standard); or as far as the measurement of rubberless liquids are concerned by diluting electrolyte solutions with benzene, as they were diluted with sol just before. At the

[1]) H. R. KRUYT and H. G. DE JONG, Z. physik. Chem. **100**, 250 (1922).

[2]) M. VON SMOLUCHOWSKI, Koll. Z. **18**, 190 (1918). We prefer the term electro-viscous to quasi-viscous, which v. SMOLUCHOWSKI uses, but which may give rise to misunderstanding.

[3]) H. R. KRUYT, Koll. Z. **31**, 338 (1922).

[4]) Batavia 1920.

beginning and at the end of every series the electrolyte-free mixture was measured, and when there was a difference, a correction was applied to the intermediate values. The measurements have been performed in an OSTWALD viscosimeter [1]) and at 25°.

In the subjoined tables the concentrations given are end-concentrations, the viscosity of benzene is put at 1.000, η_e is the viscosity of an electrolyte solution, η_{s+e} that of a rubber sol with equal electrolyte concentration. The relation of these quantities is given under $\frac{\eta_{s+e}}{\eta_e}$, after the said correction for the time reaction has been applied.

TABLE I.

Influence of benzoic acid on the viscosity of rubbersols.

Conc. Benzoic acid mMol p. L.	Viscosity benzene + benzoic acid η_e	Viscosity rubber + benzoic acid η_{s+e}	$\frac{\eta_{s+}}{\eta_e}$ corrected
0	1.000	1.698	1.698
6	— [2])	1.635	1.633
12	— [2])	1.601	1.598
24	— [2])	1.584	1.577
48	1.010	1.565	1.552
96	1.018	1.559	1.533
192	1.036	1.574	1.522
0	1.000	1.695	1.698

In fig. 1 these results are represented graphically. Corresponding determinations have been carried out with acetic acid, hydrochloric acid, sulphuric acid, sulphuretted hydrogen and mercury chloride. Essentially the results are the same, the viscosity reducing action alone is different; most for hydrochloric acid, in which already $1^1/_4$ mMol per litre reduces the viscosity from 1,573 to 1,486

Ammonia shows a very remarkable behaviour; the results are recorded in table II.

[1]) With observance of all precautions according to H. G. BUNGENBERG DE JONG, Rec. Trav. chim. Pays Bas **43**, 1 (1923).

[2]) Interpolated between the values for 0 and 48 mMol per l.

The viscosity of the NH_3-benzene mixtures (η_e) did not appreciably differ from that of benzene.

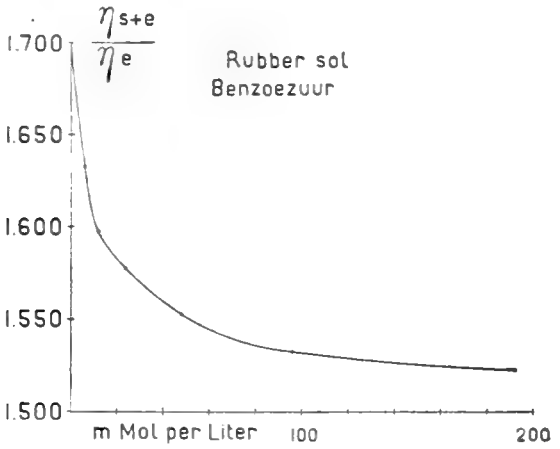

Fig. 1.

TABLE II.
Influence of ammonia on the viscosity of rubbersols.

Conc. ammonia mMol p. L.	Viscosity rubber + NH_3 η_{s+e}	$\frac{\eta_{s+e}}{\eta_e}$ corrected
0	1.608	1.608
0.37	1.616	1.616
0.75	1.622	1.621
1.49	1.625	1.624
2.98	1.622	1.620
5.96	1.620	1.618
11.92	1.620	1.618
23.85	1.621	1.618
0	1.611	1.608

It appears from this that the viscosity of the sols rises by addition of ammonia, reaches a maximum, and then descends.

3. The great change of viscosity by an added substance in so small a concentration as is the case with the acids, certainly makes the impression of an electro-viscous effect.

In fig. 2 a graphic representation is given of the results of all examined electrolytes, but only for concentrations below 6 mMol per litre. We have always taken the relative viscosity of the electrolyte-

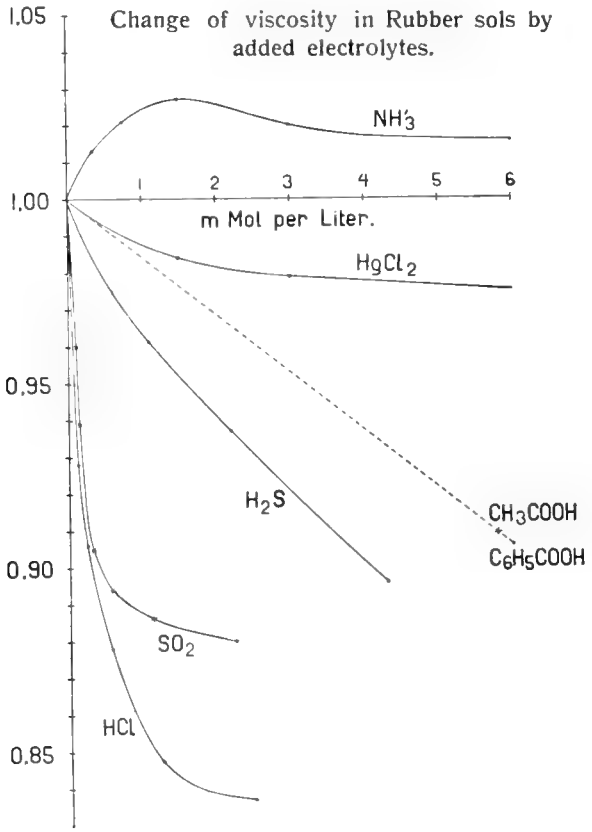

Fig. 2.

free sol as unit, and then plotted the relation $\frac{\eta_{s+e}-1}{\eta_s-1}$ as ordinates.

The behaviour of NH_3 is in striking agreement with this explanation: for it has appeared in all investigations on capillary-electric phenomena that alcalic substances give a higher potential to a negatively charged wall, lowering it again on further addition [1]). In harmony with this researches in this laboratory by Mr. LIER confirmed the occurrence of an increasing electro-viscosity by hydroxyl

[1]) See e.g. G. VON ELISSAFFOF, Z. physik. Chem. **79**, 385 (1912); R. ELLIS, Z. physik. Chem. **80**, 597 (1912); H. R. KRUYT and A. E. VAN ARKEL, Koll. Z. **32**, 29 (1923).

ions in casein, those by Dr. RUNGENBERG DE JONG did so in amylum [1]). Especially this positive effect is, therefore, a forcible argument in favour of our view.

4. Two objections may, however, be raised. The first is: is NH_3 and also is SO_2 in benzene an electrolyte? For the formation of an OH-ion from NH_4OH, resp. an H-ion from H_2SO_3 the presence of water is required. If it is, however, calculated how much water is required with the very small concentrations in question, values are found which are only a small part of the solubility of water in benzene, a quantity that is certainly always present in benzene that has not been dried with particular precautions.

A second objection might be supplied by the question whether the electrolytes in benzene are sufficiently dissociated to put these phenomena to their account. WALDEN's [2]) investigations, however, may reassure us in this respect. Equal dissociation in two solvents is attained at dilutions that are to each other as the third powers of the dielectricity constants, i.e. for the relation bezene-water $4{,}7 \times 10^4$. Hence the succession of the strengths is the same in two solvents. If now according to WALDEN the α is calculated for HCl in benzene, conc. 1 mMol per litre, taking into account that the constant from OSTWALD's law of dilution varies proportional to the concentration of the undissociated molecules, the value $\alpha = 0{,}32$ is found. Here there is, therefore, a considerable ionisation. With a weak acid, as benzoic acid, the dissociation is, indeed, more greatly lowered by benzene than in the case that the substance is dissolved in water, but in the concentrations in question here, it is yet not less than $^1/_{200}$ of that in water.

There is, however, a striking difference between the electro-viscous phenomena in water and those in benzene. In water the curves for cations of equal valency coincide, but this is not the case for our curves, though they all have the H-ion as discharging ion (with the exception of $HgCl_2$). It makes the impression that the real H-ion concentration plays a part: for the anorganic acids discharge in the order of their strength. The two organic acids are, indeed, stronger than H_2S, but organic anions always counteract the discharge through their greater absorbability, the aromatic ion more strongly than the aliphatic one, thus compensating its greater strength. The exceedingly

[1]) Still unpublished; compare however for casein W. PAULI, Kolloidchemie der Eiweisskörper. 81 et seq. (Dresden-Leipzig 1920) and for amylum M. SAMEC Koll. Beih. **4**, 132 (1913), **5**, 141 (1914) etc.

[2]) P. WALDEN, Z. physik. Chem. **94**, 363 (1920).

weakly ionized $HgCl_2$ has accordingly the smallest discharging power.

5. In conclusion we wish to draw attention to a consequence of the stated electro-viscous character of part of the viscosity in the rubbersol. It has often been tried to compare the quality of different samples of rubber by measuring the viscosity of benzene solutions of the same concentration. The choice of this property for a comparison is not unlogical, as in the first instance the viscosity may be considered as a measure for the solvation, and this can be taken into account as a real colloid characteristic. Experience now actually teaches that there exists a certain correlation between the viscosity of the sol and the mechanic properties which determine the quality; it is, however, no more than a vague correlation. It has, however, appeared above, that part of the viscosity is *not* in connection with the solvation, but is of electric origin, and has, therefore, a perfectly casual character, dependent on the soluble components which accompany the rubber and which have no influence on the mechanic properties in these minimum concentrations. If it is, therefore, desired to detect a functional relation between viscosity and the properties of the quality of the rubber, it will be necessary to eliminate beforehand the electro-viscous effect by judicious addition of the electrolyte.

We consider the knowledge of these electro-viscous phenomena of importance from the standpoint of pure colloid chemistry, because they open a way to the study of the electric relations in non-aqueous sols.

Utrecht, VAN 'T HOFF-laboratory, 1922.

Physics. — "*Isotherms of di-atomic substances and their binary mixtures.* XX. *The critical curve of oxygen-nitrogen mixtures, the critical phenomena and some isotherms of two mixtures with* 50% *and* 75% *by volume of oxygen in the neighbourhood of the critical point.*" By J. P. KUENEN †, T. VERSCHOYLE and A. TH. VAN URK. Communication No. 161 from the Physical Laboratory at Leiden. (Dr. KAMERLINGH ONNES, holding his deeply regretted friend in affectionate memory, is glad to perform the honourable task of presenting for the Proceedings a paper by the late Dr. KUENEN which was made almost ready for the press),

(Communicated at the meeting of November 25th, 1922).

§ 1. *Introduction.*

This work is a continuation of that of KUENEN and CLARK [1]), the investigations, however, being carried out in such a way as to allow of the construction of complete isotherms, which involved a slight modification of the apparatus then used. The mixtures on which measurements were made, contained respectively 50% and 75% by volume of oxygen, and it was found that these gave sufficient data for the construction of the critical curve. This was found to be almost a straight line, while both critical constants proved to be an almost linear function of the composition.

§ 2. *Preparation of the mixtures.*

A simple mixing apparatus was employed, consisting essentially of a measuring-bulb of about a litre capacity, surrounded by a water-jacket and connected with an open manometer tube, in which the gases to be mixed were measured, and of a mixing bulb of some 2 litres capacity. Through 3-way taps the bulbs could be put in connection with each other, the source of gas, the piezometer to be filled, and a vacuum pump, as might be desired. In measuring the relative quantities of the gases to be mixed, the mercury was

[1]) J. P. KUENEN and A. L. CLARK. These Proc. XIX (2) pg. 1088. (Febr. 1917.) Leiden Comm. N°. 150*b*.

always brought to a mark on the stem below the measuring-bulb, when the pressure-difference was read with a cathetometer, allowing for the height of the barometer, while the temperature of the waterjacket was observed.

The oxygen was prepared from pure potassium permanganate [1]) and the nitrogen from solutions of sodium nitrite and ammonium chloride [1]). In the preparation of nitrogen the air was first driven out by carbon dioxide; to free the nitrogen from the remaining carbon it was frozen out in liquid oxygen boiling under reduced pressure, before being used. The first part of the evaporating nitrogen was pumped off, and the next part used to fill the apparatus; the residue was also removed.

§ 3. *Apparatus.*

a. Piezometer.

The usual type of piezometer used in the laboratory at Leiden was employed. The volume of the large reservoir was some 500 cm³. and that of the small reservoir about 1 cm³. The form of the latter differed from that used in Comm. N°. 150*b* as an other method of stirring was adopted, to avoid the difficulties mentioned there. It consisted of a capillary about 50 cm. long *C*, with a bore of $\pm$ 0.2 mm., and a cylindrical bulb *R*, about 10 cm. long, which was carefully rounded at the lower end, so that the stirrer *r* contained within (a short length of iron wire enclosed in a glass tube) could reach the extreme end, thus avoiding any dead space. The bulb *R* was graduated; and, as its section and the volume of the stirrer were known by previous calibration, it was possible to estimate the volume of any liquid formed in it. The reservoir was calibrated as a whole after its construction had been completely finished.

b. Manometer.

Pressures were read on the closed hydrogen manometer M. 60 [2]). For the arrangement of the pressure connections and of the piezometer, reference may be made to Comm. N°. 97*a*. (Plate I.)

c. Cryostat.

The cryostat contained, besides the small reservoir *R* two platinum resistance thermometers *W* for reading the temperature, a stirrer *B* for the cryostat liquid (in this case ethylene), and the usual auxiliary

[1]) H. Kamerlingh Onnes, C. Dorsman and G. Holst. Proc. XVII (2) pg. 950. (Dec. 1904). Leiden. Comm. N°. 145*b*.

[2]) These Proc. IX p. 754 (Dec. 1906). Leiden Comm. N°. 97*a*.

capillary H, a simple helium thermometer[1]) for ascertaining the mean temperature of the portion of the small reservoir capillary within the cryostat, but above the liquid, as well as the necessary tubes for the introduction and removal of the ethylene.

Fig. 1.

The connections, required for the regulation of the pressure within the cryostat may be seen in Plate I of Comm. N°. 97a.

The method of stirring the gaseous and liquid phases within the small reservoir bulb was as follows: round the bulb was fitted a soft iron sheath, which could be moved vertically up and down by means of an electro-magnet E above the cryostat, in the same way as the liquid stirrer is worked.[2]) In the lowest position of the sheath m (which is of about the same length as the reservoir R, and about three times as long as the stirrer), the lower edge was at the level of the bottom of the bulb R; in the highest position of m its lower edge was raised to fully two thirds of the height of the bulb R. In order to be able to obtain a clear view of the whole length of the bulb R, two slits were cut out of opposite sides of the sheath m, and the latter so arranged that these slits were in line with the clear strips in the silver surfaces of the vacuum-glasses. Round the outer glass a ring electro-magnet m was placed with the bottom surfaces slightly above the level of the top of the bulb R. By a correct adjustment of the current circulating through this and the weight of the iron sheath m, it was possible to raise and lower the latter, which carried the small stirrer r with

[1]) These Proc. IX pg. 754. (Dec. 1906). Leiden Comm. N°. 97a.

[2]) These Proc. XX (2) pg. 991 (June 1917). Leiden Comm. No. 152a (§ 3).

it, so that the gas and liquid phases in the bulb could be stirred as required.[1])

§ 4. *Observations and precautions.*

Before commencing the work at low temperatures, isotherms were experimentally determined for the two mixtures at 20°, the calculations being carried out on the lines of Comm. N°. 78. The values of the expansion coefficients for the mixtures

$$\alpha = \frac{1}{v_A}\left[\overline{\left(\frac{\partial v_A}{\partial t}\right)_p}\right]_0^{20}$$

required in the calculation of the isotherms were interpolated as linear functions of the composition from the corresponding values of the pure gases, the error involved being negligible. These values were, in the case of oxygen, those found by KAMERLINGH ONNES and HYNDMAN[2]) and, in the case of nitrogen, calculated from isotherms determined at 0° and 20° by one of us, which are not yet published. These normal temperature isotherms were determined with small reservoirs ± 5 cm³ volume. For greater certainty a second series of points were determined for the 75 % oxygen mixture using the small reservoir of ± 1 cm³ of the piezometer used in the critical zone as a leak occurred during the first series, and consequently only the normal volume determined at the end could be used in the calculations. The agreement of this control is satisfactory. Isotherms were made over a range starting about 5 degrees above the temperature of the critical point of contact, and extending as low as the proportions of the piezometer allowed, i.e., 6 degrees below that temperature for the 50 % mixture, and 2½ degrees for the 75 % mixture. The temperature intervals were in general some 2 degrees, but, in the neighbourhood of the zone, were reduced to $^1/_{10}$ degree or less. All observations were made with rising pressure, the importance of which fact is insisted on in Comm. 150*b*; and, after finishing any series, the pressure is completely released, and gas in the piezometer well mixed by successively raising the pressure to 10 atmospheres or so and lowering, before proceeding to a new series. When only one phase was present, the pressure steps were of the order 2—3 atmospheres, but, when two were present, and near the critical zone, they were reduced to a few tenths of an atmosphere and sometimes the raising was accomplished by even

[1]) A. VAN ELDIK. Amsterdam Akad. Versl. Mei—Juni 1897. Leiden Comm. N°. 39.
[2]) These Proc. IV pg. 761. (Maart 1902). Leiden Comm. N°. 78.

smaller steps. As soon as two phases are present, the equilibrium becomes extremely sensitive to the smallest change in pressure or temperature, and therefore the quantities that determine the conditions of equilibrium must be kept as constant as possible. In the critical zone, an alteration of a hundredth of a degree in the temperature will cause the mercury in the stem of the piezometer to rise or fall by millimeters. Although the *end*-points of condensation could be fairly accurately observed, provided the pressure-increases were made with extreme care the tendency of the liquid phase to remain out, despite vigorous stirring, did not allow of accurate observation

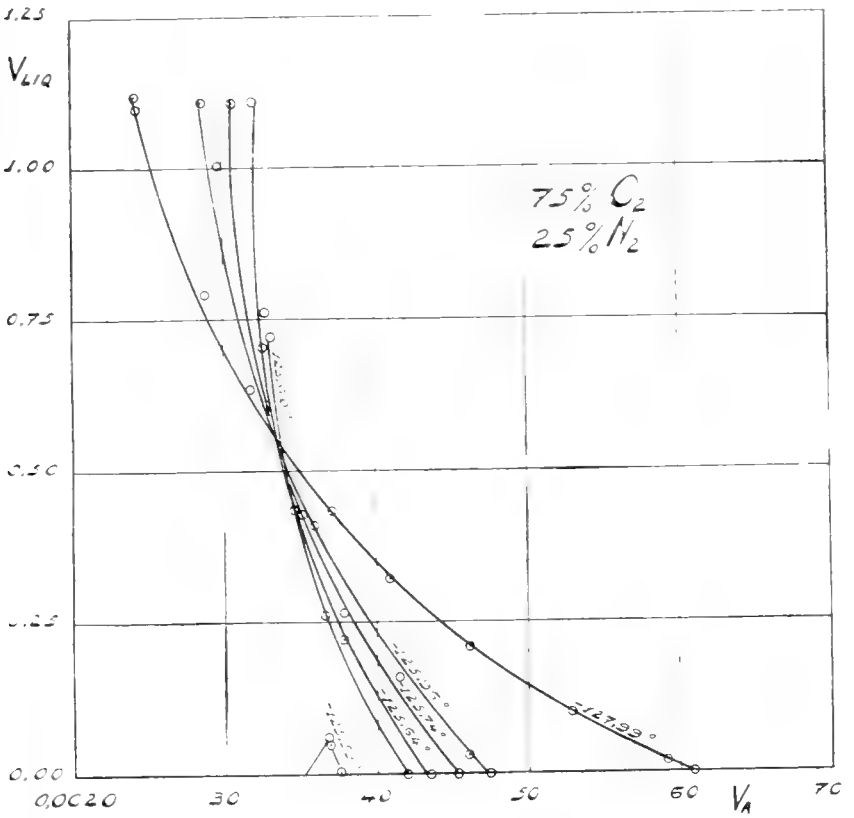

Fig. 2.

of the *beginning*-point. Both points were accordingly graphically taken from the isotherms by finding the intersection of the one-phase and two-phase portions of the latter. Even in this way only very approximate results can be obtained in the critical zone.

The critical phenomena were well observed, both as regards the typical opalescence at and near the plait-point, and the process of

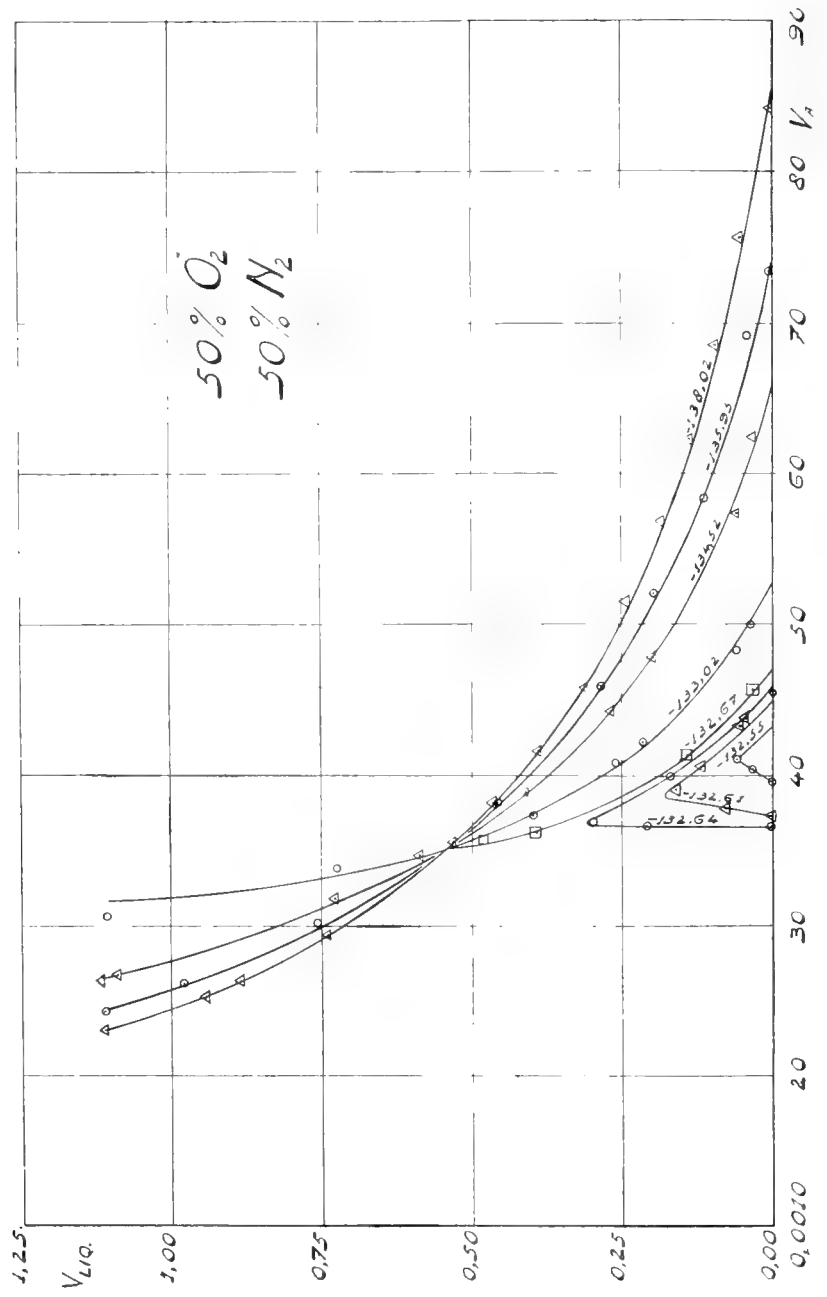
50% O2
50% N2
V Liq.
1,25
1,00
0,75
0,50
0,25
0,00
0,0010
20
30
40
50
60
70
80
90
V v
-138,02
-135,93
-134,54
-133,02
-132,67
-132,55
-132,51
-132,64

Fig. 3.

retrograde condensation, although the latter was limited to a range of 0.13 degree at most.

As an illustration of the perfection to which the regulation of pressure and temperature has been brought in the Leiden Laboratory, the 50 % mixture was maintained under the plait-point conditions for over an hour, the blue opalescence being continually there, while an indefinite meniscus alternately appeared and disappeared in the middle of the bulb on stirring. From the results a p, v_A graph for each mixture is constructed, and the points of beginning and end condensation determined as previously stated: the border curve is drawn through these. It was found that the two-phase line during the period of observation is to all intents and purposes a straight line, although, in the case of the 50 % mixture, the first points determined after condensation lie below this line on every isotherm. This can hardly be explained by any delay in the appearance of the liquid phase (which would give the reverse effect), and the deviation is far greater than any error of observation.

The accuracy of the pressure determination is at least 1 in 5000 [1]); that of the temperature reading within 0.02 of a degree, while the probable observation error of the volumes is not greater than 1 in 2000 when one phase was present, and 1 in 200 when two phases were present — apart from a possible constant calibration error of 1 in 500. To eliminate the last error it would have been necessary to measure a few points of the isotherm of 20° C. of hydrogen with this piezometer, and to compare the results with the accurate isotherm of SCHALKWYK. But as such accuracy was of little importance in our case, this was not done.

The results for the two mixtures are given below with:

p = pressure in atmospheres.

v_A = volume, expressed in the normal volume.

V_L = volume of liquid, expressed in volume of the small reservoir.

θ = temperature on the provisional intern. Kelvin scale, reduced by 273.09.

The condensation points, as found from the p,v_A graphs, are plotted on a p,t ($t = \theta$) graph; the results of KUENEN and CLARK being included on the same graph. (Fig. 4). The vapour pressures of pure oxygen [2]) and nitrogen [3]) are also plotted, and the critical

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[1]) C. A. CROMMELIN and Mej. E. J. SMID. These Proc. XVIII (1) pg. 472. Leiden. Comm. Leiden. N°. 146*c*.

[2]) H. KAMERLINGH ONNES, C. DORSMAN and G. HOLST. l. c.

[3]) C. A. CROMMELIN. These Proc. XVII (2) 959 (Dec. 1914.) Leiden Comm. N°. 145*d*.

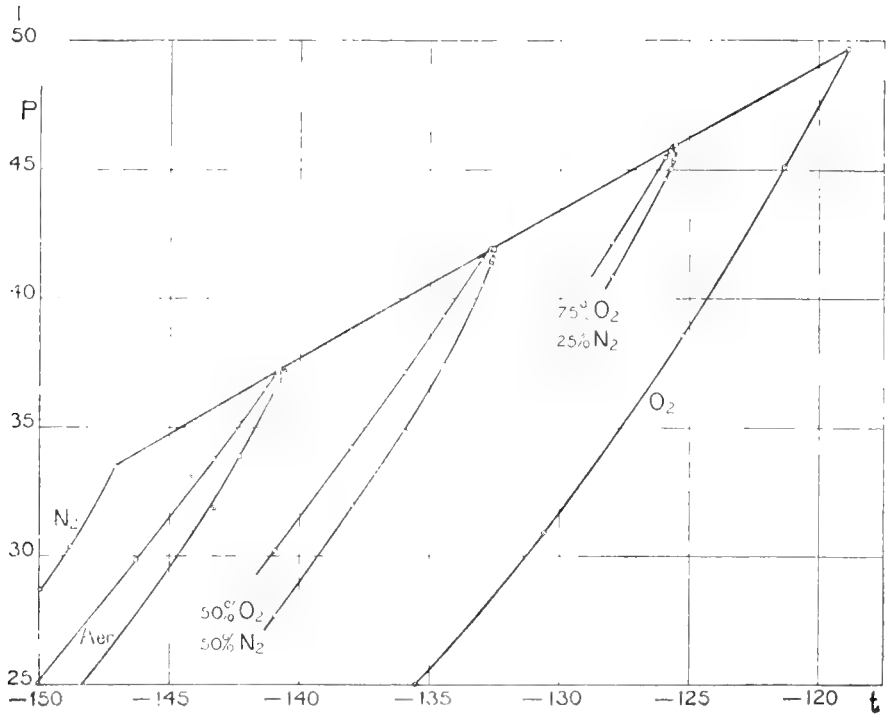
P
50
45
40
35
30
25
−150
−145
−140
−135
−130
−125
−120
t
N2
Air
50% O2
50% N2
75% O2
25% N2
O2

Fig. 4.

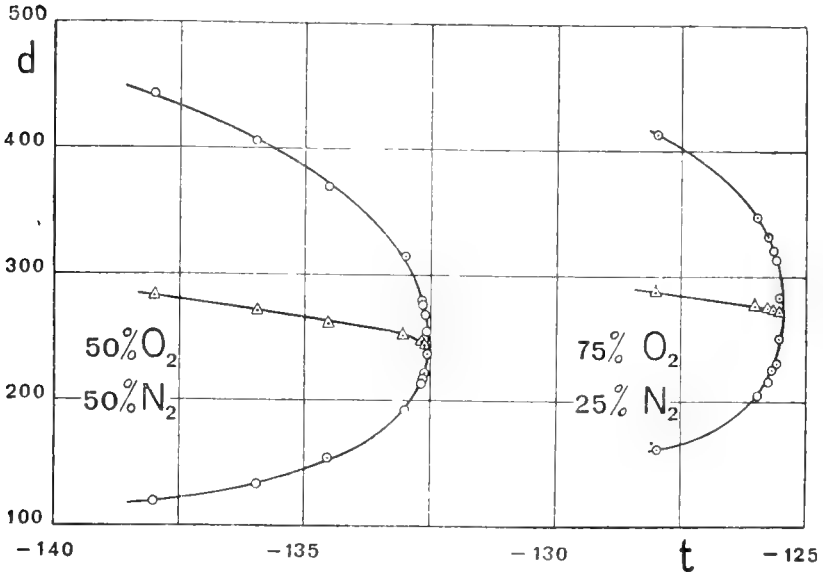
d
500
400
300
200
100
−140
−135
−130
−125
t
50% O2
50% N2
75% O2
25% N2

Fig. 5.

curve drawn tangential to the various border curves, touching those in the plait points.

By plotting V_L the volume of the liquid against v_A a series of curves are obtained which clearly show the process of retrograde condensation in the case of the 50% mixture [1]).

A peculiarity of the last mixture is that all the lines in this graph go through the point V_L = half the volume of the small reservoir, which means that the corresponding line of constant division of volume is a line of constant v_A, therefore in the p,v_A diagram it runs parallel to the p-axis [2]).

If a d_A, t graph is drawn, a diameter is obtained which is rectilinear (as for a pure substance), but which is strongly curved towards the temperature axis at the extreme end, though in this zone, the position of the point as found must be necessarily rather qualitative than quantitative.

The plait-point constants were found to be:

	50% O_2 *Mixture.* (series XIV. 4.)	75% O_2 *Mixture.* (series IX. 5.)	
p	41.90	45.89	(observed)
v_A	0.00358	0.00336	(from p,v_A graph)
θ	—132°.66	—125°.60	(observed)

The critical point of contact constants were found to be:

	(series IX. 4.)	(series X. 3).	
p	41.90	45.86	(from p,v_A graph)
v_A	0.00404	0.00375	(from d_A,t graph)
θ	—132°.53	—125°.53	(observed)

For the critical point of contact temperature it was found that, at 0.01 of a degree above it no condensation was, of course, observed, and at 0.01 below there was a momentary, but very evident condensation.

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[1]) J. E. VERSCHAFFELT. These Proc. I. pg. 288 (Dec. 1898.) Leiden Comm. N°. 45.

[2]) Leiden Comm. Suppl. N°. 23, p. 51. Enc. Math. Wiss. V 10.

Results for the mixture 50 % O_2—50 % N_2.

Isotherm of 20° *C.*

Point.	p	d_A	pv_A	Point.	p	d_A	pv_A
1	34.24	32.39	1.0573	7	52.34	49.82	1.0507
2	37.55	35.56	1.0561	8	46.25	43.91	1.0533
3	41.39	39.25	1.0548	9	41.12	38.99	1.0546
4	46.41	44.07	1.0531	10	37.28	35.28	1.0567
5	51.88	49.37	1 0508	11	34.03	32.17	1.0579
6	51.85	49.35	1.0507				

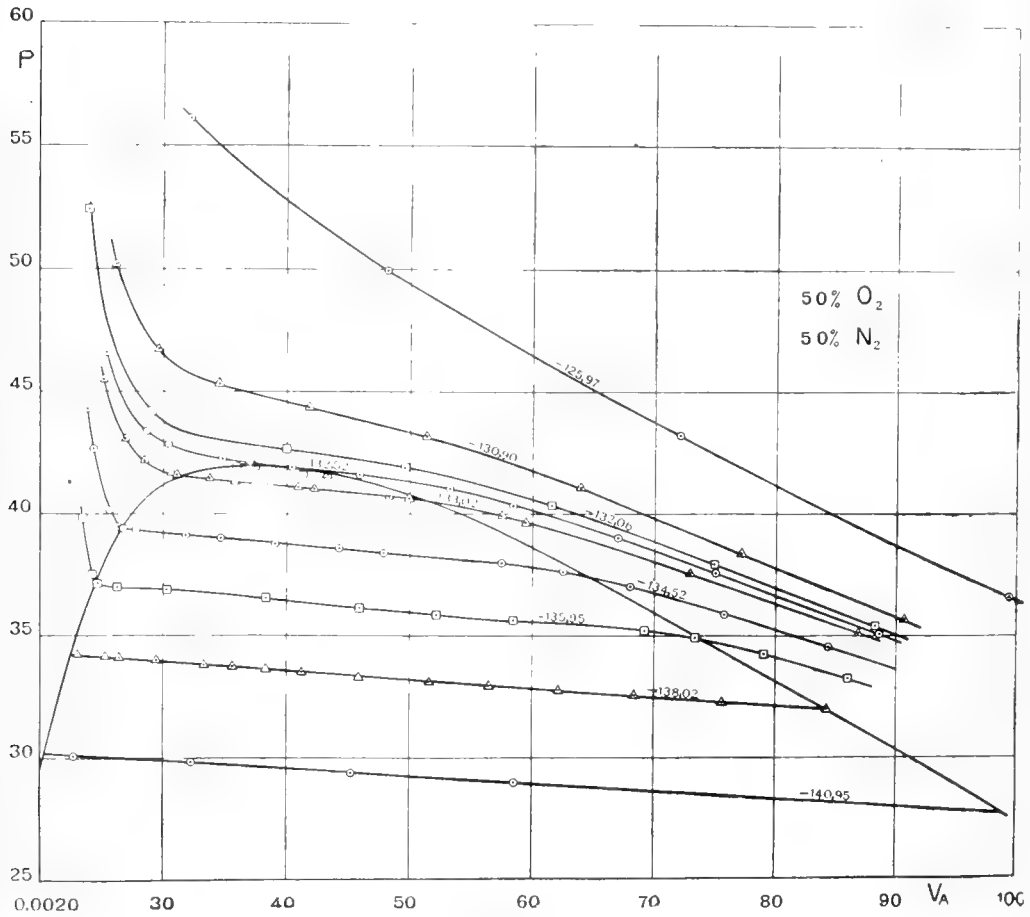


Fig. 6.

Isotherms at low temperature.

Point	p	v_A	V_L	θ
I. 1	37.12	0.01080		−120°.76
2	44.81	00793		8
3	54.30	533		6
II. 1	36.57	0.00994		−125°.97
2	43.23	721		7
3	49.95	481		8
4	56.13	322		7
III. 1	35.65	0.00907		−130°.90
2	38.38	771		0
3	41.03	639		89
4	43.16	513		91
5	44.35	418		0
6	45.26	345		0
7	46.73	295		0
8	50.18	261		0

Point	p	v_A	V_L	θ
IV. 1	35.41	0.00882		−132°.06
2	37.95	749		6
3	40.33	615		6
4	41.91	495		7
5	42.67	398		6
6	44.17	290		6
7	52.44	239		6
IX. 1	35.08	0.00886		−132°.51
2	37.57	751		1
XV. 1	39.03	670		2
IX. 3	40.34	584		1
XV. 2	41.02	532		2
3	41.60	458		2
IX. 4	41.90	404		3
5	42.13	369		0
XV. 4	42.25	347		2
5	42.78	303		2
IX. 6	43.38	285		1
XV. 6	46.60	253		2

Point	p	v_A	V_L	δ
X. 1	41.84	0.00410	0.051	−132°.56
2	41.89	403	029	5
3	41.93	395	000	5
XII. 1	41.60	0.00436	0.042	−132°.61
XI 1	41.62	433	046	0
XII. 2	41.77	406	107	0
XI. 2	41.85	392	138	1
XII. 3	41.92	379	068	0
XI. 3	41.95	372	000	1
XII. 4	41.97	373	000	0
XIII. 1	41.46	0.00455	0.000	−132°.64
2	41.75	399	151	4
3	41.89	369	268	4
4	41.91	366	186	4
5	41.91	366	000	4
XIV. 1	41.40	0.00455	0.026	−132°.67
2	41.61	413	123	7
3	41.89	362	350	7
4	41.90	358	430	6
V. 1	35.12	0.00867		−133°.01
2	37.58	728		1
3	39.67	594		3
XVI. 1	39.92	574		3
VI. 1	40.62	499	0.031	2
XVI. 2	40.65	527		2
V. 4	40.73	482	54	1
VI. 2	41.03	421	192	0
XVI. 3	41.08	408	235	2
VI. 3	41.29	373	356	−132 .99
V. 5	41.34	358	472	−133°.03
XVI. 4	41.46	336	648	2
5	41.62	310		2

Point	p.	v_A	V_L	δ
VI. 4	41.69	0.00306	1.000	−132°.00
V. 6	42.16	284		3
XVI. 6	43.13	268		1
7	45.78	251		2
XVII. 1	34.52	0.00843		−134°.50
2	35.89	757		2
3	37.03	680		1
4	37.60	624	0.030	1
5	37.98	574	054	3
6	38.41	478	177	2
7	38.58	442	240	2
8	38.80	390	369	2
9	39.01	346	524	2
10	39.14	318	653	1
11	39.33	277	916	2
12	39.40	267	978	3
13	39.46	263	1.000	2
14	40.15	256		1
15	42.64	242		2
16	44.25	237		2
XVIII. 1	33.21	0.00860		−135°.98
VII. 1	34.32	790		1
XVIII. 2	34.85	733	0.002	8
VII. 2	35.21	692	037	1
3	35.66	584	101	3
XVIII. 3	35.82	521	175	7
4	36.14	459	255	7
VII. 4	36.60	382	408	1
XVIII. 5	36.90	302	677	7
6	37.09	262	877	6
VII. 5	37.19	246	990	2
6	37.56	242	1.000	2
XVIII. 7	39.77	234		7

Point	p	v_A	V_L	δ
XIX. 1	32.02	0.00842	0.000	−138°.02
2	32.31	755	048	2
3	32.57	684	082	2
4	32.78	621	123	2
5	32.98	564	165	2
6	33.16	514	219	1
7	33.37	457	279	2
8	33.54	415	349	2
9	33.65	382	414	2
10	33.75	355	480	2
11	33.84	331	540	2
12	33.99	293	663	1
13	34.11	263	792	1
14	34.16	253	843	1
15	34.24	230	1.000	2
VIII. 1	27.69		0.000	−140°.95
2	28.99	0.00585	186	5
3	29.41	452	316	5
4	29.85	322	544	4
5	30.09	227	901	6

Results for the mixture 75 % O, —25 % N.

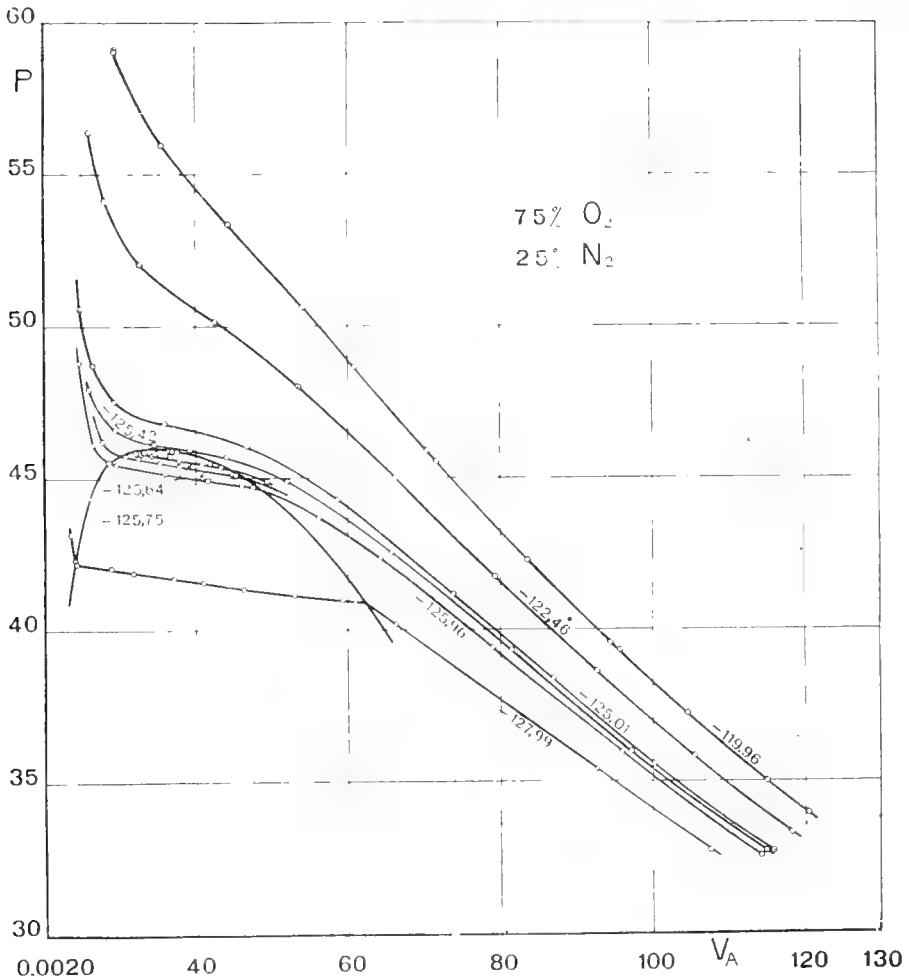
Isotherm of 20° C.

Point	p	d_A	pv_A	Point	p	d_A	pv_A
I. 1	51.68	49.49	1.0441	II. 1	36.91	35.08	1.0521
2	45.18	43.15	1.0471	2	42.90	40.89	1.0492
3	40.12	38.20	1.0503	3	48.79	46.63	1.0464
4	35.91	34.14	1.0520	4	56.40	54.08	1.0429
5	32.69	31.01	1.0542				
6	28.89	27.34	1.0564				

Isotherms at low temperature.

Point	p	v_A	V_L	θ
I. 1	33.93	0.01206		−119°.95
7	35.00	1151		7
2	37.20	1045		5
3	39.31	0956		5
8	39.55	944		6
4	42.27	835		5
5	45.54	715		4
9	45.88	704		6
6	48.66	608		4
10	50.64	540		7
11	53.36	443		8
12	55.97	355		6
13	59.13	294		7
II. 1	33.29	0.01183		−122°.47
2	35.82	1055		6
3	38.60	0926		6
4	41.72	793		6
5	45.00	661		7
6	48.02	535		6
II. 7	50.16	0.00426		−122°.47
8	52.05	327		6
9	54.15	280		6
10	56.41	260		6
III. 1	32.71	0.01159		−125°.00
2	35.59	1006		0
3	38.40	0869		4.99
4	41.19	737		5.01
5	44.28	587		1
6	46.04	467		1
7	46.83	359		1
8	47.51	293		1
9	48.73	265		1
10	50.60	248		2

Point	p	v_A	V_L	δ
VI. 1	32.68	0.01149		—125°.42
2	35.97	0973		3
3	39.30	813		2
4	42.54	657		2
5	44.89	520		1
6	45.67	440		2
7	45.93	382		2
8	46.14	346		3
9	46.59	294		2
10	47.95	261		2
X. 1	45.89	0.00369	0.041	—125°.53
2	45.90	345	53	3
3	45.86	375	000	3
IX. 1	45.49	0.00419	0.000	—125°.60
2	45.72	366	236	0
3	45.81	347	394	0
4	45.85	333	648	59
5	45.89	336	1.000	60
VIII. 1	44.89	0.00394		—125°.64
2	45.36	433	0.000	4
3	45.46	408	78	4
4	45.59	378	199	3
5	45.75	343	455	4
6	45.80	328	682	4
7	45.83	321	1.000	5
VII. 1	44.43	0.00523		—125°.75
2	45.09	453		5
3	45.12	448	0.000	4
4	45.38	378	239	6
5	45.50	354	384	4
VII. 6	45.65	0.00326	0.631	—125°.73
7	45.70	307	1.000	4
8	46.19	277		3
IV. 1	32.53	0.01143		—125°.96
2	35.93	0959		6
3	39.41	789		8
4	42.37	641		7
5	43.68	561		7
6	44.67	474	0.000	7
7	44.75	460	025	6
8	44.91	415	142	6
9	45.15	361	368	6
10	45.43	297	901	6
11	45.51	287	1.000	6
12	46.11	267		6
13	48.83	248		6
V. 1	32.74	0.01076		—127°.99
2	35.41	0926		8.00
3	38.10	785		0
4	40.19	663		0
5	40.89	609	0.000	7.99
6	40.95	592	020	9
7	41.14	529	090	9
8	41.38	462	186	9
9	41.55	409	289	9
10	41.70	371	389	8.00
11	41.88	318	569	0
12	42.04	288	709	7.98
13	42.19	244	994	9
14	42.28	242	1.000	9
15	43.17	236		8.00



In conclusion it is our pleasant duty to thank Miss H. VAN DER HORST and Mr. J. D. A. BOKS for their careful regulation of the temperature, and Mr. L. OUWERKERK and Mr. C. F. L. KRAANEVELD for the technical skill with which they helped us during the whole course of the measurements.

Anatomy. — "*Contributions to the knowledge of the brain of bony fishes.*" By Prof. KYOZO KUDO, Mukden (Manchuria). (Communicated by Dr. C. U. ARIËNS KAPPERS).

(Communicated at the meeting of January 27, 1923).

I. *The Tr. olfactorio-opticus.*

NILS HOLMGREN found in Osmerus eperlanus with the CAJAL-method, but also with methylene-blue colouring, a bundle which, long before the middle of the telencephalon, separates from the tr. olfactorius lateralis, then, following the sulcus externus, extends as far as the opticus, into which it enters, and can be traced (in the opticus) for some distance towards the eye. He called the bundle *tr. olfactorius lat. optici* (op. cit., p. 188). With Callionymus lyra he found a similar bundle, but lying in the medial olfactory tract (l.c., p. 188, Anmerkung).

This discovery should be considered most remarkable. Being able to test and confirm the latter case (the fibres in the tr. olfactorius med.) with various Teleosts also by WEIGERT-preparations, I will describe it here more fully.

With the WEIGERT-colouring these fibres, connecting the tr. olfactorius with the tr. opticus, seem fairly coarse; they are nearly always scattered and mixed only with the tr. olfactorius medialis, never with tr. olfactorius lateralis. In the bony fishes, which I examined they run always the same way. As these relations are the most distinct in Ammodytes tobianus, I take this fish as example.

With this fish the tr. olfactorius med. consists of two sorts of medullary fibres, a thin one and a much coarser one. The fibres divide into three parts

The *pars dorsalis* is that part of the tr. olfactorius med. that on a quite frontal level turns towards dorsal. It consists for the greater part of thin fibres that radiate in dorso-lateral direction and disappear rather soon, already on the level of commissura anterior. A few coarse fibres, however, also belonging to this portion, run further caudad, always following laterally the tr. olfacto-hypothalamicus med., but strongly contrasting with these by their coarseness. They cannot be traced accurately from the place where they medially

pass the fibrae ansulatae of Bellonci, caudo-ventrally descending by and by with the accompanying tractus olf.-hypothal. med.

These fibres form the *dorsal* group of the coarse olfactory fibres.

The *pars intermedia* of the tr. olfact. med. consists half of the thinner, and half of the coarser fibres. The first form a small bundle and cross in the commissura anterior (the so-called comm. interbulbaris).

The *pars ventralis* is formed by coarse fibres exclusively. They at first join to a bundle, but gradually they separate into several small bundles. These, together with the fibres from the pars intermedia form the *ventral* group of the coarse olfactory fibres. The bundles soon arrange in regular order dorso-ventrally in the praethalamus, thereby dorsally touching the tr. olfacto-hypothalamicus med., ventrally the tr. opticus, into which they are taken up gradually (fig. 1). A few fibres that are in an exceptional dorsal position, enter into the just rising commissura minor.

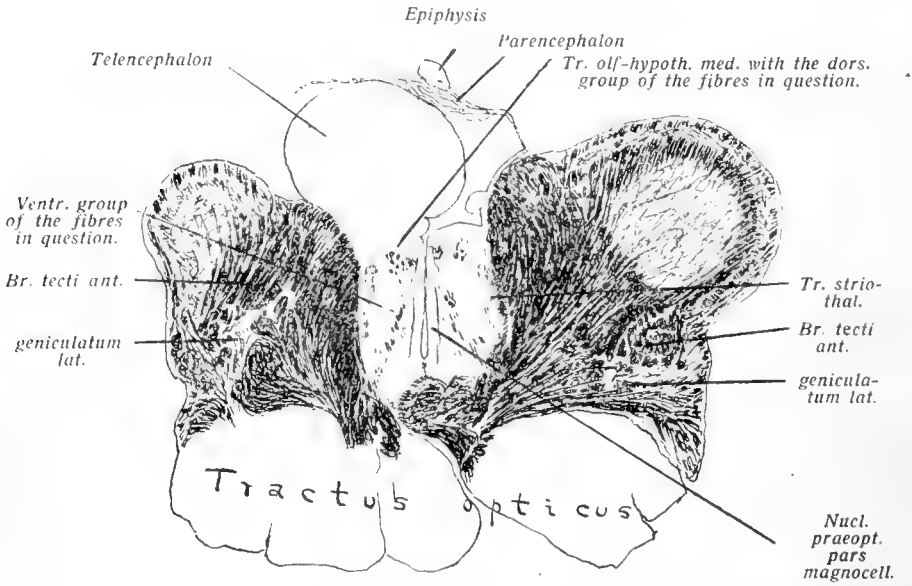


Fig. 1. Ammodytus tobianus.
(Preparation of the Central Institute for Brain-research, Amsterdam).

With other bony fishes the course of these coarse olfactory fibres is nearly the same. Only the relation to the commissura minor (and to the fasciculus medialis n. optici) does not always exist, this being most distinct, besides in Ammodytes, with Rhombus, Hippocampus and Morone.

I have been able to find these fibres with Catostomi, Percesoces,

Acanthopterygii and Plectognathi, but not with Malacopterygii, Ostariophysi, Symbranchii and Haplomi. They probably are neither to be found with Anacanthini and Pediculati (at least not medullated).

The quantity of the fibres varies according to the kind of fishes. It often is so small that the fibres can be easily counted. In any case the fibres are generally not very numerous. The *ventral* group is always superior in number to the *dorsal*, which I even cannot find in Solea and Cottus.

With Callionymus lyra, the very bony fish that HOLMGREN examined, the matter is somewhat different.

Inasfar as it may be judged by WEIGERT-PAL series these fishes show no tr. olfactorius lat. The tr. olfactorius med. is fairly coarse and is found as usual on the medio-ventral side of the frontal half of the brain. Caudally it soon descends a little ventrally already before the level of the commissura minor, and comes in touch with the opticus fibres and especially with those that arrive here from lateral. I think it fairly probable that fibres are being exchanged between these two tracts, that is to say that tr. olfactorius gives away a part of its fibres to the tr. opticus, but in exchange receives more fibres from the latter tract during its descension (see below) and so becomes visibly coarser. The fibre tract now runs medially along the dorsal opticus root in a ventral direction. [1]) More frontal a small part of the tr. olfactorius is separated from the chief bundle on the spot where this tract begins to descend in order to come into touch with the tr. opticus. This separated little bundle runs independently in the praethalamus also ventrad, about in the middle of the inner and outer surface of same, and finally joins again the principal bundle. [2])

When the thus formed olfacto-optic bundle has at last left the tr. opticus, it runs lateral, viz. between the tr. strio-thalamicus med. and the nucl. anterior thalami [3]). More caudally it wedges between the nucl. praerotundus [3]) and the tr. tubero-dorsalis of GOLDSTEIN. A little way back nerve cells begin to appear in the bundle and finally take up the area of the bundle nearly entirely. This nucleus takes its place quite superficially of the lateral surface of the hypothalamus, close above the lobus inferior (see fig. 87 of HOLMGREN's

[1]) The formation of the very extensive tectum plate and accordingly the topography of the opticus roots are in this bony fish different from others.

[2]) So with Callionymus nearly all medullary olfactory fibres seem to run as far as into the praethalamus.

[3]) I have not yet succeeded in identifying these nuclei free from objection with this fish.

work, showing the nucleus in question, medial of "tr. olf. tect. sem."). The cells of this nucleus are small.

From this nucleus proceeds a new fibre tract, running in a curve in the torus semicircularis, parallel to and inward of the tr. isthmo-tectalis mihi (see below), yet it seems to end in the torus itself. That besides this, fibres should come from the said tract in the path of the tr. tecto-bulbaris or tr. tecto-isthmicus into the tectum, as HOLMGREN seems to presume, is improbable to me[1]).

As to the origin and the end of my coarse olfactory fibres, I am quite unable yet to say anything definite. That frontally they are connected with the bulbus olfactorius is undoubtedly sure, but their caudal destination remains quite uncertain. HOLMGREN seems to hold the opinion that his tr. olfactorius lat. optici runs centrifugally in the opticus (l.c., p. 188). I myself am more inclined to believe that the fibres of the *ventral* group tend through the path of the commissura transversa towards the tectum or, less probable, towards the nucl. praerotundus of the other side, the spot where they penetrate into the opticus root just corresponding with the most frontal level of this commissure. This, however, is a mere supposition.

It is harder still to say anything of the *dorsal* group of my fibres.

Concerning the curious olfacto-optic bundle in Callionymus, it can only be said that the part of tr. olfactorius med., that enters frontally into the opticus, corresponds fairly certainly with the *ventral* group of my fibres. But about the other part of the tract, the associating opticus fibres, the peculiar nucleus and the bundle originating from it, I cannot give an opinion, as — till now — I have not seen anything similar with the other bony fishes.

Summarizing we find in the brain of the Teleosts a remarkable fibre system, connecting the tr. olfactorius with the tr. opticus, consisting in the more primitive forms (Osmerus eperlanus) of thin fibres (that can only be exposed by the impregnation method) running in the tr. olfactorius lat., whereas in the more highly organised forms it has coarse, medullary fibres as components and is mixed between the fibres of the tr. olfactorius med. At the present I must be content with confirming HOLMGREN's finding, leaving the arising questions to later investigations.

II. *The Tr. tecto-praerotundus.*

I think I have discovered in the brain of bony fishes a medul-

[1]) HOLMGREN, namely in fig. 87 of his work, has indicated this newly forming fibre tract as „tr. olfacto-tectalis et semicircularis", without further referring to it in the text.

lary fibre tract, not yet described as far as I know, which probably connects the tectum with the nucleus praerotundus. This tract appears in transverse sections on the level where the fasciculus medialis nervi optici, swinging across the tr. strio-thalamicus, joins the lateral opticus root. It there appears as a bundle, running dorso-ventrally, medially along the dorsal opticus root. Dorsally it lies in separate bundles between the just-mentioned opticus root and those fibres that branch off (more frontally) from the tr. opticus, run dorsad in the post-habenular region and finally lateral into the deep medullary layer tectum. From these fibres (probably corresponding to the fibrae tectales n. optici of KRAUSE) the tract is distinguished by the smaller caliber of its fibres and its steeper course. Somewhat more caudally it bends in a lateral direction and enters also into the deep medullary layer of the tectum.

Ventrally it crosses with the fasciculus med. n. optici directly medially to the praetectal nucleus, then runs laterally down from the tr. strio-thalamicus, to finally join the fibres of the commissura transversa (see fig. 2).

Often, however, it does not run laterally to the tr. strio-thalamicus but medially, together with the fasciculus med. n. optici towards medioventral, consequently in this case delusive of a commissura minor in the sense of ARIËNS KAPPERS (one-sided in a specimen of

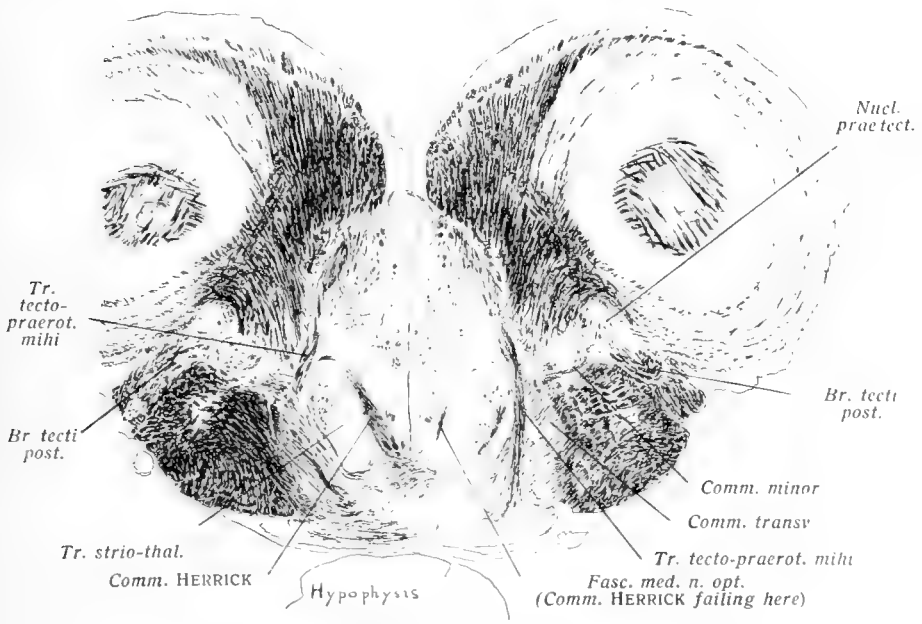


Fig. 2. Mugil chelo.
(Preparation of the Central Institute for Brain-research, Amsterdam).

Gasterosteus (see fig. 3), double-sided in a specimen each of Belone and Exocoetes). However, I want to consider this course as aberrating, for in another specimen of Gasterosteus and also of Exocoetus it runs laterally to the tr. strio-thalamicus, as with the other Teleosts.

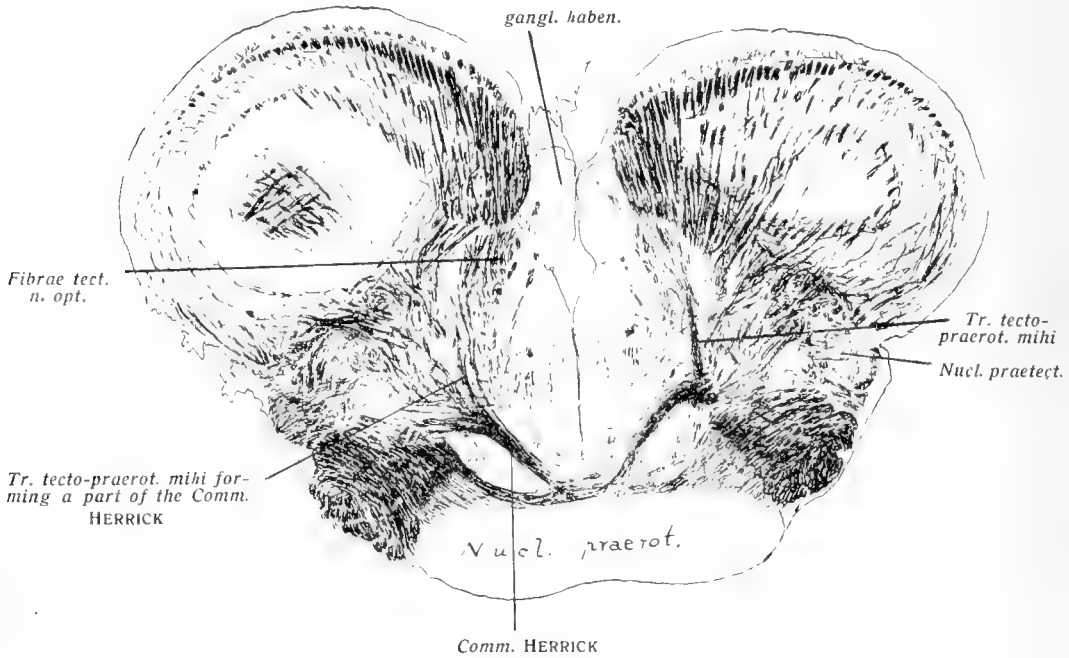


Fig. 3. Gasterosteus aculeatus.

(Preparation of the Central Institute for Brain-research, Amsterdam).

Ventrally it mixes, as mentioned, among the fibres of the commissura transversa, in two ways. One time it runs nearly horizontally towards medial as the most caudal part of this commissure, close to the ventral periphery of the midbrain. Here it soon cannot be traced any further among the commissure fibres. Yet it seems plausible to me that here it crosses the medial line. This I find with Trutta, Syngnathus?[1]), Ammodytes?[1]), Mugil, Ophiocephalus, Morone, Osphremenus, Pleuronectes platessa?[1]), Rhombus, Hippoglossus?[1]), Solea?[1]), Cyclopterus, Agonus?[1]), Trachinus and Tetrodon.

Another time it forms the most dorsal part of the commissura transversa and crosses directly under the ventricle. I was able to ascertain this course, besides in the cases where it is delusive of the commissura minor (Gasterosteus, Belone and Exocoetus), with

[1]) The question-marks denote that with these Teleosts I do not find the matter quite clear.

Clupea?[1] and Pleuronectes limanda. Also with Gadidae this tract can easily be traced from the subventricular crossing as far as the deep medullary layer of the tectum.

FRANZ has erroneously indicated this bundle in Gadus, in fig. 6 of his work, as "fibrae tect. n. opt." The correctness of my observation, however, can be tested not only in Gadus itself, but very easily in Lota and Motella. In fig. 24 of the same work FRANZ has called the same tract correctly "ascending decuss. transversa" (the upper line of his reference, at the bottom, left hand).

Although the exact way of origin and termination of this bundle is not yet clear to me, I should like to call it provisionally *tr. tecto-praerotundus* and introduce it under this name.

III. *Tr. isthmo-praetectalis.*

FRANZ describes the course of his *tr. isthmo-opticus* as follows (op. cit., p. 414): "a tract which, together with the opticus, first appears ventrally to the midbrain roof, ascends on the inside of the midbrain roof, reaches the torus semicircularis and here curves round to the ganglion isthmi — here often difficult to distinguish from the fibres of the commissura transversa. It may be possible that part of the fibres remains already in the torus semicircularis."

He further is of opinion, supported by experiments (on which fishes?) that the fibre tract arises undoubtedly from cells of the ganglion isthmi and sends its neurites centrifugally into the eye (l.c., p. 415).

My investigation confirms the presence of such a bundle and its caudal course, as described by FRANZ. I only want to remark that I could not find this bundle with all bony fishes. A few fishes, such as Salmonids, Siluroids, Misgurnus, Symbranchidae, Esox, Ammodytes?[1]), Gadidae, Lophius?[1]) and Tetrodontidae?[1]) seem not to possess such a fibre tract.

Its further frontal course, however, is not as supposed by FRANZ, but different according to the Teleosts examined.

FRANZ seems to be right, in so far as the bundle, after its characteristic curve in the torus semicircularis, gathers its fibres at the lateral basal ridge of the midbrain, then by bundles enters into the lateral opticus root and disappears from examination under the fibres of the latter, as is the case with Megalops (with all)[2]), Gastero-

[1]) The question-marks indicate here that the matter concerned is not quite clear in these fishes.

[2]) The brackets refer to the quantity of fibres, entering into the opticus root.

steids (few)[2]), Scombresocids (part)[2]), Mugil (the greater part)[3]), Ophiocephalus (the greater part)[3]), Morone (greater part)[3]), Osphromenus (part)[2]), Cottids ?[1]), Cyclopterus ?)[1]), Agonus ?)[1]) and Trachinus (with the smaller part of the fibres)[3]).

On the other hand there are bony fishes in which the bundle in question can be distinctly followed as far as the frontal tectum section, as Clupea, Cyprinidae, Syngnathidae, Osphromenus, Pleuronectidae (Solea excepted) and Callionymus. I therefore will give here a minute description of the course of this fibre tract in some specimens of fishes.

Just with Pleuronectes, where FRANZ believed to have his tr. isthmo-opticus fully established, we can clearly prove that the bundle in question actually enters frontally into the tectum, in the most superficial medullary fibre layer of it. The small bundles namely gather on the lateral surface of the midbrain to a roundish bundle that protrudes into the optic ventricle and then takes a wholly sagittal course. (see fig. 6 of Hippoglossoides). At first it runs ventrally to the bundles of tr. tecto-bulbaris and directly dorsal of the ventral point of the tectum plate. Some way more frontal it takes the shape of a curved medullary plate and encircles about half of the nucl. praetectalis from the lateral side. During this course its position corresponds to the bottom of the tectal furrow characteristic of this fish and, situated caudally deep inside, it frontally more and more approaches the outer surface; at last, quite frontal on the level of the geniculatum, it comes close under the molecular layer[3]) of the tectum. Till there it never touches the opticus root. Now part of the bundle radiates caudo-laterally of the geniculatum, towards mediodorsal into the dorsal part of the tectum[4]), whereas the other part runs further frontad and finally enters latero-ventrally into connection with the ventral part[4]) of the tectum, which extends a little further frontad than the dorsal one. At this radiation of the bundle I could quite clearly distinguish its small bundles from those of the brachium tecti (KAPPERS) and the optic root, and prove that the most superficial narrow layer of the so-called opticus fibre layer of the tectum consists of the little bundles of the fibre tract in question and the next much broader layer of medullary fibres (of the opticus

[1]) See note 1 p. 71.

[2]) See note 2 p. 71

[3]) This designation has its explanation in the above-mentioned treatise „On the torus longitudinalis etc.".

[4]) The frontal part of the tectum plate is divided into two parts by the aforementioned furrow.

fibre layer), separated from the first by a strand of grey substance, consists of the small bundles of the brachium tecti and the tr. opticus (see fig. 4).

Also with Clupea, examined likewise by FRANZ and drawn by him in fig. 9 of his treatise, I could trace the fibre tract as one or two coarsefibred bundles above and partly through the lateral opticus root, lateral of the praetectal nucleus and the geniculatum, as far as in the upper fibre layer of the most frontal tectum (fig. 5).

With the Cyprinidae its course is about the same as with Clupea. The small bundles gather medio-dorsally to the lateral root in one or two bundles, then run frontad, at first between the nucl. praetectalis and the lateral opticus root, then between the nucl. anterior thalami and the ganglion geniculatum. Here a part of the fibres branches off into the tectum, but the remainder runs, dorsally to the geniculatum, further frontad, then turns medio-dorsad and finally also reaches the tectum.

By the way I want to remark that with these Teleosts (Pleuronectids, Clupea and Cyprinids) the bundle in question is accompanied in its entire frontal course by the tr. geniculo-myelencephalicus of HOLMGREN. The two bundles, however, can easily be distinguished, the fibres of the first bundle being far coarser than those of the latter.

With Osphromenus the bundle shows a peculiarity. Part of its fibres enter, as mentioned above, already caudally into the lateral opticus root. Here again I expected to be able to trace the small bundles through the complex of the opticus fibres right into the upper fibre layer of the tectum. The other part, however, turns in the subependymal layer gradually medio-dorsad, above the comm. horizontalis fibre group[1]), and finally enters into the fibre system of the most medio-frontal tectum part.

Therefore I am fully convinced that there is no doubt but the bundle in question, that is to say what FRANZ calls tr. isthmo-opticus, does not run in the opticus to the eye, as FRANZ presumes, but to the frontal tectum, and there becomes part of the opticus fibre layer. Also in those cases where the fibres concerned enter already caudal into the lateral opticus root, it may be assumed that we have the same state of affairs as I have been able to prove in Osphromenus. In the following I shall call the bundle provisionally *tr. isthmo-tectalis*, although the direction of its course (ascending or descending) is not have proved yet.

[1]) Complex of commissura horizontalis and tr. tecto-cerebellaris (cf. fig. 5).

This *tr. isthmo-tectalis* mihi was traced already fairly minutely by MAYSER, the sharp observer. He called it "the outer (back) layer of the stratum zonale" of the torus semicircularis; but made it originate frontal in the corpus geniculatum externum (nucl. praetectalis of the authors) and terminate caudally between "numerous

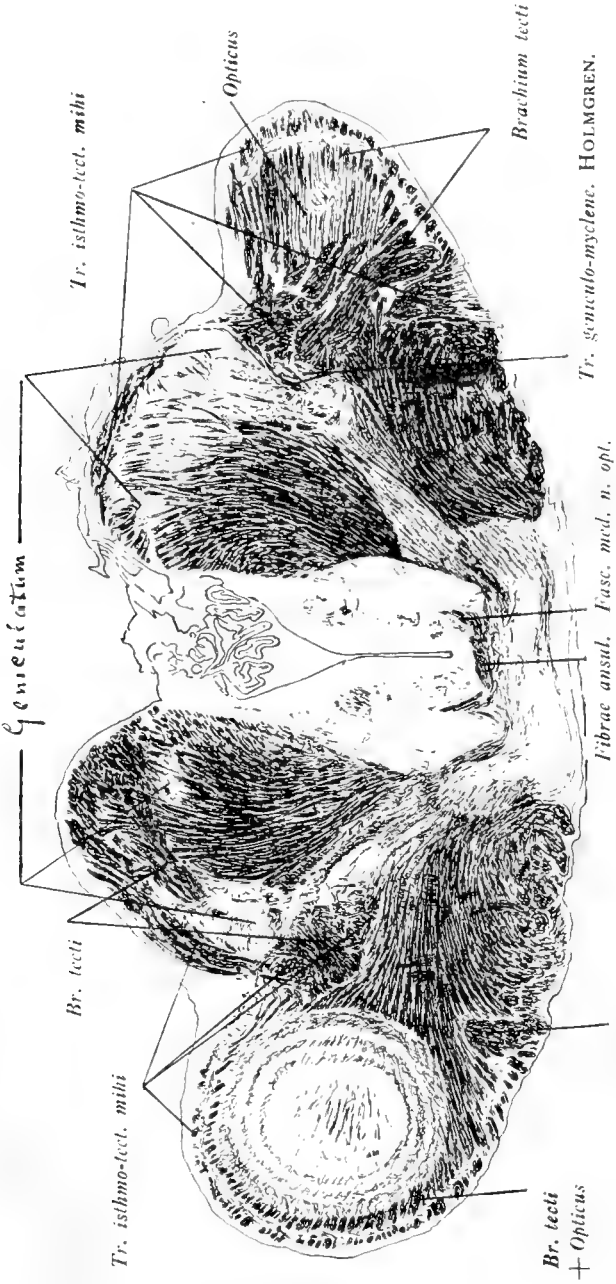


Fig. 4. Pleuronectes platessa.
(Preparation of the Central Institute for Brainresearch Amsterdam.)

small nerve cells chiefly in the caudal and upper part of the torus" (op. cit., p. 342—343 and 348), without yet having applied to the latter a special name (ganglion isthmi).

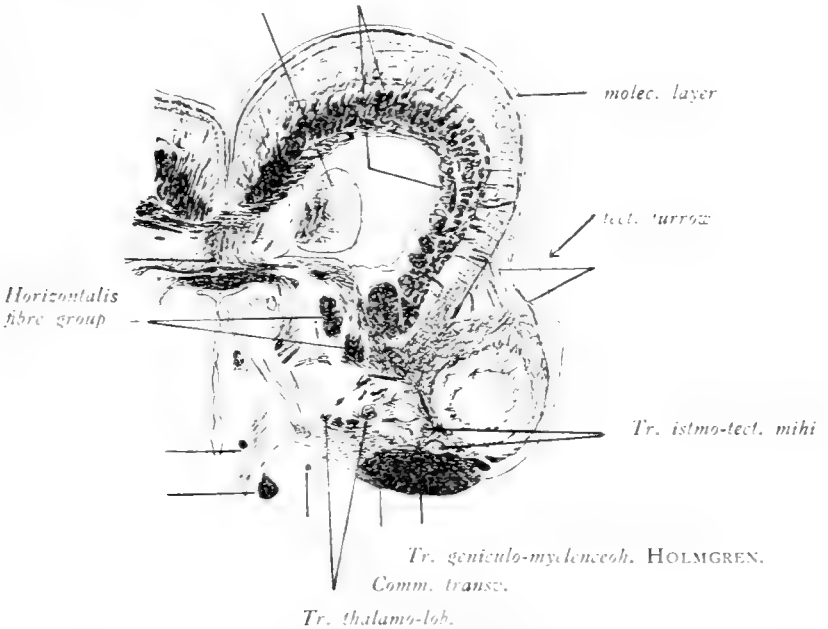


Fig. 5. Clupea harengus.
(Preparation of the Central Institute for Brain research, Amsterdam

After MAYSER no authors had occupied themselves, as far as I know, with this interesting fibre bundle, till FRANZ discovered it again.

Through the result of this investigation the relation between the tectum opticum and the ganglion isthmi with Teleosts gets closer. The connection is twofold, one time through the tr. tecto-isthmicus of FRANZ, another time through the tr. isthmo-tectalis of myself, and we may presume that the first leads tectofugal, the latter tectopetal.

Although MAYSER's opinion that the bundle in question originates from the corpus geniculatum externum s. l. (i.e. from nucl. praetectalis of the authors) cannot be confirmed by me as to the Cyprinoids (s. above), yet I saw in a specimen of Exocoetus a small part of the fibres of my tr. isthmo-tectalis end in the nucl. praetectalis, the other larger part running further frontad past this nucleus. The same I have been able to find with Cyclopterus, Trachinus, with some doubt also with the Gasterosteids, Syngnathidae, Belone and Solea. As just on the spot where these fibres should enter and end

in the nucl. praetectalis, the brachium tecti, whose fibres are as coarse as the others, leaves this nucleus, it is quite possible that a misinterpretation can arise. All the same I can maintain this point as being quite certain, at least with Exocoetus. When this is the case, these fibres correspond with those of CATOIS from the nucl. praetectalis, of which he writes: "Les autres (fibres)[1]) descendent presque verticalement de la partie inférieure du noyau prétectal, se recourbent ensuite en arrière et se dirigent vers la région basale du mésencéphale" (op. cit., p. 97). But when he supposes "qu'elles doivent servir à établir des connexions entre le thalamus et la moelle spinale" (l.c., p. 97—98), this remains a mere supposition, for according to my investigations we can say for certain that they are connected with in the ganglion isthmi and consequently form a *tr. isthmo-praetectalis*,[2]) (or *praetecto-isthmicus*).

IV. *Pars praetectalis Comm. posterioris.*

The component part of the so-called *"stratum zonale"* of the torus semicircularis, (for convenience' sake I here use this long abandoned nomenclature) is not at all exhausted with the above tr. isthmo-tectalis mihi and the commissura transversa. With most bony fishes there is namely a fibre connection between this stratum and the commissura posterior. One bundle from about the middle of this commissure runs latero-ventrally and at the same time frontad, joins the tr. isthmo-tectalis in the most caudal level of the nucl. praetectalis (fig. 6) or, if this is missing, directly the commissura transversa and then joins in the characteristic curved course of the torus semicircularis. To the first category, where there is a tr. isthmo-tectalis, belong the Gasterosteids, Belone, Mugil, Ophiocephalus, Morone, Osphromenus, Pleuronectids, Gobius, Cottus, Cyclopterus, Agonus, Trachinus and Callionymus, to the latter, where the tr. isthmo-tectalis is not present, belong Symbranchidae?, Esox, Ammodytes, Gadids, Solea, Lophius and Tetrodontidae. The fibres of the latter bundle are thinner than those of the tr. isthmo-tectalis. On the other hand there are fishes that have no such connection, e.g. Megalops?, Clupea, Cyprinoids, Syngnathidae, Exocoetus and Zoarces?.

This bundle differs from the other components of the commissura posterior by its finer fibres and mostly also by its compactness. As to its position it is about in the middle of the commissure, in sagittal

[1]) Bracketed by myself.

[2]) Cf. also HOLMGREN's drawings of Callionymus: fig. 87, 88 and 89: Tr. ist.-praet.

as well as in dorso-ventral direction. In WEIGERT-preparations it often has a greyish colour, which makes me suppose that it consists of medullary as well as unmedullated fibres.

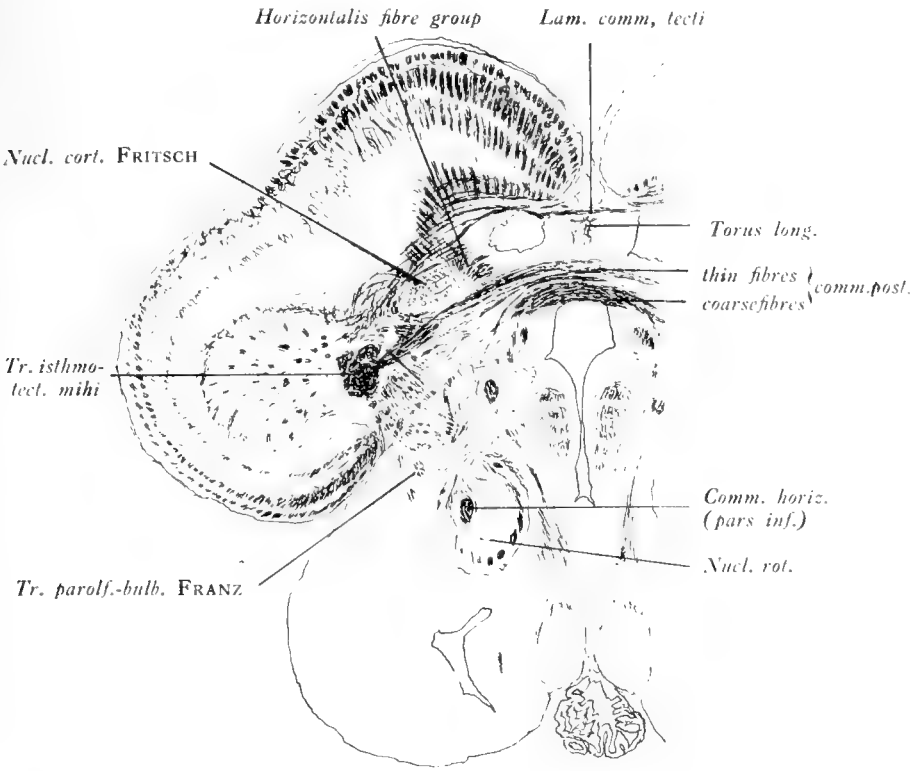


Fig. 6. Hippoglossoides platessoides.
(Preparation of Prof. RÖTHIG's Collection, Berlin.)

ARIËNS KAPPERS in his work on Ganoids (op. cit., p. 475) has expressed the supposition that the middle part of the commissura posterior partly enters into a connection with the geniculatum (i.e. thé nucl. praet-ectalis of the authors), partly passes over this nucleus, bends backwards and ends in the tegmental region, just under the torus semicircularis. Also in his treatise on the brain of Chimaera (p. 158) and of late in his manual (p. 818) he considers it as highly probable that this "lateral part" of the commissura posterior originates in the geniculatum (i.e. in the nucl. praetectalis) of the one side and extends caudad on the other side (extremely clear with Pleuronectidae, as he emphasizes).

HOLMGREN calls the relative part of the commissura posterior after EDINGER commissura praetectalis, but is also of opinion that it is "not excluded that praetectalis fibres, that were traced till in the

commissura posterior, may go on the other side to an other final station than in the nucleus praetectalis" (op. cit., p. 262).

In my opinion one of the two formations is to be considered as the caudal destination of this bundle, viz. the torus semicircularis or the ganglion isthmi, the first with greater probability than the latter. About its frontal extremity I cannot express an opinion for the present, although the nucl. praetectalis of the other side seems to be the most probable. In any case further investigations on this point are most desirable.

It is difficult yet to say anything about the relation of the bundle concerned, to the commissura praetectalis (or pars praetectalis of the commissura posterior) of EDINGER. However, I have been able to state with a specimen of Leuciscus rutilis that, although with no other of the Cyprinoids which I examined, I could prove a well-marked fibre connection between the commissura posterior and the tr. isthmo-tectalis mihi or even the "stratum zonale" of the torus semicircularis, with the said fish there existed a well characterized, closed commissura bundle between the nuclei praetectales of both sides.

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Chemistry. — "*n.α-Sulfobutyric acid and its optically active components*". By Prof. H. J. BACKER and J. H. DE BOER. (Communicated by Prof. F. M. JAEGER).

(Communicated at the meeting of January 27, 1923).

After it had been shown that α-sulphopropionic acid can be separated into its optically active components [1]), we tried to effect this resolution also for norm. α-sulphobutyric acid. At the same time the occasion was taken to study this acid, which has been known already since 1875, but hitherto had not been obtained in a pure crystallised state.

The acid is formed by direct sulphonation of n.α-butyric acid or of butyric anhydride [2]).

Just as in the case of the propionic acid, the sulphonic acid group is attached to the α-carbon atom, as proved by its relation to α-bromobutyric acid, of which the structure is fixed.

HEMILIAN caused the ester of this acid to react with ammoniumsulphite and we have applied this reaction to the free α-bromobutyric acid; in both cases the same sulphobutyric acid was formed as by direct sulphonation.

We also obtained the sulphobutyric acid in a good yield (70 %) from ethylmalonic acid, which by sulphonation loses one molecule of carbon dioxyde. Besides, this formation may serve as an argument for the structure, the active hydrogen atom of the ethylmalonic acid having the greatest chance of being substituted by the sulphonic acid group.

As a method of preparation we used the sulphonation of the carefully fractionated n. butyric acid with sulphur trioxyde. In the cold butyrylsulphuric acid is formed, which on heating passes into sulphobutyric acid:

$$C_2H_5 . CH_2 . CO_2 . SO_3H \rightarrow C_2H_5 . CH(SO_3H) . CO_2H.$$

The acid was separated in the form of its barium salt, which was purified by crystallisation, and from which sulphuric acid liberated again the organic acid.

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[1]) FRANCHIMONT and BACKER, These Proceedings **17**, 653 (1914); Recueil d. trav. chim. **39**, 751 (1920).

[2]) HEMILIAN, Ann. d. Chemie **176**, 2 (1875). FRANCHIMONT, Recueil d. trav. chim. **7**, 27 (1888). VAN PESKI; Recueil **40**, 736 (1921).

Sulphobutyric acid was hitherto only known as a viscous liquid. We succeeded in obtaining the acid in the crystallised state by leaving a concentrated pure solution for a long time in vacuo over phosphorus pentoxide.

The *a*-sulphobutyric acid forms colourless hard crystals. Like sulphoacetic and sulphopropionic acids it contains one molecule of water of crystallisation and is extremely hygroscopic. The melting point, determinated by the aid of a formely described apparatus[1]), was found to be 66°.

Since sulphoacetic acid melts at 84—85° and sulphopropionic acid at 100.5°, we have here perhaps the beginning of an alternating series of melting points, as shown by the fatty acids.

From sulphobutyric acid we have prepared some salts with aromatic amines.

The acid sulphobutyrate of aniline forms small glistening crystalline plates with the melting point 175°.

The acid sulphobutyrate of p-toluidine, which is separated by ether from its alcoholic solution in the form of an ethergel, may be obtained as a white crystallised substance of the melting point 163°.

The acid salts of p-anisidine and p-phenetidine were obtained in a crystallised state, but not pure and colourless.

If these sulphobutyrates are heated with an excess of the corresponding amines, the carboxyl group is changed into amide through loss of water, the sulphonic acid group remaining combined with a molecule of the amine.

In this way aniline formed the *butyranilide-a-sulphonic acid salt of aniline*

$$(C_2H_5)(CONHC_6H_5)CH \cdot SO_3H \ldots NH_2C_6H_5,$$

which crystallises from water in concentrically grouped featherlike needles, occasionally 5 cM. in length, which melt at about 253°—256°.

From the other above mentioned aromatic amines well crystallised amides were also obtained, viz.

butyro-p-toluidide-a-sulfonic acid salt of p-toluidine, m.p. 260—263°,
butyro-p-anisidide-a-sulfonic acid salt of p-anisidine m.p. 242°,
butyro-p-phenetidide-a-sulfonic acid salt of p-phenetidine, m.p. 264—266°.

When heated with aromatic o-diamines, sulphobutyric acid, just as sulphopropionic acid, loses two molecules of water and gives derivatives of benzimidazole.

The sulphobutyrate of o-phenylenediamine, for instance, formed

[1]) Chem. Weekbl. **16**, 1564 (1919).

on heating at 180° *benzimidazole-2-propylsulphonic acid* (I), whilst from the sulphobutyrate of 3,4-diaminotoluene was formed in the same way *methylbenzimidazole-2-propylsulphonic acid* (II).

$C_6H_4<^{N}_{NH}>C\,.\,CH<^{C_3H_5}_{SO_3H}$ (I) $CH_3\,.\,C_6H_3<^{N}_{NH}>C\,.\,CH<^{C_3H_5}_{SO_3H}$ (II)

These imidazoles were obtained as white crystalline substances. They are almost insoluble in the common solvents, have a very high melting point, and, notwithstanding the presence of a sulphonic acid group, they do not combine with aromatic amines and they are not hygroscopic. All these properties indicate, that the sulphonic acid group forms an internal salt with the basic function of the imidazole (III) and they completely recall the properties of taurine, for which an analogous structure is assumed (IV).

$$C_6H_4<^{N}_{NH}>C\,.\,\underset{\vdots\ldots\ldots HO_3S}{CH}\,.\,C_3H_5 \quad \text{(III)} \qquad \begin{matrix} CH_2 - CH_2 \\ | \quad\quad | \\ SO_3H\,..\,NH_2 \end{matrix} \quad \text{(IV)}$$

With strong bases, such as baryta, these imidazolesulphonic acids give well crystallised salts. From the barium salt and copper sulphate a green solution is formed, which, however, decomposes immediately when heated and also when kept for a long time at the ordinary temperature, so that the copper salt could not be separated in a crystallised pure state. It deserves attention, that, in spite of many efforts, also no copper salt of taurine has been obtained.

The resolution of racemic *α*-sulphobutyric acid was attempted with the aid of strychnine, by reason of previous experience with sulphopropionic acid, and the attempt was successful.

The acid strychnine salt of the d-acid is less soluble than the salt of the l-acid, just as in the case of sulphopropionic acid.

After three or four crystallisations the acid strychnine salt of the d-sulphobutyric acid is entirely free from the other component. It crystallises with two molecules of water in small glistening needles.

On concentration, the first mother liquor slowly gives a crop of the acid strychnine salt of l-sulphobutyric acid, which by repeated crystallisation from alcohol is obtained in a pure state.

Decomposition of the strychnine salts by baryta gives the barium salts of the active acids.

These barium salts crystallise from water in long needles which contain $2\frac{1}{2}$ molecules of water, in contradistinction to the racemic barium salt, which separates in small glistening leaflets with two molecules of water of crystallisation.

The direction of the rotation of the neutral barium salts is, as in the case of sulphopropionic acid, opposite to that of the free acids.

The molecular rotatory power depends on the concentration; on dilution it rises a little. The barium salts, for instance, give in a $2\frac{1}{2}\%$ solution for sodium light a molecular rotation of 32.2° and in a 5% solution a rotation of 29.9°. In a $2\frac{1}{2}\%$ solution the presence of 10% of barium chloride lowers the molecular rotation to 29.3°.

This indicates, that the rise of the molecular rotation on dilution may be ascribed to an increasing of the ionisation, a phenomenon, which is perhaps connected with the fact that the sign of rotation of the neutral salts is opposite to that of the free acids.

The molecular rotation of the free salts for sodium light is 7.8°.

The acid salts rotate the plane of polarisation in the same direction and to about the same amount as the free acids.

In this respect also, the behaviour of sulphobutyric acid is therefore analogous to that of sulphopropionic acid.

The investigation is being continued and will be published later in greater detail.

Groningen. 13 Jan. 1923. *Organic Chemical Laboratory of the University.*

Chemistry. — *'The second dissociation constant of sulphoacetic and α-sulphopropionic acids.''* By Prof. H. J. BACKER. (Communicated by Prof. F. M. JAEGER).

(Communicated at the meeting of January 27, 1923).

The determination of the second dissociation constant of a dibasic acid H_2A from the concentration of the hydrogen ions in the solution of an acid salt readily suggests itself.

However, A. A. NOYES [1]) has shown, that generally these data will not suffice.

Suppose that the ionisation of the acid sodium salt (reaction I) is nearly complete, and that the concentration of the HA' ions, which according to reaction II are partly split further, may be identified with the concentration of the acid salt dissolved, we must nevertheless remember that the number of hydrogen ions will decrease by combination with the ions HA' (reaction III).

$$NaHA \rightarrow Na^{\cdot} + HA' \text{ (I)} \quad HA' \rightleftarrows H^{\cdot} + A'' \text{ (II)} \quad HA' + H^{\cdot} \rightleftarrows H_2A \text{ (III)}$$

This last reaction will be especially noticeable, when the acid is weak, which is indeed the case with all organic acids examined in this respect.

The sulphocarboxylic acids, however, are examples of strong dibasic acids, which at small dilutions are already well ionised. Therefore, we may expect, that the consumption of hydrogen ions for formation of the free acid will only have a small influence, so that from the concentration of the hydrogen ions the degree of dissociation of reaction II may be determined, and further the dissociation constants.

NOYES has given the following general formula for the acid salts of dibasic acids:

$$k_2 = \frac{(k_1 + c + H)\,H^2}{k_1\,(c - H)}.$$

k_1 and k_2 are the first and second dissociation constants of the acid, c is the original concentration of the acid salt (in gram molecules per litre) and H is the concentration of hydrogen ions (in gram ions per litre).

[1]) Z. f. physik. Chemie **11**, 495 (1893).

If k_1 is large compared with c and H, we may write:

$$k_1 = \frac{H^2}{c-H}.$$

Now, this expression is identical with OSTWALD's dilution law, $k = \alpha^2/(1-\alpha)v$, as shown by substitution of $\alpha = H/c$ and $v = 1/c$.

This simplification will be permissible in the case of sulphoacetic and sulphopropionic acids, for which, in a previous paper[1]) the first dissociation constants were found to be 0.58 and 0.57 respectively.

Now, the solutions of the acid salts of these compounds in various concentrations were compared, by the aid of indicators, at room temperature, with the buffer solutions of SÖRENSEN and of CLARK.

In the following table v is the numbre of litres, containing 1 gram molecule of the acid salt, p is the hydrogen exponent ($p = -\log H$) H is the concentration of the hydrogen ions in gram ions per litre, α is the degree of dissociation of reaction II ($\alpha = Hv$) and the equilibrium constant derived therefrom is $k_2 = \alpha^2/(1-\alpha)v$.

	v	p	H	α	k_2
Sulphoacetic acid	16	2.65	0.00224	0.0358	8.3×10^{-5}
	32	2.8	0.00158	0.0506	8.4
	64	2.95	0.00112	0.0717	8.5
	128	3.05	0.00089	0.114	11.5
	256	3.25	0.00056	0.143	9.4
	512	3.4	0.00040	0.205	10.3
Sulphopropionic acid	16	2.7	0.00200	0.0320	6.6×10^{-5}
	32	2.85	0.00141	0.0451	6.7
	64	3.0	0.00100	0.0640	6.8
	128	3.2	0.00063	0.0806	5.5
	256	3.4	0.00040	0.102	4.5
	512	3.55	0.00028	0.144	4.7

The concordance of the constants at various dilutions is very satisfactory, as the indicator method does not allow a great accuracy.

However, for great concentrations a correction might be made

[1]) These Proceedings **25**, 359 (1922).

according to NOYES' formula. For this the values of k_2 must be multiplied by $(k_1 + c + H) / k_1$.

This correction only affects the dilutions 16, 32 and 64.

Thus, the following values are found:

	$v =$ 16	32	64	128	256	512	mean value.
sulphoacetic acid	$k_2 =$ 9.1	8.9	8.8	11.5	9.4	10.3	9.7
sulphopropionic acid	$k_2 =$ 7.2	7.1	7.0	5.5	4.5	4.7	6.0

Little differences in the colorimetric determinations of p have in this method a great influence on the value of k_2.

In a simpler way the second dissociation constant of a dibasic acid may be measured by examining a mixture of a neutral and an acid salt [1]).

If α'' is the degree of dissociation of the neutral salt Na_2A and α' the degree of dissociation of the acid salt NaHA, then the second dissociation constant of the acid may be represented by:

$$k_2 = \frac{\alpha'' \times [Na_2A]}{[\alpha' \times NaHA]}.$$

Since these degrees of dissociation for salts are not much smaller than 1, the factor α''/α' may be neglected in a first approximation.

For the sake of simplicity a solution was taken containing an equal number of molecules of the acid and of the neutral salt, so that $k = H$, and this solution was examined at various dilutions.

The concentration of hydrogen ions was again determined by means of the indicator method.

In next table v is the number of litres containing one molecule of the neutral salt together with one molecule of the acid salt.

The variations of the constant due to dilution are not considerable, but it is remarkable that they are all in the same direction. By dilution the degree of acidity of the solution decreases.

This behaviour indeed agrees with the theory, since for the sodium salt of a dibasic acid the dissociation on diluting increases more than for the sodium salt of a monobasic acid. Therefore the value of α''/α', which for infinite dilution must amount to 1, is smaller for the greater concentrations.

The value of α'' follows from the conductivity of the neutral sodium salt at various dilutions, published in the previous paper, and for α' the above mentioned values may be taken.

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[1]) In this way I. M. KOLTHOFF has measured the second dissociation constants of a number of dicarboxylic acids. (Der Gebrauch von Farbenindicatoren, p. 102).

	v	p	$H = k_2$
Sulphoacetic acid	32	4.0	10.0×10^{-5}
	64	4.05	8.9
	128	4.1	7.9
	256	4.15	7.1
	512	4.25	5.6
Sulphopropionic acid	32	4.25	5.6×10^{-5}
	64	4.3	5.0
	128	4.35	4.5
	256	4.4	4.0
	512	4.4	4.0

When this correction is made, the following constants are found:

	$v =$	32	64	128	256	512	mean value :
sulphoacetic acid	$k_2 =$	8.7	7.9	7.3	6.7	5.4	7.2
sulphopropionic acid	$k_2 =$	4.8	4.4	4.1	3.7	3.8	4.2

In the preceding paper the second dissociation constants of both acids are calculated from measurements of the conductivity of the acid salts.

In the present paper these constants have been obtained colorimetrically first from the p_H of the acid salt and then from the p_H of mixtures of neutral and acid salts.

The mean results of the various methods are collected in the following table.

Methods	Sulphoacetic acid	Sulphopropionic acid
I. Conductivity of acid salts.	$k_2 = 7.4 \times 10^{-5}$	$k_2 = 4.8 \times 10^{-5}$
II. Hydrogen ion concentration of mixtures of acid and neutral salts.	7.2	4.2
III. Hydrogen ion concentration of acid salts.	9.7	6.0

In judging these figures it should be remembered, that each of the methods used here only gives approximative values, which is also evident from the deviations in each series of measurements.

However, the order of magnitude is the same for the constants determined in various ways.

Thus from this research we may conclude, that the second dissociation constant of sulphoacetic acid amounts to about 1×10^{-4} and that the constant of sulphopropionic acid is about one third smaller.

Groningen, January 1923.

Organic chemical laboratory of the University.

Zoology. — "*Experimental Budding in Fungia fungites*". By Dr. H. Boschma. (Communicated by Prof. C. Ph. Sluiter).

(Communicated at the meeting of January 27, 1923).

A large number of the Fungiae to be found on coralreefs display anomalies mostly arising from the destruction of part of the living tissue. In many cases the destroyed stretches of living tissue are attacked by small algae, which penetrate to a considerable depth, and gradually spread into the living tissue. Such decaying spots often stimulate the adjacent tissue, which consequently exhibits a more energetic growth-activity than usual. The result then is that something like a raised rim arises on the border between the living and the defunct part. In many cases this greater activity is also manifested even in the formation of buds. In a previous publication I discussed this budding in adult Fungiae [1]). Here I also pointed to the fact that algae-parasitism is one of the chief causes of budding in adult corals. Generally the destruction of only a small part of the living tissue suffices for the vicinity to be stimulated to a more energetic growth-activity.

This induced me to endeavour to develop buds experimentally in *Fungia fungites*. My material for this experiment consisted of specimens of *Fungia fungites* from the reef of the island of Edam near Batavia. The most normal corals devoid of buds or other anomalies were selected. To destroy part of the tissue a small piece of putty was pressed into the central region of the oral surface of some fifty specimens on the 18th and the 19th of August 1921. The putty was held fast on either side of the mouth by the septa. The corals were then restored to their original places.

In this experiment, I expected the destruction of part of the central tissue to extend to the mouth in most of the specimens, as this would most likely bring about a strong reaction to the lesion, so that budding would soon ensue. True, the ingest of food would hereby be slightly impeded. But considering that Fungia feeds only partly on organisms other than zooxanthellae, and considering moreover

[1]) H. Boschma, "On Budding and Coalescence of Buds in Fungia fungites and Fungia actiniformis." Proceedings Kon. Ak. van Wetensch. Amsterdam. Vol. XXIV, 1922.

that the basal portions of the axial groove were not entirely covered, the impediment was not of a serious nature. This experimenting method was most suitable for achieving results in a short time.

After the lapse of nearly four months the putty could still be seen unaltered as to shape, as a hardened substance above the mouth. Some corals had already developed buds. On the 11th of December 1921 five specimens were brought back, one of which (N°. 464) was preserved in formalin and the other four were left dry (N°. 460—463). The changes resulting from the experiment are summarized as follows:

N°. 462. About one fifth of the upper surface is defunct. Beneath it buds have developed on the under surface, smaller ones at the margin, larger ones more towards the centre.

N°. 463. Half of the upper surface is defunct. Only few septa in this destroyed part exhibit in the margin residues of living tissue. Portions of the margin of the under surface, under the defunct part of the upper surface, are also defunct. The rest is still covered with living tissue. On the upper surface some large buds and many small buds at the margin. (Fig. 1—3).

N°. 464. Two opposite quarters of the upper surface devoid of living tissue. The destruction of the soft parts has extended round the margin of the coral, so that here also some portions are defunct. On the under surface a few large buds, a few smaller ones in the margin.

N°. 460. On the upper surface the living tissue was quite lost, on the under surface only in some places at the margin. Here a few small buds are to be recognised, while in the more central part a few larger ones have developed.

N°. 461. Upper surface quite defunct, under surface still covered with living tissue. In the margin of the under surface many small buds, in the centre a few larger ones.

In all specimens a stretch of the tissue nearest to the putty first died away. This process progressed along the septa to the periphery so that the defunct part assumed the form of a sector of a circle. The decay of the living tissue now spread along the margin on the lower surface, the consequence of which was that the environing tissue was stimulated to greater activity and accordingly developed buds.

At the living corals the larger buds, which were located at some distance from the margin, were most conspicuous. (Fig. 3). The diameter of the basal part of these buds varied from 2 to 12 mm. The mouth was invariably small and the height inconsiderable. The spines of the costae of the parent coral were often visible through the thin living portions of the bud. In these large buds the skeleton is still very incomplete. The theca and the first septa are only little developed; on the other hand the columella is already distinguishable in the form of a large number of irregular trabeculae.

In the smaller buds, which were generated principally in the marginal regions of the under surface the development of the skeleton can easily be traced, as the buds differ very much in age. They are of a much more regular structure than the larger ones.

In the youngest buds, with a diameter of about 0.5 mm., nothing of the skeleton is visible except the theca, which appears as a thin wall, stretching obliquely upward and consequently looks like a truncated cone. (Textfig. *a*).

Fig. *a*. × 45.

Fig. *b*. × 45.

Fig. *c*. × 45.

The theca has no perforations, which come forth only in much older buds. Soon after this the first cycle of six septa spring up. They proceed from the theca further towards the centre of the bud. (Textfig. *b*). The upper rim of the theca rises above the septa. The columella also develops in this phase as short projections in the basal parts of the bud. In buds of this size there are never more than six septa. They originate almost simultaneously, buds with a smaller number of septa occurring only very seldom. The number of similar buds with less than six septa is too small to ascertain whether the septa arise in a definite order.

The next cycle of septa can only be observed in buds of about 1 m.m. in diameter. In them the septa of the first cycle have already considerably increased in size and in thickness, and are already provided with some dentations. (Textfig. *c*). Likewise the columella has grown larger in this stage. The septa of the second cycle are distinguishable at first sight from those of the first cycle by their being less developed and being shorter. The bud has now attained the length of the youngest stage described by BOURNE[1]),

[1]) G. C. BOURNE, On the Postembryonic Development of Fungia. Transact. Roy. Dublin Soc. Vol. V, 1893.

to which it bears great resemblance. The further development of these buds resembles that of the buds of an anthocormus.

With the exception of the five specimens that were brought back in December 1921, the other Fungiae remained on the reef during nearly nine months. On the 2nd of September, when the experiment had been going on for more than a twelvemonth, the specimens that could still be found, were collected. The putty was still in the central part of the oral surface; in the majority of cases the form was unaltered.

In most corals at least some part of the oral surface had lost its living tissue, in a few cases only the plug of putty had caused little or hardly any change. The aspect of the Fungiae was now as follows:

Nos. 507. 510, 519 and 520. The aspect of the corals was very normal, without defunct parts. No budding.

N°. 518. Living tissue normal. The central parts of some septa have risen and have longer dentations. This is owing to the occurrence of new mouths by the side of the old mouth, as was easy to see in the two following specimens.

N°. 509. No parts of the living tissue destroyed. The central extremities of many septa have grown higher in those places which were in contact with the putty and new mouths have been developed beside these elevations of the septa. The new mouths are now entirely surrounded by septa; on the one side by long regular ones (the original septa of the parent-coral) on the other side by higher parts of recent origin. These parts are somewhat irregular in shape; also the dentations are longer than those of the original septa.

N°. 508. Covered all over with living tissue. On either side of the old mouth-fissure a few young buds had developed, whose mouths lay between the normal longer parts of the septa and the higher irregular parts that originated later on. (Fig. 5). This specimen is very much like N°. 509, in which the young septa between the new mouths and the putty are also provided with long dentations.

N°. 521. Few alterations. The living tissue has disappeared only from the central parts of some contiguous septa. Budding is absent.

N°. 512. Some adjacent septa devoid of living tissue, further no alterations. No budding.

N°. 506. Upper surface with two defunct parts, the larger of which covers nearly one fifth of the surface; the smaller part is a narrow streak from the mouth to the margin of the coral. The larger part of destroyed tissue overlaps the margin and covers a small portion of the under surface. At the margin two stemmed young buds have taken origin. The diameter of the disc is respectively 2,5 en 3 m.m. In the defunct part on the under surface there are a few smaller buds.

N° 501. Almost half of the upper surface defunct, just as a smaller part of the under surface, especially the margin under the destroyed portion of the upper surface. On the boundary between the living and the destroyed part of the under surface, five buds have developed still completely encircled by living tissue of the mother-coral. They are very regular and distinctly stemmed. The diameter of the disc, which in all of them is already broader than the stem, amounts to 6, 7.5,

10.5, 8.5 and 6 m.m. In the defunct marginal part of the under surface there are some smaller younger buds (diameter 1 to 3m.m.), which, however, have lost their living parts.

N°. 500. Along the shorter diameter of the corallum a broad band of the upper surface has lost its living tissue. In the living part some septa exhibit a more energetic growth of the central part; however, new mouths could not be distinguished as yet. The parts of the margin of the under surface contiguous to the defunct part of the upper surface had lost their soft portions. In their vicinity buds had developed in the living tissue, five on one side and two on the other (Fig. 4). These buds are less regular in form than those of N°. 501. Their stages of development differ. The dimensions are: 13×8, 7.5×7, 4.5×4, 6×5.5, 5×4, 13.5×8.5, and 10×7 m.m. They are fixed to the parent coral by a broad base, The septa of the youngest buds, which are still little developed, are distinguishable from the spines of the costae of the mother-coral by their flattened shape. In the basal parts of most of these buds the spines of the mother-coral are still unaltered. In the destroyed part of the margin, with the five buds, a stemmed bud has developed (diameter of the disc 4.5 m.m., of the stem 3.5 m.m., height 3.5 m.m.). Besides these there are still remains of a number of smaller ones, whose living tissue has, however, disappeared.

N°. 511. Only one third of the coral was covered by living tissue on the upper surface as well as on the lower surface. In the defunct portion of the lower surface a great many buds had arisen, most of which were still alive. The diameter of these buds ranges from 1 to 3 m.m.

N°. 502. Of the upper surface only a small part of the margin was still covered with living tissue; of the under surface almost one fourth was still alive. In this part there are in the vinicity of the defunct region four large buds, only two of which possess well-developed septa. The dimensions are 10×7, 7×6.5, 9.5×7.5. and 10×9 m.m. The buds are not yet stemmed, so that the basal parts of the septa are still fixed over their whole length to the skeleton of the mother-coral. The septa of the youngest buds are distinguishable from the spines of the costae of the mother-coral only by their flattened shape. Besides these large buds there are at the margin, now surrounded by the destroyed region, two stemmed buds with a disc, 3 and 2.5 m.m. in breadth. Moreover a few smaller ones are also visible in the marginal part

N°. 514. Upper surface without living parts. However, the tissues of a fourth part had died off quite recently, the skeleton of this part still being little overgrown with algae and other organisms, in contradistinction to the remaining part. At the margin of the part that died off long ago some few young, stemmed buds have developed, which however, have likewise lost their living tissue. The under surface still possessed rests of living tissue beneath that portion of the upper surface, which kept alive longest. Then follows a broad edge from which nearly all soft parts had disappeared. Here some large buds have developed (diameter up to 6.5 m.m.). Little is to be seen as yet of the skeleton In the remaining part of the under surface, which had been defunct longer, the remains of many small buds are visible, none of which were alive any more.

N°. 516. Upper surface devoid of living tissues in the margin a few short-stemmed young buds. Under surface still covered with living tissue. In the margin a few young buds of small dimensions, still completely encircled by living tissue of the parent-coral.

N°. 513. Living tissues entirely disappeared from the upper surface; on the under

surface about one third defunct. In the marginal stretches, where the soft parts have disappeared, a few young buds, most of which are stemmed. Diameter of the disc of these buds up to 3 m.m. In the part of the under surface, which is still covered with living tissue, there occur a large number of buds in all stages of development. The size ranges between 0.5 and 3.5 m.m. The stages represented in the textfigures are also perceptible in many buds.

N°. 505. The upper surface as well as the under surface without living tissue. At the margin some buds occur; the disc of the largest bud has a diameter of 7 m.m. On the under surface of the coral many young buds in different stages of development.

N°. 517. This specimen happened to lie upside down. It had lost its living tissue on both sides. On the aboral surface (now the upper surface) no buds had formed, on the oral surface there are eight buds, some of which are already stemmed. The diameter of the disc of these buds varies from 2.5 to 5.5 m.m.

It appears from the foregoing that the results are very different. In some cases the destruction of part of the living tissues had an influence only on the immediate vicinity, where the tissue was consequently brought to greater activity. This appeared from the formation of new mouths beside the old one which had got lost, and of small septa between the new mouths and the defunct part.

Owing to the experiment a smaller or a greater part of the remaining living tissue of the *Fungia* had been destroyed. This process began invariably at central parts of one or more septa, i.e. beside the putty. When the central part of a septum has lost its living tissue, this process progresses towards the periphery and farther along the margin to the under surface of the coral. Of the decaying tissue some isolated parts keep alive and buds issue from them. At the margin of the mother coral these buds are small and of a regular shape; they develop like buds of an anthocormus.

Regarding the development of the skeleton a few remarks may follow here. According to BOURNE[1]) the twelve first septa of Fungia originate simultaneously, as is also the case with *Astroides*. In the former, however, the six septa of the first cycle come first and then those of the second. Since the development of the buds is so very regular and the older stages are quite similar to those of the buds of an anthocormus, it may be expected that the first stage of development of the skeleton of the young Fungiae, which arise from planulae, is similar to that of the youngest buds here described.

Moreover the youngest stages of *Fungia patella* described by GARDINER[2]) possess no more than six septa, while the older stages

[1]) loc. cit.

[2]) J. STANLEY GARDINER, On the Postembryonic Development of Cycloseris. Willey's Zoological Results. Pt. II, 1899.

bear a striking likeness to the young Fungiae, described by BOURNE. VAUGHAN [1]) also points out that it has not yet been proved that the first twelve septa of Fungia appear simultaneously.

In the development of *Caryophyllia* [2]) there is one stage in which the skeleton agrees very much with the stage illustrated in Textfig. *b*. However, the preceding processes differ in the two corals; whereas in *Caryophyllia* the septa are formed prior to the theca, the reverse takes place in *Fungia*. In *Caryophyllia*, therefore, the septa have outgrown the theca much sooner than in *Fungia*.

So while a great number of small buds appear at the margin, and several large ones on the under surface, the tissue is dying off by slow degrees. The result is a defunct specimen with a large number of living buds of different age. Many authors [3]) look upon such buds on defunct specimens of the same species as having originated from larvae.

In a previous paper I advocated my view that these young Fungiae must be considered as buds [4]). My experiment yielded all sorts of intermediate stages between normal specimens and defunct ones with buds. The large buds that may arise on the aboral surface, are in their earliest phase so large already (up to 12 mm. in diameter) that it is a priori highly improbable that they should have been formed from larvae. Besides the lateral tissues of the bud are connected with those of the parent, while the basal living parts of the bud overlie the skeleton of the old coral, which results from the way in which the columella is formed in these buds. The trabeculae of their columella namely are generated between and on the unaltered spines of the costae of the parent coral.

In the above description the young individuals, which resulted

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[1]) T. WAYLAND VAUGHAN, Recent Madreporaria of the Hawaiian Islands and Laysan. Smithsonian Institution, U. S. Nat. Museum, Bull. 59. 1907.

[2]) G. VON KOCH, Entwicklung von Caryophyllia cyathus. Mitt. Zool. Stat. Neapel, Bd. XII, 1897. (The stage alluded to is reproduced in Fig. 14).

[3]) S. STUTCHBURY, An Account of the Mode of Growth of Young Corals of the Genus Fungia. Trans. Linn. Soc. London, Vol. XVI, 1833.

H. N. MOSELEY, Notes by a Naturalist on the Challenger. London, 1872.

L. DÖDERLEIN, Die Korallengattung Fungia. Abh. der Senckenb. naturf. Ges. Bd. XXVII, 1902.

Also the youngest stages of *Fungia patella*, described by GARDINER (loc. cit.) are probably buds of a specimen, of which the remaining part of the living tissue had been destroyed.

[4]) loc. cit. SAVILLE KENT (The Great Barrier Reef of Australia. London, 1893) also deems it most probable that these young Fungia are buds, originating from the remains of the living tissue.

from the destruction of stretches of living tissue, have been called buds. Theoretically however, none of these individuals can be considered as buds. In budding the parent remains intact, the buds are generated through a local intensified growth at the body of the parent (DEEGENER [1])). The animal, on which the young individuals grow, is now only a remnant of what it was before. The process of development of the young individuals under consideration, is rather to be defined as a fragmentation, as it has been termed by KORSCHELT and HEIDER [2]). Small portions of the tissue of the body are apt to develop into new independent individuals. That these portions are not detached from the parent coral but remain fixed to the skeleton does not take away from the theoretical significance of separation.

KORSCHELT and HEIDER point to the fact that fragmentation is originally not a phenomenon of itself, but the effect of processes of fission or budding.

The processes in Fungia, dealt with in this paper are undoubtedly related to budding. Sometimes daughter-individuals are found on the aboral surface of specimens, whose oral surface presents no anomalies. These daughter individuals are true buds. They have the same outward appearance and are attached to the parent-coral in the same way as the buds which were developed experimentally. Daughter-individuals can also be developed from that part of the living tissue of a mother-coral, which is contiguous to a small region of the margin of which the living tissue has been destroyed. The mother-coral will then remain alive, although it is slightly injured, and the young individuals, derived from a portion of the living tissues, are buds also in this case.

The evidence produced shows that any part of the tissue may develop into a complete animal. This, however, occurs only when the interconnection between the living parts of the original animal ceases to exist in consequence of destruction of part of the tissue.

The place where the young individuals develop is very different. They may arise at the top of the costae or between two costae or, when they are larger, on several costae together (Fig. 1). In corals that were inverted while the tissue was being destroyed, young individuals may develop between the septa and in the vicinity of the mouth, i.e. on the oral surface.

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[1]) P. DEEGENER, Versuch zu einem System der Monogonie im Thierreiche. Zeitschrift f. Wiss. Zoologie. Bd. 113, 1915.

[2]) E. KORSCHELT und K. HEIDER, Lehrbuch der vergleichenden Entwicklungsgeschichte der wirbellosen Thiere. 1 u. 2 Aufl. Allgemeiner Theil. 4 Lief. 2 Hälfte. 1910.

Some of the experimental animals could have survived in a slightly altered form. They are the corals, in which new mouths had been formed round the destroyed central part of the oral surface. Most specimens however had altered their shape completely: the ultimate result would ever have been a defunct disc with a number of young living individuals, chiefly on the under surface and at the margin. The young individuals on the under surface were in unfavourable conditions for further development, although some were already rather large (Fig. 4). The young Fungiae at the marginal regions, would have developed into a stemmed specimen if the corals had remained on the reef, When their disc has grown to a certain size, it falls off and at the upper extremity of the stem a new disc forms. These young Fungiae, originated from the last living residues of a defunct specimen, develop further in the same way as young individuals do, which are generated from fertilized ova.

Leyden, Jan. 1923.

Zoological Laboratory of the University.

[redacted], at any rate less acid than boric acid [redacted] substance had a smaller conductivity, while it could be ascertained by determinations of the freezing-point that it had not entirely split up into its components in aqueous solutions.

The discovery of this compound made a renewed investigation of the boro-pyro-catechates necessary.

[1]) Recueil **37**, 184 (1917).

[2]) Cf. These Proc. following communication.

Plate I.

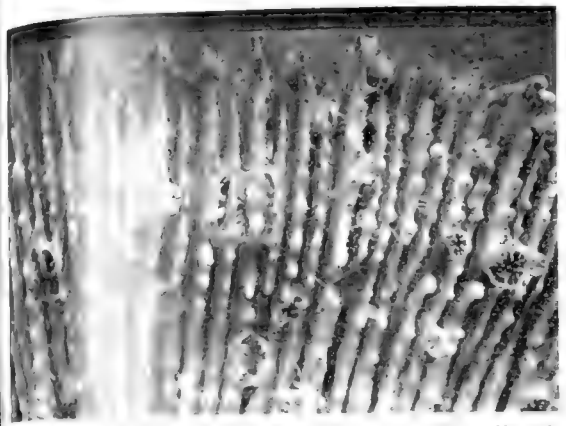


Fig. 1. Part of the margin of the aboral surface of *Fungia fungites* No. 463 living. Many small buds and a few larger, less regular ones. Magnified $\times 5$

Plate II.

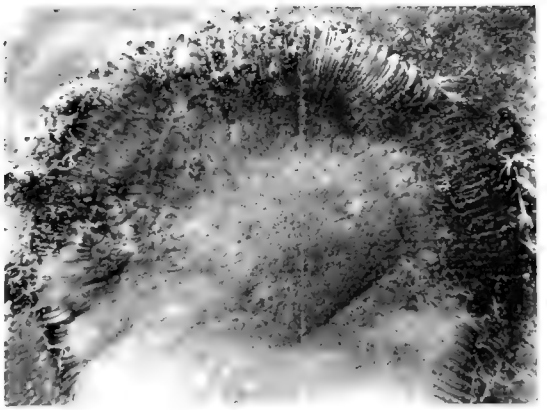


Fig. 3. Lower surface of *Fungia fungites* No. 463 living. A number of large buds, whose living parts are connected with the unaltered tissue of the lower surface. Natural size.

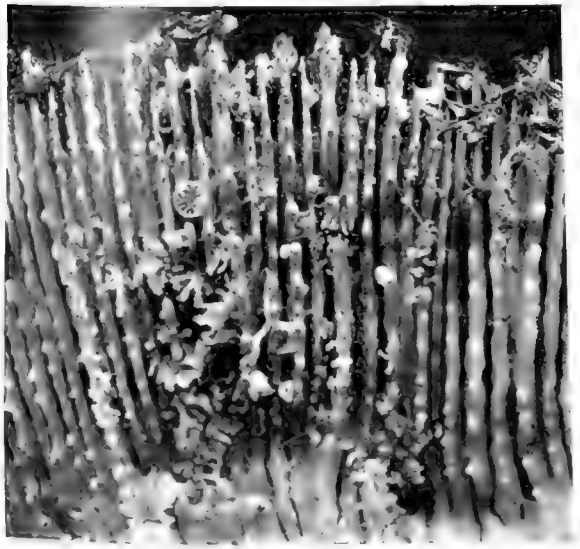


Fig. 2. Other part of the margin of the aboral surface of the same specimen. Besides a few smaller buds also a large number of irregular elements of the skeleton, especially of the columella of larger buds, are observable. Magnified $5\frac{1}{2}$

The photographs for figs 1 and 2 have been taken by Mr. G. F. BLEY of Batutulis near Buitenzorg.

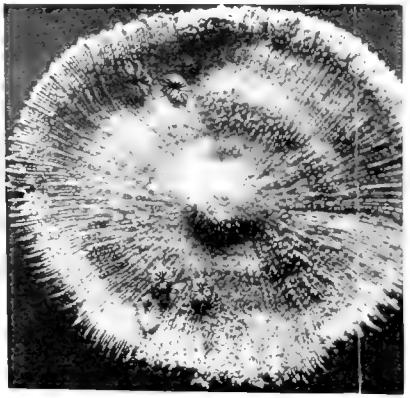


Fig. 4. Lower surface of *Fungia fungites* No. 500. Buds in the living part adjacent to a portion of the margin where the soft parts have died off. $\frac{1}{4}$ nat. size.

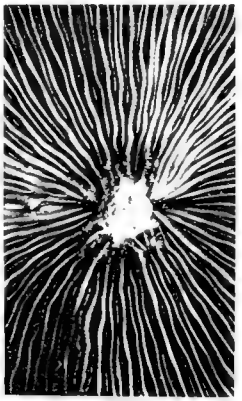


Fig. 5. The central portion of the oral surface of *Fungia fungites* No. 508. By the side of the plug of putty new mouths had been generated, which, towards the central part of the parent-coral, are encircled by raised portions of the old septa with larger dentations. $\frac{2}{3}$ nat. size.

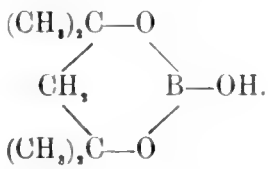
Chemistry. — "*The Valency of Boron*". By Prof. J. BÖESEKEN.

(Communicated at the meeting of December 30, 1922).

As the complex organic boric acid compounds have gradually acquired a great significance for the determination of the composition of a number of organic compounds and for the knowledge of the configuration and of the state of motion of the molecules in space, it was felt as a serious deficiency that the existence of these complex compounds had so far been exclusively derived by an indirect way, and that no compound had as yet been separated, the composition of which had been entirely made clear.

Some years ago we had, indeed, succeeded [1]) in obtaining some well crystallized salts of pyrocatechol boric acid, but they seemed to be built up in such a complicated way that no accurate conception could be formed of their composition.

Now it chanced that Mr. HERMANS [2]), who was engaged in an investigation of the equilibria in the system glycol + acetone $\gtrless$ glycol acetone + H_2O, and also examined the behaviour of the glycols towards boric acid, obtained a compound that crystallized beautifully from tetra-methyl-propane-diol-1.3 and boric acid, which according to analysis and properties possessed the following cyclic composition :



Against our expectation this compound, which had a delicate saffron odour, was hardly acid, at any rate less acid than boric acid itself, as a solution of this substance had a smaller conductivity, while it could be ascertained by determinations of the freezing-point that it had not entirely split up into its components in aqueous solutions.

The discovery of this compound made a renewed investigation of the boro-pyro-catechates necessary.

[1]) Recueil **37**, 184 (1917).

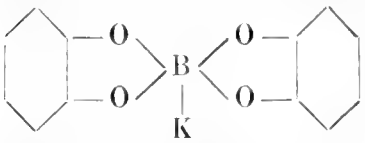
[2]) Cf. These Proc. following communication.

Mr. HERMANS, who undertook this investigation (c.f. following communication) soon succeeded in clearing up the composition of these compounds.

The empirical formula $KBO_4(C_6H_4)_2$ applies to the beautifully crystallized potassium salt; the volatile ammonium salt is $NH_4BO_4(C_6H_4)_2$, aniline salt $C_6H_5NH_2HBO_4(C_6H_4)_2$, from which through careful heating in vacuum, the free acid $HBO_4(C_6H_4)_2$ (prepared and analysed by Mr. MEULENHOFF) was obtained.

There are, accordingly, two pyrocatechol rests bound to the boron atom, in which an *entirely new type of compounds originates,* as the potassium salt hardly reacts alkalically, and, as has been known for a long time already, the relatively strong acid nature of the hydrogen derivative manifests itself in aqueous solution by increase of the conductivity.

In view of the empirical constitution and this modification of properties the below-given structural formula naturally suggests itself, in which we must imagine the anion of a relatively strong acid to have arisen through binding of the fourth oxygen atom to the boron. The acid is partially hydrolysed by water, but can be sublimated undecomposed in anhydrous condition. Also in its spatial structure the anion will be an antipode to the kation of the ammonium compounds; the four O atoms will lie in the four angles of a tetrahedron, and the two benzene rings then are vertical to each other.



The discovery of this type of boron compounds throws light on the composition of a great number of other boron compounds, and indirectly gives a powerful support to the recent considerations on the atomic structure in general. In this connection we must devote a few words to LEWIS [1]) and LANGMUIR'S [2]) atomic model, and to the natural system of elements according to KOSSEL [3]).

Very much simplified and somewhat modified [4]) these hypotheses come to what follows:

The atom is assumed to be a positive nucleus surrounded by different shells of electrons, in which the number of electrons must

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[1]) G. N. LEWIS. Journ. Am. Ch. Soc. **38** 762 (1916).

[2]) IRVING LANGMUIR ibid **41** 868 (1919) and **42**, 274 (1920).

[3]) Ann. der Physik **49** 229 (1916).

[4]) I wish to state here emphatically that I apply these considerations exclusively to the *first* period of the system, because I consider the atoms of the second period already to be too complicated to satisfy the simple postulates.

be equal to the excess of protones of the nucleus. The electrons which can more or less easily be shifted, and can even be removed, are found in the outer shell, and also electrons of other atoms can penetrate into this outer shell.

There is further a general tendency to gather eight electrons in this outer shell, because this represents most likely a very stable condition of equilibrium. We meet with this constellation in the noble gases, which do not possess chemical affinity. Only helium has only two electrons in its outer shell, and evidently forms an exceedingly stable whole with the nucleus.

The mono-valent metals have only one electron in their outer shell, and will easily split this off, in this way getting into the condition of the nulli-valent element, which stands one place lower down as to its rank; the elements of the seventh group, the halogens, have seven electrons in the outer shell, and will have a tendency to add one electron, passing with it into the condition of the nulli-valent element, which is one place higher in rank.

Thus an exceedingly stable substance of the type of Helium-Neon will be formed when Li and F are joined, with this difference that there exists a very strong electric field between these atoms, which is wanting in the noble gases.

KOSSEL has designated this kind of bonds by the name of *hetero-polar*, they exist between all metallic elements on one side and the non-metallic ones on the other side. When the electron of the metal has entered the shell of the non-metal, this has obtained for the metal-ion a same value as the seven already present ones, which means that the metal-ion is no longer bound to a definite place in the molecule; it can place itself opposite to each of the electrons present.

When the number of electrons in the outer shell increases, resp. decreases, they no longer get so easily quite outside, resp. the power to absorb foreign electrons has diminished; then ensues an interpenetration of the two shells, in which one electron of each of the atoms joins to a pair in the mutual shell division.

This is the *homöopolar* bond according to KOSSEL, in which the two atoms are bound to a very definite place. The hetero-polar or briefly *polar* bond gives rise to molecules which conduct the electric current e. g. in aqueous solution; the homöo- or *non-polar* bond is met with in substances that do not conduct the electric current.

As a type of the first we may name the alkali-halogenides, as a type of the second the organic compounds, but also water, boron-trichloride etc.

In the polar bond the atoms are thought separated, in the non-polar bond they penetrate into each other at definite places.

There is still a third kind of bond, which comes near to the

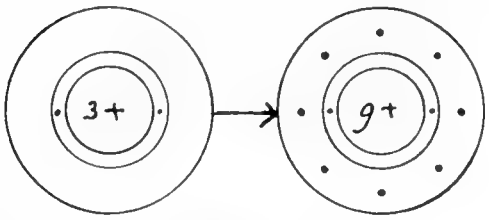


Fig. 1*a*.
Lithium fluoride.

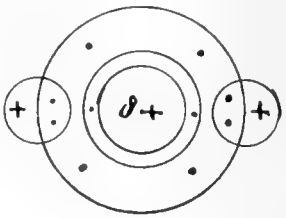


Fig. 1*b*.
Water.

non-polar bond, and is distinguished from it only in form, not in nature.

It is seen from the symbol for water that the oxygen atom has still two pair of electrons in the outer shell. These endow this molecule with the power to combine with other molecules, and especially with those of which one of the atoms lacks a few electrons in the outer shell.

Thus we must imagine that metal atoms which have ceded their electrons to acid rests on the salt formation, can get saturated with water molecules, and thus form hydrated metal ions. This kind of non-polar bond is that which was supposed to come about through by-valencies, and which is explained from the tendency to collect eight (or sometimes more) electrons in the outer shell.

It is easy to see that ammonia, though a saturated compound, can combine with a great number of substances owing to the free electrons in the outer shell. All these bonds are of quite the same nature as those that come about through the principal valencies. The penetration of these ammonia molecules into the metal atom often gives it a more pronounced electro-positive character.

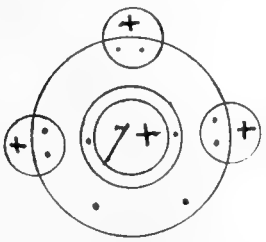


Fig. 2.
Ammonia.

That this bond is really restricted to a definite place of the molecule, follows from MEISENHEIMER's investigation [1]), in which he has succeeded in splitting up methyl ethyl aniline oxide into its optical antipodes. The four non-polar bonds, among which that where the nitrogen with its free electrons, has penetrated into the outer shell of the oxygen find a place in the angles of a tetrahedron.

[1]) Berichte **41**, 3967 (1908).

We point out that the nitrogen here behaves as a tetra-valent substance, the oxygen as a univalent one.

Ammonia, in spite of its having 8 electrons in its outer shell,

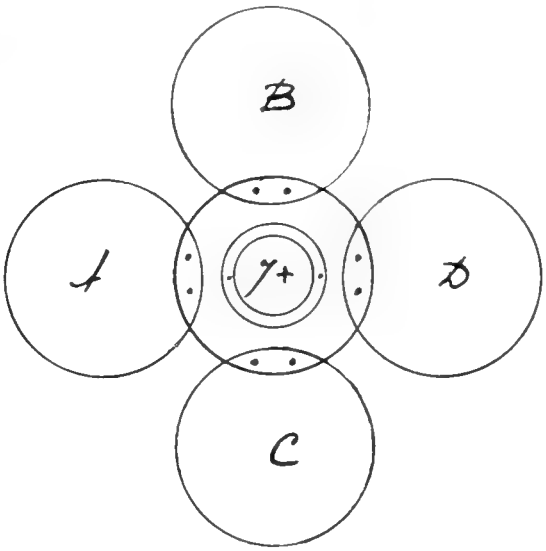


Fig. 3.
A = phenyl, B = methyl, C = ethyl, D = oxygen.
Methyl ethyl phenyl ammonium oxide.

can bind certain definite other atoms non-polarly, provided there be also an atom present that the electron, which is now in excess (and is, therefore, expelled) can take up.

This may also be expressed as follows: ammonia passes into the positive ion condition when forming a bond with a hydrogen atom, or in other words: ammonia can only receive a hydrogen *ion*, as it is saturated with electrons. Here the nitrogen does not become tetra-valent, but penta-valent. This fifth valency, however, has another character: it gives rise to a polar bond.

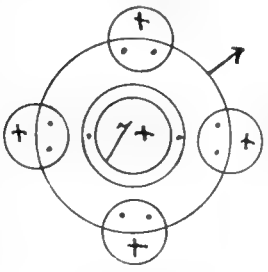


Fig. 4.
Ammoniumion.

It is this very power through which a number of atoms, which to start with, have an electro-negative character, acquire the property of an alkali-metal; we need only mention iodine and sulphur.

We may now apply these considerations to the boron atom, and examine in the first place what is the nature of the bonds in the simple derivatives of this element. The halogen compounds are the most suitable to decide this question.

These have BX_3 as constitution and entirely possess the character of acid chlorides, and not of salts. The three electrons are, accordingly, not ceded, as even the fluorium atoms are non-polarly bound.

In these compounds boron has only six electrons in the outer shell; in some respects they will, therefore, have an unsaturated character (Fig. 5). These halogen compounds can, indeed, become saturated in two ways.

The *first* way, which has been known longest and has already been explained by WERNER to a certain extent, refers to the adoption of a molecule HF. Then there is formed e. g. HBF_4, a mono-basic acid. It may now be assumed that a fourth atom F becomes non-polarly bound, which, however, is not possible, as boron has no free electron left, unless at the same time an electron (of the H) is taken up, and consequently the group BF_4 passes into the negative ion-condition (Fig. 6).

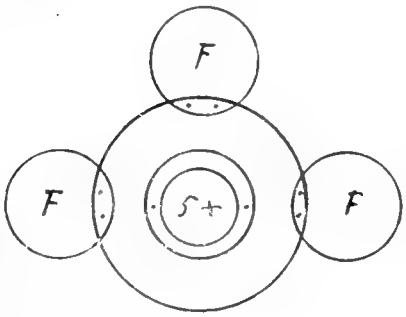


Fig. 5.
Borium fluoride.

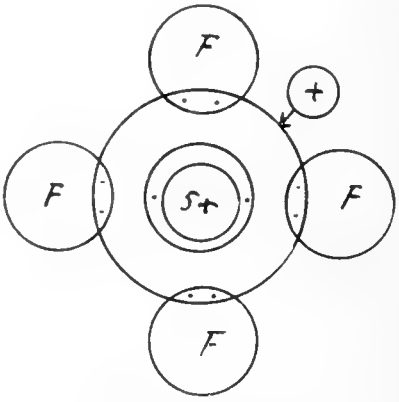


Fig. 6.
Borium fluor hydrogenic acid.

It may also be said that the polarly-bound HF-molecule enters the shell of the boron with two of the electrons of the fluorium atom, the whole BF_4-group becoming a negative ion.

For the H-ion it is entirely immaterial whether the ceded electron is attached to one of the four fluorium-atoms outside or inside the shell of the boron; as ion it has no fixed place in the molecule, and can wander all round the complex.

In view of the mono-valency of fluorium and of the complex, boron may be assumed to be penta-valent with as much reason as the nitrogen in ammonium compounds.

The *second* way in which boron fluoride can add to its electrons is: to combine with molecules of which there are two electrons

available in the outer shell of one of their atoms, without this giving necessarily rise to ionisation.

Thus BF_3 forms stable compounds with PB_3 and with ammonia,

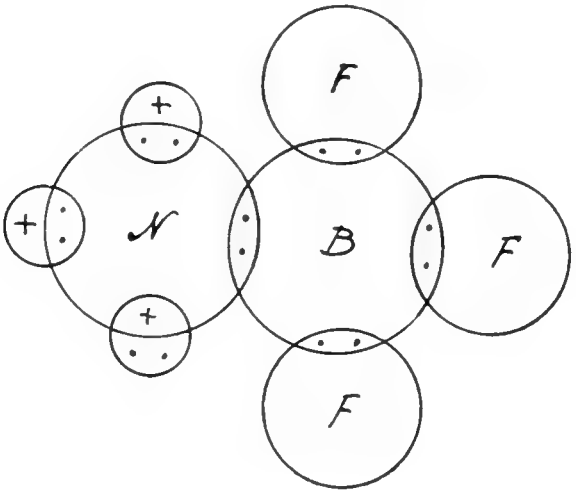


Fig. 7.
Boron fluoride ammonia.

of which the latter can be distilled undecomposed. Their constitution may be represented by the above simplified symbol; the two electrons which the N of the ammonia has in excess have penetrated into the shell of the sphere of the boron, thus forming a non-polar bond. Both atoms have eight electrons in this shell, and are mutually saturated (Fig. 7).

It is not subject to doubt that when different groups are substituted for the H-atoms at the N, a substance is formed which can be split up into its optical antipodes [1]).

As regards the valency of the boron, this may be put, like that of the nitrogen, at four, as there is no reason to assume the bond between the N and the B to be of another nature than between the B and the F (resp. between the N and the H).

Let us now proceed to the complex boric acid compounds. The very weak, volatile boric acid itself is, at least for the greater part, a derivative of the tri-valent boron, in which all the bonds are non-polar. In aqueous solution a very small part will be a derivative

1) It may cursorily be pointed out that the constitution of the addition products of $AlCl_3$ with a number of organic and inorganic compounds can be seen in entirely the same light.

of the penta-valent boron, in which one of the bonds is polar (see further).

The non-acid complexes agree with this, the acid ones, which are formed with the poly-hydroxy compounds, the hydroxyl groups

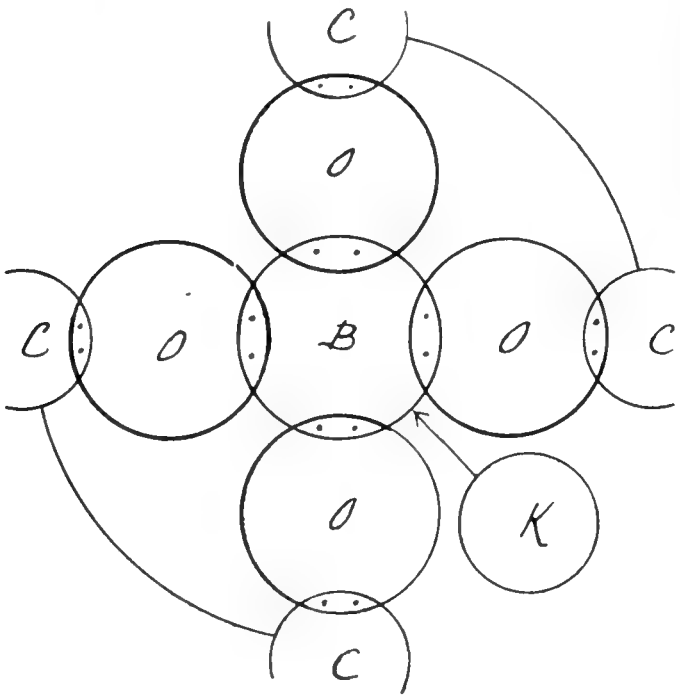


Fig. 9.
Potassium boro pyro catechate.

of which have a favourable position, are derivatives of the penta-valent element. Let us choose as an example potassium boro pyro catechate.

The four oxygen atoms of the two pyro catechol rests are bound to the boron atom. This cannot take place, however, until one electron of a metal or of an H-atom has been ceded to the complex. When this has once been accomplished, it is immaterial for the potassium (or H-) atom, where this electron is to be found in the complex; in view of the tetra-valency of the carbon, of the bi-valency of the oxygen, and of the mono-valency of the complex, the boron may here be assumed as penta-valent; one of these bonds is then polar (Fig. 9).

The four non-polar bonds, just as in the carbon atom — will be grouped as a tetrahedron, so that we may already expect optical activity in mono-derivatives of the pyro-catechol. These complex

boric acid compounds always being more or less hydrolized in aqueous solution, the splitting up into optical antipodes will be difficult.

In general the negative ion will be particularly easily formed:

1. When the hydroxyl groups of the poly-alcohols have a favourable situation.
2. When the organic rests bear an electro-negative character.
3. When the other atom easily cedes an electron.

1. The researches on the complex boric acid compounds of the last ten years have proved that the substances with a pronounced acid character from scarcely acid compounds are formed particularly easily, when the hydroxyl groups are situated in one plane with the C-atoms bound to them. It may be assumed that the first phase will be the formation of the derivative of the tri-valent boron. When this complex meets a second molecule of the organic compound, the unsaturateness of the boron will collaborate with the favourable constellation of the poly-alcohol to form the very stable derivative of the penta-valent boron.

2. When this favourable situation of the hydroxyl groups coincides with strongly electro-negative properties of the poly-oxy-compounds, as of α-hydroxy acids and aromatic ortho-hydroxy-acids, these penta-valent boric acid compounds will be exceedingly easily formed. Mr. HERMANS has actually succeeded (cf. following communication) in proving this for boro di-citric acid, and in ascertaining the constitution of the already known boro di-salicylic acid zinc from this point of view.

3. It was to be expected that especially the alkali-salts of these complex acids could be isolated, because the complexes are only realizable on adoption of an electron, and this is easily ceded by an alkali-metal. We meet here with the same influence which the metal atom in general exerts on the stability of the acid rest, which renders it possible to obtain salts of which the corresponding acid is unstable and even unknown.

This latter circumstance renders it also desirable to write the metal atom by the side of the atom to which it has ceded the electron, though in reality the whole complex becomes a charge richer, and it therefore seems indifferent to a certain extent where this metal atom is placed, since as an ion it is not bound to a definite place [1]).

[1]) That this is not quite immaterial may appear from the different behaviour of $AgNO_2$ and KNO_2 resp. AgCN and KCN towards alkyl iodides, which will be discussed later.

We are now able to bring some order in the inorganic derivatives of boron.

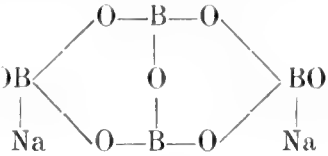
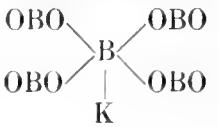
The volatile boric acid and its esters are, as was stated above, derivatives of tri-valent boron, and as such, somewhat unsaturated. It will try to supply the deficiency by complex formation.

AUERBACH's investigations [1]) have brought to light that when an insufficient quantity of a base is distributed between boric acid and arsenic acid there is formed far more borate than was to be expected according to the dissociation constant of boric acid. Complexes must be formed which are much more strongly acid than boric acid in diluted aqueous solution.

Hence in virtue of 3 the added bases cause the quantity of poly-boric acid ion to increase.

This is corroborated by an investigation of P. MÜLLER [2]), who could shake out but very little boric acid from a mixture of borate and boric acid with amyl alcohol, though the free acid is easily dissolved in it, evidently because the boric acid was bound with formation of poly-borates in consequence of the above-mentioned kation-action.

These stronger poly-boric acids will be derivatives of penta-valent boron, and accordingly in the symbol a place may be assigned to the metal atoms which promote this phenomenon, next to the boron atom.

The *metaborates* then have the composition $M(BO_2)$, *borax* has the formula :  , while *potassium penta borate* KB_5O_8 (see HERMANS, following communication), which crystallizes beautifully from formic acid, possesses the constitution  all assumed to be anhydrous.

There are described a great number of poly-borates; on the condition that the number of penta-valent boron atoms be taken the same as the number of positive metal valencies, their configuration can be easily constructed.

Boric acid anhydride is distinguished from boric acid by its slight

[1]) Zeitschr. anorg, Ch. **37** 353.

[2]) ABEGG Handbuch III. 1 p. 32 (1905).

volatility; this furnishes a sufficient ground for assuming this substance to be strongly polymerized. This may possibly be explained from the tendency of the boron atoms of one molecule to form non-polar bonds with pairs of electrons of the oxygen atoms of other molecules. It is possible to form an idea of this polymer by imagining the anhydride molecules to be built up in columns, in which alternately the oxygen atoms have penetrated into the outer shells of the boron atoms, thus contributing to the completion of the "octet". There are enough free atoms left at the oxygen atoms to render the easy hydration to boric acid comprehensible.

The boro hydrogen compounds. From the place of the boron in the system it was to be expected that the affinity of the H should be slight. The interesting investigations by STOCK and his pupils [1]) have really proved that these compounds are formed in very small quantities from magnesium boride, and are very unstable. At first B_2H_6 and B_4H_{10} were separated as gaseous boro-hydrogens, and later B_5H_9 besides higher boro-hydrogens. STOCK is of opinion that the boron must be assumed to be tetra-valent in these compounds.

He, therefore, tried to prepare halogen boron compounds BX_4, in which he did not succeed, which is, indeed, not astonishing in view of what precedes; such a combination can only be realized when at the same time an electron is added.

The B_2H_6 obtained by him is not necessarily a derivative of tetra-valent boron; the BH_3, which would have to be formed in virtue of the tri-valency of the boron, is evidently so unstable that two molecules inter-penetrate, in which, however, one of the B-atoms must more or less change into the ion-condition. It is actually immediately adopted by KOH with formation of $KBOH_2$ (propably a mixture or combination of $KBOH_2$ and $KBOH_4$) and H_2. Accordingly it is a compound with tri- and penta-valent boron, which through this makes the impression of being a derivative of the tetra-valent element (see the symbol on the following page).

Nor need the second gaseous boro-hydrogen B_4H_{10} possess a tetra-valent boron. In this two BH_2-groups can be bound to each other, each of them bearing a BH_3-group, while besides two H-atoms have passed into the kation-condition, and the rest, therefore, forms a bi-valent anion. The B_5H_9, which is, moreover, the most stable boro-hydrogen [2]), can certainly, not consist exclusively of tetra-valent boron atoms. If it is assumed that one of the boron atoms is bound

[1]) Berichte **54** A 142—158 (1921).

[2]) Berichte **54** A 155 (1922).

to four BH_2-groups, which at the same time has taken up an electron with H-nucleus, the relative stability and the fact that this boro-hydrogen dissolves in KOH without residue, evidently with formation of a salt, has been explained in a satisfactory way. Its formula is, therefore, $H[B(BH_2)_4]$ with one penta-valent and four tri-valent boron atoms.

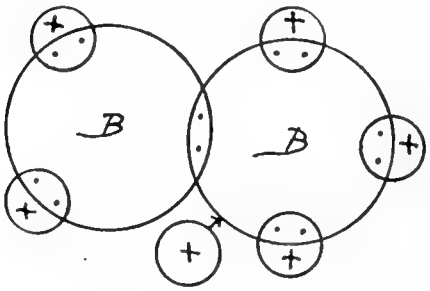


Fig. 10. Borohydrogen.

In the boro-alkyl compounds transition of an H-atom into the ion-condition is not possible: $B(CH_3)_3$ has been separated, and a polymerisation to $[B(CH_3)_3]_2$ has not been observed — also boro-triphenyl was lately prepared.

That the boro-alkyl compounds can combine with ammonia [1]) can be explained in entirely the same way as for BF_3 (cf. p. 103), there is sufficient reason in these non-polarly bound molecules to assume the boron, just as the nitrogen, to be tetra-valent.

Boro-nitrogen. BN. It has not been possible so far to melt, this substance, which forms a white powder and which is very resistant against the action of the air also at high temperature, for which reasons it has been proposed as material for fire proof receptacles; it is very interesting as far as the considerations given here are concerned. In appearance the demand of the valency has been completely fulfilled, as the tri-valent nitrogen is combined with the tri-valent boron. When, however, the properties of nitrogen compounds of other light elements, as cyanogen gas, halogen nitrogen compounds, etc. are considered, boro-nitrogen must at any rate be assumed to be very far polymerized.

When every nitrogen atom is supposed to be surrounded by three boron atoms, and these again each bound to three nitrogen atoms and so on, two electrons of every nitrogen atom remain available in the outer shell for a non-polar bond. Inversely every boron atom can be joined by a pair of electrons. This mutual saturation is here exceedingly probable, because then at the same time an exceptionally stable structure can be attained, viz. that of the carbon in diamond. The properties of boron nitrogen lead us at any rate to expect

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[1]) Berichte **54** B 531 (1922). The ammonia compound of boro-trimethyl is a volatile well-crystallizing compound, much more stable when exposed to the air than $B(CH_3)_3$ itself.

a very stable configuration. If attempts to bring it to crystallisation should succeed, a substance may be expected with a very high refractivity and very great hardness, and with a still more considerable resistance against external influences than any amorphous product known so far.

The difference with the way of binding of the carbon in diamond is this that one of the bonds at the moment of its formation is not quite equal to the other; when one considers, however, that this difference has vanished after the two elements have combined, so that it is impossible to decide which of the four was this particular bond, the expectation is the more justified that crystallized boro-nitrogen will have the character of diamond.

It is seen that when represented in this way, the idea of the valency begins to diffuse. The boron is more than tri-valent with respect to the nitrogen, because the element lacks something. And the nitrogen is more than tri-valent with regard to the boron, because in the simple compound this element has something too much. Combined they make, therefore, the impression of two tetravalent elements. Hence the valency is replaced by WERNER'S coordination value, to which a firmer foundation is given by these considerations.

If it should appear, e. g. from the Röntgenogram, that the diamond structure is applicable to the crystallized boro-nitrogen, this proves at the same time that a distinction between principal- and by-valencies is not rational, and that polar and non-polar bonds should be substituted for this, in which the non-polar bond is a connection between two atoms, which in consequence of mutual repulsion of some such bonds, has taken up a certain place in the molecule, whereas the polar bond forms a connection between one of the atoms and a rest, which will often consist of a multiple of atoms, but which, also when it consists of only one atom, is not fixed to a definite place of it.

It is self-evident that in the first periods, in which the atoms are simply composed, the number of pairs of electrons will not be greater than four, and the coordination-value will not exceed this number.

As the atoms get more complicated, the coordination-value can also increase; we see this already happen in the second period in aluminium, many compounds of which are known, in which this element is bound non-polarly to six atoms.

With regard to the other boron compounds, I will still draw attention to additional compounds of the boric acid esters with

alcoholates, e. g. $Na[B(OCH_3)_4]$, which entirely possess the character of salts in absolutely alcoholic solution — they are decomposed by water.

The boron is non-polarly bound to the four mono-valent OCH_3-groups, which is only possible through the complex having taken up one electron.

A very interesting group of compounds has been found by W. DILTHEY [1]). He found that when acetyl acetone-rests had substituted two chlorine-atoms in BCl_3, the third chlorine atom assumed the character of an anion, hence the rest of a kation. He rightly calls these substances *boronium compounds*: the considerations developed

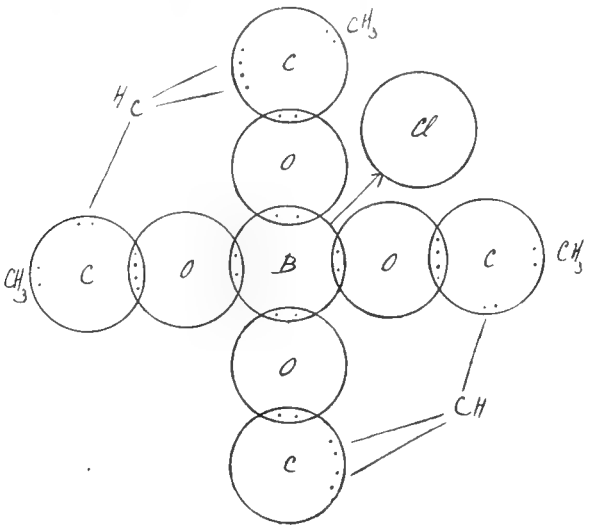

Fig. 11.
Boron di-acetyl acetone chloride.

above account satisfactorily for the phenomenon. The two acetyl acetone rests have as enol replaced two of the chlorine atoms of BCl_3, and then are bound non-polarly to the boron atom. The favourable situation of the C = O-groups with regard to the boron-atom now gives rise to the penetration of two electrons of each of the oxygen atoms into the outer shell of the boron, causing non-polar bonds; this is, however, only possible, when at the same time the third chlorine atom, which was at first non-polarly bound, passes into the (polarly-bound) anion state and the boron complex becomes a kation.

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[1]) Annalen **433**, 300 (1906).

There is certainly no need to state explicitly that only a sketch has been given in the above. It seemed, however, desirable to me to test KOSSEL's and LEWIS-LANGMUIR's hypotheses by the simplest atom that can be bound both polarly and non-polarly to other atoms, for it is to be expected here that the complex compounds will be built up in the least complicated way.

Complications occur in the elements of the second period, e. g. Al, Si, and S, as appears from the existence of compounds as $Na_3 AlF_6$, $K_2 Si F_6$ and the derivatives of the hexa-valent sulphur. In connection with the above it would have to be assumed that these atoms try to bring together *six* pairs of electrons in their outer shell, which then possibly might have to be ascribed to the influence of the electrons of the first spherical shell on those of the second. Before this can be examined more closely, the phenomena referring to the simplest elements will first have to be more fully cleared up.

In the case of boron it is, indeed clear, that as regards the formation of compounds *pairs* of electrons play an important part, and that especially the non-polar bond, i. e. the bond that does not conduct electrically, is brought about by such pairs. If it is further borne in mind that the latter kind of bonds is much less reactive than the former, it is natural to suppose that the difference between polar and non-polar bond consists in a greater closeness of the latter. The non-polar bond might be compared to an electro-magnet with a well-closed armature or a toroid, whereas in the polar bond the armature is removed or the toroid opened.

A similar image might be applied to the action of catalysts, in which it is likewise assumed that closed bonds are opened, which gives rise to a greater chance of interaction when meeting other molecules.

Delft, Dec. 1922.

Physics. — "*On the diffraction of Röntgen-rays in liquids*". II. By Prof. W. H. KEESOM and Prof. J. DE SMEDT. (Communication N°. 12 from the Laboratory of Physics and Physical Chemistry of the Veterinary College). (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated at the meeting of January 27, 1923).

§ 1. *Introducton.* The experiments on the diffraction of Röntgen-rays described in Comm. N°. 10 [1]) were all made with K_α-rays of copper. No diffraction ring was observed caused by the interference of rays scattered by the separate atoms in the molecules. F.i. in the case of oxygen this might be ascribed to the circumstance, that the distance of the centres of the systems of electrons grouped round the atom nuclei is too small to give an interference ring with rays of that wave length (viz. smaller than 0.95 Å for $\lambda = 1.54$ Å). Therefore it seemed desirable to repeat some of the experiments with rays of a shorter wave length.

We have now made several observations with K_α-rays of molybdenum ($\lambda = 0.71$ Å).

§ 2. For method and apparatus see Comm. N°. 10. The rays emitted by the molybdenum anticathode were filtered by 0.35 mm. zirconium.

§ 3. *Results of the observations on the principal diffraction ring.* We now exposed liquid oxygen, argon and nitrogen, also water and carbonic disulphide.

For oxygen, argon, water and nitrogen (investigated for the first time now) we found confirmed that the principal ring is due to neighbouring molecules, which we may consider to be distributed approximately as spheres packed together as closely as possible and filling up the space occupied by the liquid.

This time we obtained a diffraction ring for carbonic disulphide

[1]) These Proceedings 25, 1922, p. 118.

too and this gave a deviating value for the distance between the diffracting particles. This is evident from the following table. Here φ is again the half top angle of the cone formed by the diffracted Röntgen rays. M and d have been written for the molecular weight and density, while

$$a = \frac{7{,}72\,\lambda}{4\pi \sin\frac{\varphi}{2}}$$

denotes the distance between the diffracting particles. Here we again have made the assumption that the observed diffraction ring is due to the cooperation of arbitrarily orientated systems each of two particles at that distance from each other.

TABLE I.

Substance	φ ($\lambda = 0.71$ Å)	a	$1.33 \sqrt[3]{\frac{M}{d}}$
Oxygen (9 plates)	12.50°	4.0 Å	4.0 Å
Argon (1 plate)	13.0	3.85	4.1
Water (1 „)	13.44	3.73	3.6
Nitrogen (1 „)	11.34	4.42	4.4
Carb. disulph. (1 „)	13.23	3.8	5.2

Instead of formulating a special hypothesis on the deviating behaviour of CS_2 we prefer to postpone this until more substances showing a similar deviation have been investigated.

The diffraction rings obtained now are sharper than the former ones, the liquids being radiated this time in a tube of 1 mm. diameter.

§ 4. *Results of the observations on the second ring.* On six plates of oxygen and on those of argon and nitrogen the second ring is distinctly measurable. The other plates do not show this ring, probably because the obtained films are less blackened. For argon too this ring is very weak.

TABLE II.

	φ	a
Oxygen	19.5°	2.57 Å
Argon	18.9	2.65
Nitrogen	17.0	2.95

These values of a show a striking agreement with the values obtained in Comm. N°. 6a [1]) for the diameter of the molecule [2]) viz. for oxygen $\sigma = 2.65$ Å, for nitrogen $\sigma = 2.98$ Å. This supports the assumption made in Comm. N°. 10 that this diffraction ring should be due to the collaboration of two molecules touching each other.

With this wave length we also found for water at the outside of the principal ring a rather uniform blackening, rather sharply bounded at $\varphi = 24°$, which corresponds with a distance $a = 2.1$ Å.

§ 5. *For oxygen and nitrogen no diffraction by separate atoms in the molecule.* On a well blackened film of oxygen and on that of argon we found indications of a third maximum of blackening, for oxygen at $\varphi = 29°$ and for argon at $\varphi = 30.5°$. We do not pretend the existence of this third maximum to be doubtlessly fixed by these indications. We only draw the following conclusion: If this third maximum really exists, it also does so for argon, so that this maximum cannot be ascribed to the interference of rays that are scattered by the separate atoms in the molecules.

Though on several films the principal diffraction ring is blackened very intensively, no trace of an interference figure of the separate atoms in the molecule was found in these experiments. Yet with the here used wave length a diffraction ring would have been obtained for a distance of the diffracting particles greater than 0,43 Å [3]).

For a partial verification of the above we made still an exposition

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[1]) These Proceedings 23, 1920, p. 939.

[2]) In fact the smallest distance that is possible between the centres of two molecules in the gas.

[3]) According to the discussion of the band spectra the distances of the atom nuclei would be for oxygen and nitrogen resp. 0,85 and 1,12 Å: A. Eucken, Z S. f. Elektrochemie 26, p. 377, 1920. Comp. W. Lenz, Verh. D. physik. Ges. 21, p. 632, 1919.

with Cu-K_α rays (9 mA, ± 25 KV). Though this film is thoroughly blackened, only two rings have been obtained.

It may be suggested, that the rings obtained in these experiments are all due to atoms that temporarily are arranged in a crystal lattice. The values for the diameters of these rings found in this Comm. exclude a cubical arrangement [1]). The data are not sufficient to know, whether those temporary arrangements might belong to a crystal structure from an other class of symmetry [2]). Meanwhile the fact that freezing takes place suddenly at a definite temperature and the possibility of undercooling do not seem to point in the direction of such temporary crystal arrangements.

Lead by these considerations we have made still a plate of water at ± 0,5° C. The obtained interference figure perfectly agreed with that found at room temperature. At the outward side of the nearly uniform blackening only the intensity proved to be somewhat greater. In this way a second ring develops itself there, an indication of the presence of more double molecules at those low temperatures. No indication was found of the presence of more or greater crystal groups.

[1]) Comp. Comm. N°. 10 p. 122, footnote 1.

[2]) Nitrogen and argon crystallise cubically: W. Wahl, Proc. Roy. Soc. A 87, p. 371, 1912; oxygen below the melting point first hexagonally: W. Wahl, Proc. Roy. Soc. A 88, p. 61, 1913.

Bacteriology. — *"On the Bacteriophage and the Self-purification of Water"*, by Prof. P. C. Flu.

(Communicated at the meeting of Dec. 30, 1922).

In 1896 Hankin [1]) reported that the water of various rivers in India, i. a. the Yumna and the Ganges possesses the property of rapidly destroying cholera-vibriones. He was disposed to ascribe this property to a volatile substance, which he assumed to occur in the water of the said rivers.

Subsequent experimenters have demonstrated that all so-called surface-waters have the faculty of exterminating microbes, notably fortuitous pathogenic germs, at a rate depending on the nature of the water and the temperature of the environment.

Emmerich, who studied this phenomenon, the so-called selfpurification of water, believed that in this process the part of germicide must be assigned to protozoa (Rhizopods, Flagellates and Ciliates) which occur in every surface-water. This view was adhered to by nearly all inquirers, who had occupied themselves with the phenomenon.

D'Hérelle refers in his work "Le bactériophage, son rôle dans l'immunité" to the phenomenon observed by Hankin which he thoroughly believes to be merely the effect of a bacteriophage present in the water.

Now, we know that bacteriophages are inactivated at a temperature above 75° C., and that Hankin could heat water of the said rivers in a closed vessel (a sealed-up glass tube) for half an hour up to 115° C, without depriving it of its bactericidal capacity. We also know that, on heating up the Yumna, and the Ganges-water during the same interval and up to the same temperature (but in an open vessel), it really lost its bactericidal capacity.

Now, in view of these facts it will be difficult to side with d'Hérelle, although we must admit at the same time that protozoal action does not explain the phenomenon any better.

Still, it cannot be denied that after d'Hérelle's significant discovery and after the establishment of the presence of bacteriophages attacking various germs in all sorts of surface-waters, in seawater and even in the effluent from septic-tanks and from oxidation-beds,

[1]) Annales de l'Instituut Pasteur Vol. X pag. 175 and 511.

an interpretation of the self-purification of water can hardly be afforded without reckoning with the bacteriophage.

If a special inquiry in this direction were to show that bacteriophages play a more prominent part in the process of self-purification than has hitherto been assumed, we should not only have to revise and modify our conceptions of and our insight into this self-purification of water and our views concerning the action of sand-filters and oxidation-beds, but also a broad field would be opened up for studying the biological cleansing of sewage.

Like many others I also became convinced by my experimentation in India of the prominent part played by protozoa in the destruction of micro-organisms in the surface-water.

For this reason I deemed it a matter of importance to ascertain:

a. whether in surface-water, e. g. that in and about Leyden, bacteriophage could be found, and whether the self-purification of that water was in any way due to bacteriophages that might occur in it.

b. whether in surface-water, polluted intentionally with a profusion of pathogenic micro-organisms, and allowed, to purify itself, bacteriophages are to be observed that may have annihilated the germs.

c. the influence which is played on the purification by substances that kill the protozoa but do not injure the bacteriophages.

d. whether protozoa and bacteriophages combined may accelerate the process of self-purification.

To this end the following experiments were performed:

On the 2^{d} of June 100 c.c. of various samples of Leyden water were mixed every time with a concentrated broth. The mixture stood during 24 hours at 37° C. and was then filtered first through rock-meal and subsequently through a "bougie". The filtrate was mixed in quantities of 0,5; 0,2; 0,1; and 0,05 c.c. with broth, which was afterwards inoculated with an 18-hour-old Flexner-culture. For an examination for bacteriophage a smear-culture was made on agartubes of the broth thus prepared. After an incubation of 24 hours at 37° C. an estimination was made for "phages".

The result is that from the examined waters bacteriophages can be isolated that react especially to Flexner but also have an action on other intestinal bacteria.

Thus the isolated bacteriophages annihilate all the Flexner, Y, and Shiga Kruse stocks of our collection.

They also have an action on bacillus faecalis alcaligenes, on a proteus and a proteus X 19, but do not act upon Typhus, Paratyphus A. and B. or Enteridite Gärtner, neither on two coli-stocks of our collection.

Neither was any effect of the bacteriophages on cholera-vibriones at all apparent.

This result could be expected, as it is known that from the dejecta of fowls and horses a nearly always highly active bacteriophage antibacteria dysenteriae can be isolated and the surface-water in and about Leyden is being constantly polluted on a large scale by the excrements of a number of living beings, also by those of horses and fowls.

Anyhow this inquiry teaches us that bacteriophage occurs in the surface-water of Leyden.

On the 2^{d} of June quantities of 5 Liters of various kinds of Leydenwater were infected every time with two loopfuls of a 24-hour-old cholera-culture. The infected water was placed in large glass receptacles in diffuse daylight at room temperature (15° C.).

On the 21st of June we examined two quanta of 25 c.c. of water; in neither of those samples could cholera-vibriones be detected.

Of every sample of 5 L. 25 c.c. was examined for bacteriophages by mixing the water with $^{1}/_{10}$ of the volume of concentrated broth, and inoculating the mixture with a loopful of an 18-hour-old cholera-culture.

After an incubation of 24 hours at 37° C. the sample was examined in the usual way for bacteriophage anticholera-vibriones. The result was negative.

On the 24th of June three flasks were filled each with 0,5 L. of Rijnwater, in which, as our examination had proved, bacteriophage antibacteria dysenteriae was present.

Flask I was inoculated with the whole cholera-culture of a sloped agar tube; flask II in the same manner with typhus-bacilli; and flask III with Shiga-Kruse bacilli.

The fluid of each of the three flasks became very turbid and was placed at room-temperature in diffuse daylight.

On the fifth of July the fluid of each of the three flasks became lucid and was examined for bacteriophage in the ordinary way. In all the flasks we found bacteriophage antidysenteriae, which was present in the water already before the beginning of the experiment, but in the typhus-flask not any bacteriophage antityphus was found, no more than bacteriophage anticholera in the cholera-flask.

The flask infected with Shiga did not become lucid sooner than the one infected with typhus and cholera, which might have been

expected if a protozoal action had been assisted by the bacteriophage antidysenteriae present in the water.

In each flask the number of protozoa increased already two days after the inoculation with the mass of bacteria. Their number was greatest one day before the contents of the flasks became lucid, whereas it decreased after the clarification had been completed; some of them were transformed into cysts.

Again a culture, equal to the one at the beginning of the experiment was transplanted into the flasks in which the typhus-bacteria and the cholera-vibriones had disappeared. The same was repeated twice when, after about ten days the contents had clarified again.

After each new infection the number of protozoa was augmented, as with the first, reached its maximum shortly before the clarification and decreased again after it. Every time a portion of the protozoa were seen to turn into cysts.

When the contents of the flasks had become quite clear again after the fourth infection, another examination was performed for bacteriophage antityphus abdominalis and anticholera vibriones. The result was absolutely negative.

So these experiments go to show that large crowds of typhus-bacteria and cholera-vibriones may disappear without any interference whatever of bacteriophages, from water into which they were introduced fortuitously or intentionally. Even in water containing a bacteriophage anti-bacteria-dysenteriae the B. dysenteriae do not disappear quicker than other bacteria not attacked by bacteriophage.

It was nevertheless of interest to examine especially the influence of the presence or the absence of bacteriophage anti-shiga on the rate of disappearance of B. dysenteriae from the water.

Two series of experiments were accordingly carried out.

In the first series the fate of B. dysenteriae in unfiltered water was compared with that of the same bacilli in filtered water.

Protozoa cannot pass through a filter impervious to bacteria, whereas the bacteriophage is let through.

In the second series a comparison was made of the rapidity of the selfpurification process of bacteriophage containing water that was or was not mixed wit KCN.

The results of these tests, which were every time the same, are reported below.

Vlietwater, which contains bacteriophage, was used for the inquiry. Part of it was filtered through a Berkefeld-filter. A control-experiment showed that this water is free from bacteria and protozoa.

Part of the filtered, as well as the unfiltered water was infected

with another quantity of highly active bacteriophage (0,2 cc. to 10 cc. of liquid. The bacteriophage was still active in a dilution of 10^{-10}). Bacteriophage was superadded to demonstrate its influence still more conclusively than could be done with the bacteriophage already occurring in the Vlietwater.

The subjoined table shows the details of the experiment and gives a survey of the results achieved:

Contents of the tube.	Experiment begun	Lucid after how many times 24 hrs.
Filtered Vlietwater 5 c c + Flexner	23,9, '22	
" " " + Shiga Kruse	" "	After 12 × 24 hrs all still turbid, after the next sojourn of 4 × 24 hrs 28° C. all remain turbid.
" " " + K. B. [1])	" "	
" " " + Flexner + Bacteriophage 0,1	" "	
" " " + Shiga + Bact. 0,1	" "	
" " " + K. B. + Bact. 0,1	" "	
Unfiltered Vlietwater 5 cc + Flexner	" "	4 × 24 hrs lucid.
" " " + Shiga Kruse	" "	10 × 24 " "
" " " + K. B.	" "	6 × 24 " "
" " " + Flexner + 0,1 Bact.	" "	6 × 24 " "
" " " + Shiga + 0,1 Bact.	" "	9 × 24 " "
" " " + K. B. + 0,1 Bact.	" "	6 × 24 " "

The tests of the 2nd series were conducted as follows:

The fluid of two flasks, each holding 0,5 L. of bacteriophage-containing Vlietwater, was infected with such an amount of Flexner-culture as to render it quite turbid.

To the fluid of one of the flasks 20 mgms of KCN was added, after which the flask was well fitted with a rubber stopper. Both flasks were placed at room-temperature in diffuse daylight.

After a week the fluid of the flask without KCN had become quite clear, whereas the KCN-flask still contained a turbid fluid. In the former a large number of protozoa were found, which were lacking in the latter.

On the eleventh day of the experiment the KCN flask was also getting more lucid and protozoa were noticeable in it. After a fortnight the fluid in either flask was clear.

[1]) K. B. is a Flexnerstock resistant to any bacteriophage action.

The phenomenon exhibited in the KCN flask is to be interpreted by the fact that at the beginning of the experiment the KCN destroys the vegetative forms of the protozoa and consequently they are prevented from clearing away the germs present in the water. The cysts of the protozoa are not killed by KCN. After a week so much of the KCN has been decomposed through contingent chemical processes, that the cysts again grow into vegetative protozoa, which devour the Flexner bacilli, present in the water.

CONCLUSIONS.

When summarizing our results it must be concluded that the significance of the bacteriophage for the selfpurification of water is no doubt only small. I for one did not succeed in establishing the slightest influence.

The purification is effected in the absence of the bacteriophage, whereas its presence does not accelerate the process, nor render it more complete.

The experiments again yield conclusive evidence for the prominent rôle played by protozoa in the self-purification of water.

When, under such circumstances as the laboratory enables us to establish, we eliminate the protozoa, the self-purification of water is entirely arrested even though bacteriophage be added to the water.

(*From the Laboratory for Tropical Hygiene of the Leyden-University*).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXVI

Nos. 3 and 4.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vols. XXXI and XXXII).

CONTENTS.

Mathematics. — "*A Null System* (1, 2, 3)." By Prof. JAN DE VRIES.

(Communicated at the meeting of February 24, 1923).

1. We consider as given a congruence $[\varrho^3]$ of twisted cubics with the base points C_1, C_2, C_3, C_4, C_5 [1]) and the crossing straight lines a and b.

Through a point N there passes one curve ϱ^3; let r be the tangent at N and t the transversal of a and b through N. We conjugate $\nu \equiv rt$ to N as a *null plane*.

The curves ϱ^3 touching a plane ν have their points of contact in a conic ϱ^2. The transversal t lying in ν, cuts ϱ^2 in the *null points* N_1 and N_2 of ν.

If ν revolves round the straight line l, t describes a scroll $(t)^3$ and ϱ^3 a cubic surface through l. The locus of N is accordingly a twisted curve λ^5, which has evidently l, hence also a and b, as trisecants.

We have therefore a *null system* with the characteristic numbers $\alpha = 1,\ \beta = 2,\ \gamma = 3$.

2. The points C_k are *singular*; for C_k carries one straight line t but ∞^2 straight lines r. The null planes of C_k form a pencil of planes round t as axis.

Also the points A of a and B of b are *singular*. For each of them carries ∞^1 straight lines t which are combined to a plane pencil. The null planes of each of these points form a pencil of which the axis lies in the tangent r. These axes form two cubic scrolls $(r)^3$.

Other *singular points* S can only arise through coincidence of the straight lines t and r. Now the tangents of the curves ϱ^3 form a complex of the 6^{th} order and this complex has a scroll $(n)^{12}$ in common with the bilinear congruence $[t]$. On each straight line n there lies a point S to which any plane through n corresponds as null plane.

As l is intersected by 12 straight lines n, the corresponding curve λ^5 contains 12 points S.

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[1]) The principal properties of this congruence are to be found for instance in R. STURM: *Die Lehre von den geometrischen Verwandtschaften*, Part IV, p. 470.

3. The null points of the planes passing through the point P, lie on a surface $(P)^4$. For P is the null point of one definite plane of the sheaf and on a straight line l through P there lie the null points of three planes through l.

The intersection of the surface $(P)^4$ and $(Q)^4$ consists of the curve λ^5 corresponding to PQ, the straight lines a and b, and a curve σ^9 which is the locus of the *singular points* S and which passes evidently through the 5 base points C_k.

Three surfaces $(O)^4$, $(P)^4$ and $(Q)^4$ have in the first place the curve σ^9 in common. The points which they have further in common, are apparently the points of intersection of $(O)^4$ with the curve λ^5 corresponding to PQ. To them there belong the 12 points S on λ^5 and the 2×3 points A and B on λ^5; the remaining two are the null points of the plane OPQ.

4. Any plane α through a is *singular*; it contains a plane pencil (t) and each ray t cuts the conic ϱ^2 (§ 1) in two null points. Analogously the planes β through b are *singular*.

Also the ten planes σ each containing three base points C_i are *singular*. For in σ_{123} there lies a pencil of conics of which each individual is combined with the straight line $C_4 C_5$ to a curve ϱ^3; they cut the straight line t in σ_{123} in an involution of null points.

The surface $(P)^4$ contains the conics α^2 and β^2 lying in the planes Pa and Pb, and the intersection p of these planes. The straight line p is singular in this respect that it is a null ray for *each* of its points. The *singular null rays* p form the bilinear congruence with the director lines a and b.

Also the ten straight lines $C_k C_l$ are *singular*; for through each point on such a straight line r_{kl} there passes one straight line t, while r_{kl} may be considered as a tangent.

Mathematics. — *"A Congruence* (1,0) *of Twisted Cubics"*. By Prof. JAN DE VRIES.

(Communicated at the meeting of February 24, 1923).

1. The twisted cubics through four points C_1, C_2, C_3, C_4 cutting the straight line b twice, form a linear congruence $[\varrho^3]$; for through any point there passes one ϱ^3. The base points C are the *cardinal points*, b is a *cardinal chord*.

If d is a chord of one of the ϱ^3, $d\,(C_1\,C_2\,C_3\,C_4) = b\,(C_1\,C_2\,C_3\,C_4)$. The chords d form therefore a *tetrahedral complex*; a ray l not belonging to this complex, is not cut twice by any ϱ^3: the *class* of the congruence is *zero*.

Together with C_k and b a chord d defines a hyperboloid; on this there lie ∞^1 curves ϱ^3 and these define on d an involution; d is consequently a *tangent* to *two* curves.

The tangents meeting at a point P, lie on the complex cone of P; their *points of contact* form a twisted curve of the 5th order, ϱ^5, passing through P.

2. Let B_4 be the point of intersection of b with the plane $\gamma_{123} \equiv C_1\,C_2\,C_3$. Each *conic* ϱ^2 through the points C_1, C_2, C_3, B_4 is a component part of a degenerate ϱ^3; the transversal t_4 through C_4 resting on b and ϱ^2 is the second component part. The straight lines t_4 form the *pencil of rays* through C_4 in theplane $C_4\,b$. There are therefore *four* pencils of rays formed by *singular straight lines*.

The pairs of lines of the pencil (ϱ^2) produce three figures each consisting of three straight lines, e.g. the combination of $C_1\,C_2$, $C_3\,B_4$ and the straight line t_4 resting on $C_1\,C_2$. There are evidently *twelve* figures consisting of *three straight lines*.

3. With a view to finding the order of the surface A formed by the ϱ^3 cutting a straight line l, we determine the intersection of A with the plane γ_{123}. It consists of two conics of the pencil (ϱ^2); the former cuts l, the latter is a component part of the ϱ^3 which is defined by the transversal through C_4 of b and l. Hence A is a *surface* of the 4th order; the cardinal points C are apparently *double points* of A^4. A ϱ^3 not lying on A^4, can only cut this surface in the points C and on the cardinal chord b; from this there follows that b is a *double straight line*.

On A^4 there lie 9 straight lines and 8 conics.

The straight lines resting on b and l, determine a representation of A^4 on a plane.

A straight line l_1 through a point C cuts A^4 in two more points outside C; from this follows that the ϱ^3 cutting l_1, lie on a hyperboloid; this is entirely defined by l_1, b and C_k. Analogously the ϱ^3 resting in a fixed point on b or on a straight line intersecting b, form respectively a quadric cone or a hyperboloid.

4. A plane λ through l cuts A^4 along a curve λ^3 which has a double point on b. In each of the three points of intersection of λ^3 with l, λ is touched by a ϱ^3. Hence the curves ϱ^3 touching a plane σ, have their *points of contact* on a *curve* σ^3.

Let B be a point of b; the ϱ^3 through the five points B and C_k touching σ, form a surface of the 10th order with sextuple points in B and C_k [1]). There are accordingly $4\,\varrho^3$ through B and C_k which have b as a chord; consequently b is *quadruple* on the locus Δ of the ϱ^3 touching the plane σ and belonging to the congruence (1,0). Also it appears that Δ has *quadruple points* in C_k. Accordingly an arbitrary ϱ^3 of the (1,0) has 24 points in common with Δ, i. e. Δ is a *surface* of the 8th order.

5. A^3 has the curve of contact σ^3 and a conic σ^2 in common with the plane δ. The curves σ^3 and σ^2 touch each other in 3 points; there are therefore *three* curves ϱ^3 which *osculate* the plane σ.

If revolves round l. σ^3 describes a surface of the fourth order with the single straight line l.

On the curve ϱ^3 cutting l in R, the pencil of planes (σ) defines an involution; l is therefore cut by two tangents of ϱ^3. Consequently through l there pass two planes in which R is a point of the "complementary" curve σ^2. Hence σ^2 describes a surface of the fourth order with the double straight line l.

Let us now consider the relation between the points P and Q which the curves σ^3 and σ^2 in a plane σ have in common with l. Through P there passes one ϱ^3; the tangent at P defines the plane σ, hence two points Q. Through Q there pass two ϱ^3, hence two curves σ^2, and two planes σ each containing a curve σ^3; six points P are therefore associated to Q. If two homologous points P and Q coincide, there arises a double coincidence of the (6,2), for at that point a ϱ^3 is osculated by the plane σ. On l there lie therefore four points N for which the plane of osculation ν passes through l.

[1]) This is easily seen from the intersection of this surface with γ_{123}, which consists of 2 conics and 3 double straight lines.

6. If we consider N as the null-point of r, there arises a *null-system* with the characteristic numbers $\alpha = 1$, $\beta = 3$, $\gamma = 4$ (§ 5).

If r continues to pass through a point P, the locus of N consists of a surface $(P)^5$ and the four pencils of rays round the points C_k in the planes $C_k b$ (§ 2).

If r revolves round the straight line l, r describes a curve λ^7 and the four singular rays through C_k which rest on l.

The surfaces $(P)^5$ and $(Q)^5$ have in common the curve λ^7 corresponding to PQ, and the 18 singular straigth lines $C_k C_l$ and $C_k B_l$.

With a ϱ^3 $(P)^5$ has in common the 3 points of which the planes of osculation pass through P; the remaining 12 common points lie in the cardinal points C; these are therefore triple points of $(P)^5$. The planes of osculation in C_k envelop accordingly a cone of the third class.

Mathematics. — "*A Representation of the Line Elements of a Plane on the Tangents of a Hyperboloid.*" By Prof. JAN DE VRIES.

(Communicated at the meeting of March 24, 1923).

1. In order to arrive at a representation of the line elements (P,l) of a plane α, we consider a hyperboloid H which touches α in A, and which cuts it along the straight lines a_1 and a_2. Let R be the projection of P on H out of the point O of H, ϱ the tangent plane at R, r the intersection of ϱ with the plane Ol; we consider r as the *image* of the *line element* formed by P and l.

If, inversely, r is a tangent of H, R the point of contact, P the projection of R, l the projection of r, the line element (P,l) has the tangent r for image [1]).

We shall call the straight lines of H which cut each other in O, b_1 and b_2; b_1 cuts α in a point B_1 of a_2, b_2 passes through a point B_2 of a_1.

2. If l passes through B_1 and P coincides with B_1, R is the point of contact of the plane $b_1 l$, and any tangent r lying in this plane, may be considered as the image (B_1,l). Hence (B_1,l) is a *singular* element and its image is the plane pencil (r) round R. If l revolves round B_1, the plane pencil (r) describes the parabolic bilinear congruence with the directrix b_1, formed by the tangents which have their points of contact on b_1. Analogously the line elements (B_2,l) are *singular*.

If B is an arbitrary point of the straight line $b \equiv B_1B_2$, R lies in O. The line element (B,b) is therefore also *singular* and is represented by the plane pencil (o) of the straight lines that touch H in O and lie in the tangent plane ω.

Hence, inversely, any tangent o is *singular*, as it represents all elements (B,b). But at the same time it is the image of all the elements of which the point P lies in the intersection of o with α, for r is projected out of O by any plane which contains r. The

[1]) A fine representation of the line elements of α on the points of space may be found in the thesis of Dr. G. SCHAAKE. (*Afbeeldingen van figuren op de punten eener lineaire ruimte*, P. Noordhoff, 1922).

plane pencil (O,ω) is accordingly the image of the null system N (0,1), in which N lies on b.

Let g_1 be a straight line of H cutting b_2 and a_2 so that its projection $\bar{g}_1$ passes through B_2. As any point of g_1 may be considered as a point of contact R, P is an arbitrary point of $\bar{g}_1$ and g_1 is the image of all line elements lying on $\bar{g}_1$. The straight lines of the scrolls (g_1) and (g_2) are therefore *singular tangents*.

3. Let the symbol (λ, π) indicate a system of line elements (P,l) in which the points P lie on a curve of the order π and the straight lines l envelop a curve of the class λ.

The image of a *plane pencil* (1,0) is apparently a *plane pencil* of tangents. If P lies in A, the plane pencil (r) coincides with the plane pencil (A, l). The plane pencils (B_1, l) and (B_2, l) are represented by congruences (1,1) (cf. § 2).

The image of a *system* (0,1) consists of the tangents of a conic λ^2 lying in the projecting plane of the fixed straight line l.

A *system* (1,1) consists of the line elements of which P lies on a straight line c and l passes through a point D. If P moves on the straight line c, R describes a conic γ^2 (through O) and ϱ envelops the tangent cone which has the pole of the plane γ of γ^2 as vertex. The plane $\sigma \equiv Ol$ revolves round $d \equiv OD$ and describes a pencil which is projective with the system of the tangent planes ϱ (index 2). The image lines r describe accordingly a *cubic scroll* of which d is the double directrix and γ^2 a director curve.

The intersection of this scroll $(r)^3$ and the plane γ consists evidently of the conic γ^2 and the tangent o which rests on c and is the image of the line element (B, b) belonging to (1,1). The points of intersection of γ^2 with c lie on the straight lines a_1 and a_2; the line elements to which they belong, are represented by the tangents of $(r)^3$ which, apart from o, rest on c. To $(r)^3$ there belong two straight lines of H; they cut each other on d, and are the images of the line elements for which l passes through B_1 or B_2.

4. Let a *system* (λ,π) be given. The curve (P) which is of the order π, is projected out of O by a cone of the same order, which cuts H along a curve (R) of the order 2π (with a double point in O). The polar plane of the point F, chosen at random, contains accordingly 2π points R; hence the tangent planes ϱ envelop a surface of the class 2π. To each plane ϱ there corresponds one plane (Ol); inversely to one plane Ol (containing π points P) there

are conjugated π planes ϱ. The planes Ol and the tangent planes ϱ define on any straight line a correspondence with characteristic numbers $\lambda\pi$ and 2π. Through each coincidence there passes one image line r; accordingly the *system* (λ, π) is represented by a *scroll* of the order $(\lambda + 2)\,\pi$.

A system (λ, π) contains 2λ straight lines l passing through B_1 or through B_2. As each of them carries π line elements, the scroll contains 2λ *straight lines of the hyperboloid*, each of which is a π-fold straight line of the scroll.

The system $(1,\pi)$ in which the points P form a curve (P) of the order π which has a $\varkappa$-fold point D and where all straight lines l meet in D, has to be examined separately. For here a plane Ol contains only $(\pi—\varkappa)$ points P and defines therefore only $(\pi—\varkappa)$ planes ϱ. The characteristic numbers of the correspondence between the points of a straight line are in this case $(\pi—\varkappa)$ and 2π, so that the *system* $(1,\pi)$ is represented by a *scroll* of the order $(3\pi—\varkappa)$ on which the straight line OD is evidently 2π-fold.

A system $(1,\pi)$ of the kind in question is found in a *null system* $N(\mu,\nu)$ which is the locus of the null points of the rays of a plane pencil round a point D. For this *null curve* is a curve of the order $(\mu + \nu)$ with a μ-fold point D, so that the line elements form a system $(1, \mu + \nu)$.

5. A *null system* $N(\mu,\nu)$ is represented by a congruence of rays $[r]$. The straight line a_1 is a null ray for ν of its points P and the straight line r representing (P,a_1), coincides with a_1. Hence a_1 and a_2 are ν-fold rays of the congruence; the field-degree of $[r]$ is accordingly 2ν.

Let Q be the central projection of the point F. The null curve of Q is projected by a cone of the order $(\mu + \nu)$ and this cone has $2(\mu + \nu)$ points R in common with the conic which is the intersection of H and the polar plane of F. From this follows that the sheaf-degree of the congruence is $2(\mu + \nu)$. The image of an $N(\mu,\nu)$ is therefore a *congruence* $(2\,\mu + 2\,\nu, 2\,\nu)$.

Accordingly a *bilinear null system* $N(1,1)$ is represented by a congruence $(4,2)$. The *singular points* S_1, S_2, S_3 define three points R_1, R_2, R_3 on H; these are the vertices of three plane pencils (r_1), (r_2), (r_3), representing the plane pencils round the points S, hence *singular points* of the congruence $[r]$. The line elements on the three *singular straight lines* $s_1 \equiv S_2 S_3$, s_2 and s_3 are represented by the tangents of three conics σ_k^2 through O. Their planes σ_k are *singular planes* of the congruence. Also the plane $\sigma \equiv R_1 R_2 R_3$ is *singular*

for it contains one ray of each of the plane pencils (r_k). All tangents of the conic σ^2 along which H is intersected by σ, belong therefore to $[r]$. On σ^2 there lies one point $B_1{}^*$ of b_1 and one point $B_2{}^*$ of b_2. These two points are also *singular*, for the tangent to σ^2 at $B_1{}^*$ is the image of the line element of $N(1,1)$ that has its null point in B_1; but this line element is represented by *any* ray of the plane pencil (r) round $B_1{}^*$.

The null point of the straight line b is represented by the plane pencil (O,ω); hence also O is a *singular point* of the congruence (4,2).

6. The *enveloping cone* with vertex F is the image of a system of ∞^1 line elements of which the points P lie on the conic π^2, which is the central projection of the conic ϱ^2 in the polar plane of F. The straight lines l pass through the projection Q of F. Any line l is the projection of a conic through O and contains therefore two points P, corresponding to the two points R of ϱ^2 in Ol. The cone round F has accordingly a system (1,2) for image. The conic π^2 passes through B_1 and B_2, the point Q is to be counted double, being the class curve of l.

If F describes the straight line f, the corresponding tangent cones form a *congruence* (2,2) with *directrix* f. The curves of contact ϱ^2 pass through the intersections $S_1{}^*$, $S_2{}^*$ of H with the polar line of f, and rest on b_1 and b_2. Hence the curves π^2 form a pencil with the base points B_1, B_2, S_1, S_2, which are *singular null points*. Through a point P there passes one line l; for the corresponding point R carries one tangent r that rests on f and has the straight line $l \equiv PQ$ for projection.

A straight line l defines a point Q of the projection q of f, hence a point F, and through this there pass two tangents r to the conic in Ol. The *congruence* in question (2,2) is therefore represented by a *null system* $N(1,2)$.

The line f cuts the tangent plane $\omega \equiv b_1 b_2$ in a point F^*, the projection S of which lies on b and is a *singular null point* because the tangent OS represents all line elements round S.

The intersections $F_1{}^*$ and $F_2{}^*$ of f and H are singular for the congruence (2,2); their projections F_1 and F_2 on α are therefore *singular null points*.

In this way the *seven singular null points* which $N(1,2)$ must have [1]), are indicated.

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[1]) Cf. e.g. my paper on plane linear null systems. These Proceedings Vol. XV p. 1165.

Through F_1^* there pass two straight lines g_1 and h_1 of H, through F_2^* two straight lines g_2 and h_2. These four lines form a skew quadrilateral; g_1 and g_2 cut each other in S_1^*, h_1 and h_2 in S_2^*; g_1 and h_1 rest on b_2, g_2 and h_2 on b_1. The projections $\bar{g}_1$, $\bar{h}_1$, $\bar{g}_2$, $\bar{h}_2$ of these lines are evidently *singular null rays* and form a quadrilateral which has the singular null points S_1, S_2; F_1, F_2; B_1, B_2 as angular points. For $B_1 \equiv \bar{g}_2\,\bar{h}_2$, $B_2 \equiv \bar{g}_1\,\bar{h}_1$; $F_1 \equiv \bar{g}_1\,\bar{h}_2$, $F_2 \equiv \bar{g}_2\,\bar{h}_1$; $S_1 \equiv \bar{g}_1\,\bar{g}_2$, $S_2 \equiv \bar{h}_1\,\bar{h}_2$.

The plane Of cuts H along a conic, the tangents of which belong to $[r]$; hence the straight line q, (the projection of f) is a *singular null ray*. On q lie the singular null points F_1, F_2 and S. But S is the intersection of a tangent o, therefore also a point of the *singular null ray* $b \equiv B_1 B_2$. Accordingly the singular elements of $N(1,2)$ form the figure of the angular points, the diagonal points and the sides of a *complete quadrangle*. This null system is therefore of the same kind as the $N(1,2)$ which arises if to each straight line there are conjugated as null points its intersections with the conic in which it is transformed by an involutory quadratic correspondence.[1])

7. Five tangents r define a *linear complex* A; this has a congruence (2,2) in common with the complex of the tangents of H. The representation on α is again a *null system* $N(1,2)$; for a point P defines a point R and in ϱ there lies one ray of the plane pencil which in A has the null point of ϱ as vertex; and a line l defines on H a conic of which two tangents belong to the linear complex.

This complex has two straight lines in common with each of the scrolls of H; they form a skew quadrilateral $g_1 g_2 h_1 h_2$, the angular points of which are *singular points* for the congruence (2,2). For in A the point $g_1 g_2$ is the null point of the plane ϱ defined bij g_1 and g_2, so that any tangent at that point belongs to both complexes. Consequently the points $\bar{g}_1\bar{g}_2$, $\bar{g}_2\bar{h}_1$, $\bar{h}_1\bar{h}_2$, and $\bar{h}_2\bar{g}_1$ are *singular null points* of the null system (1,2) in α.

As g_1 and h_1 rest on b_2, $\bar{g}_1$ and $\bar{h}_1$ pass through B_2; hence B_1 and B_2 are *singular null points*. Also here the six null points are the angular points of a complete quadrilateral the sides of which are *singular null rays*. The plane pencil (O, ω) contains one ray of A which therefore also belongs to the congruence; its intersection S is the *seventh singular point* of $N(1,2)$. As S lies on b and B_1 and B_2 are singular, also b is a *singular null ray*.

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[1]) The general null system (1,2) has no singular null rays (l.c. p. 1167).

8. With a complex of the n^{th} order, Γ^n, the complex $\{r\}$ of the tangents has a *congruence* $(2n, 2n)$ in common which has for image a *null system* $N(n, 2n)$. Γ^n has $2n$ straight lines in common with any scroll of H; hence the null system has $4n$ *singular straight lines*, $2n$ of which pass through B_1 and $2n$ through B_2. B_1 and B_2 are therefore *singular null points*. The straight line b is evidently a *singular null ray*.

Anatomy. — "*The ontogenetic development of the Corpus striatum in birds and a comparison with mammals and man*". By Dr. C. U. Ariëns Kappers.

(Communicated at the meeting of November 25, 1922).

In the last ten years the corpus striatum has been a centre of interest as well for anatomists as pathologists, the latter chiefly after the researches of Kinnier Wilson.

There are however great differences in the intraventricular growths to which this name is given in different vertebrates.

Though I shall deal here chiefly with the corpus striatum in birds, mammals and man, I will start with making some introductory remarks on the intraventricular growths in fishes since the same principle which we shall meet in the amniota is already observed here: viz. the fact that the so called striatal parts do not only arise from the base of the forebrain but also from the mantle.

If one looks at the forebrain of a teleost or ganoid, it seems as if only the basal part of the forebrain consisted of nervous tissue, whereas the dorsal part merely consists of a choroid membrane.

This however is only seemingly so.

As a matter of fact, the two primordia generally observed in forebrains, the basal one and the dorsal one (from the latter of which the mantle arises), are both present also in embryo's of Teleosts and Ganoïds.

Whereas however the dorsal part in other fishes enlarges in a mantle-like way, increasing chiefly in surface and folding inward, the mantle primordium in Teleosts developes in a quite different way. Instead of increasing in surface it increases in thickness, thus narrowing the ventricle of the forebrain in which it protrudes.

This increase in thickness even goes so far that the pallial part bulges outward, pushing the dorsal wall latero-ventrally, in consequence of which the roof membrane is stretched and widely extended from left to right.

Thus an everted pallium is formed in these fishes, in contrary to the inverted mantle of other animals.

This process of development is seen in all larvae of Teleosts, and clearly demonstrated by a study of Lepidosteus osseus (a bony ganoid)

of which I give here some pictures. In the first figure (Lepidosteus larva of 5 cM.), the limit between the basal primordium (from which the palaeostriatum arises), and the dorsal (pallial) primordium is indicated by a line, the dorsal point of which might even be drawn somewhat more laterally (to coincide with the fiss. endorrhinalis interna). The basal *point de repère* of this line lies in the fissura endorrhinalis externa, only slightly indicated in this stage.

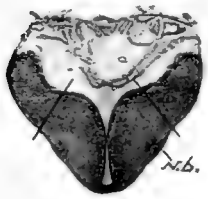

Fig. 1. Transverse section of the forebrain of a 5 c.M. larva of Lepidosteus.

N.b. = basal nucleus or nucleus peduncularis anterior.

The pallial part is very small in this stage. In a later stage, the pallial part however increases considerably. In fig. 2 and 3 I have given transsections of a 10 cM. larva and a full grown animal (1.20 M long). These two latter figures represent a more frontal level than figure 1, so that the olfactory bulb is cut, in order to show the reader that here we have really to do with a pallial part, (p.), which however in these fishes does not grow like a real mantle, but merely increases in thickness. The insertion of the roof membrane is at the place of the *X* in fig. 3, from which results that nearly all the mantle substance has an intraventricular position.

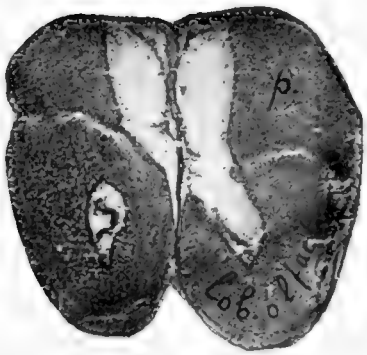

Fig. 2. Transverse section of the forebrain of a larva of Lepidosteus (10 c.M.).
p = pallium.

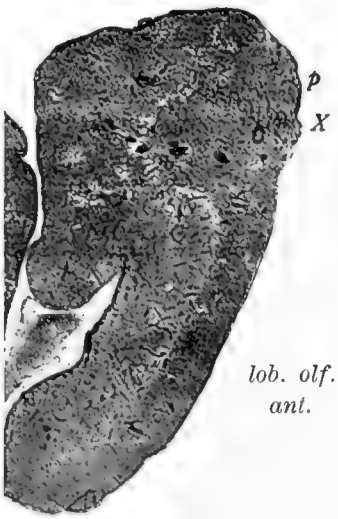

Fig. 3. Transverse section of the forebrain of an adult Lepidosteus (right half) *x* = insertion of the roof membrane, *p* = nervous pallium.

This increase in thickness gives rise — a little more caudally than fig. 3 — to a large mass of nervous tissue, extending over the palaeostriatum (which itself is derived from the basal part) and therefore has been named by EDINGER *epistriatum*.

EDINGER himself thought that this *epistriatum* is an outgrowth of the striatum. I have however been able to show that it really is caused by a medial thickening of the pallium extending over the palaeo-striatum. Also by studying its fibre-connections — which appear to be homologous to the fibre connections of the selachian mantle — I have been able to show this homology. [1]) Referring for further details concerning the Teleostean brain to the works of JOHNSTON [2]), SHELDON [3]), VAN DER HORST [4]) and HOLMGREN [5]), I will only call the attention to the fact that this epistriatum of fishes has chiefly primary olfactory functions, viz. that it receives chiefly fibres of the tr. olfactorius (fibrae bulbo-epistriaticae). In this sense it is a *primary epistriatum*.

A primary epistriatum also developes in Amphibia but it remains very small there (receiving only tr. olfact. fibres from the bulbus accessorius [6])) since the surface growth of the mantle is so considerable in Amphibia. This primary epistriatum of Amphibia developes entirely independently from the palaeo-striatum or basal nucleus, in front of it, from the side wall of the forebrain.

In Reptilia the primary epistriatum is superposed by a much larger *secundary epistriatum* or *archistriatum* i. e. by an ingrowth of the mantle which does not receive bulbo-epistriatic fibres but lobo-epistriatic, i. e. secundary olfactory fibres from the primary olfactory cortex (palaeocortex mihi; cortex praepiriformis BRODMANN).

Notwithstanding its enormous development and intraventricular

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[1]) The structure of the Teleostean and Selachian brain. Journ. of Comp. Neur. Vol. XVI, 1906. Zur vergleichenden Anatomie des Vorderhirns der Vertebraten, Anat. Anzeiger Bnd. XXX, 1907.

[2]) The telencephalon of Ganoids and Teleosts. Journ. of Comp. Neur. Vol. XXI, 1911 and the Teleostean Forebrain, Anat. Record. 1912.

[3]) The olfactory tracts in Teleosts. Journ. of Comp. Neurology Vol. XXII, 1912.

[4]) The forebrain of the Symbranchidae. Proceedings of the Kon. Akademie v. Wetensch. Amsterdam, 1920.

[5]) Zur Anatomie und Histologie des Vorderhirns und Zwischenhirns der Knochenfische, Acta Zoologica, Bnd. I, 1920.

[6]) HERRICK. The morphology of the forebrain in Amphibia and Reptilia. Journ. of Comp. Neurol. Vol. XX, 1920.

DE LANGE. Das Vorderhirn der Reptilien, Fol. Neurob. Bnd. V, 1911.

ARÏENS KAPPERS und HAMMER. Das Zentral-Nervensystem des Ochsenfrosches (Rana Catesbyana) Psych. en Neur. Bladen 1918.

(hypopallial, Ell. Smith [1]) growth, extending far backward, where it is continuous with the piriform and ammoncortex, this archistriatum keeps its contact with the olfactory area in front of the Foramen Monroi, near the primary "Anlage" of the epistriatum (nucl. tr. olfact. lateralis in Reptilia: Crosby [2]).

One might be inclined to ask, how it is possible to ascribe this hypopallial growth to neurobiotaxis — as Ell. Smith does — if the majority of aferent fibres (tr. cortico epistriaticus) comes from the periphery

Such fibres indeed cannot account for this mode of growth. But the archistriata (sec. epistr.) of both sides are connected by a very strong commissure, which thus provides them with medial impulsus and moreover it receives aferent fibres from the basimedial grey by the taenia terminalis fibres. Both systems must be made responsible for the medial intraventricular growth of the archistriatum.

Whilst this archistriatum which is thus derived from the innerside of the mantle (*hypopallium* Ell. Smith [3]) forms the larger part of the intraventricular mass in Chelonia (where the paleaostriatum is but small) a new striate substance which is only very small in turtles, becomes evident in Lacertilia, Ophidia and Crocodilia: the *neostriatum*. Moreover the paleaostriatum, the original basal nucleus of the forebrain, enlarges considerably in these animals (*palaeostriatum augmentatum* or *mesostriatum*).

Whereas the *palaeostriatum augmentatum* is really an increase from the same matrix from which the palaeostriatum primitivum arises, and from its immediate surrounding (corresponding approximatively with the tuberculum parolfactorium) the neostriatum is an entirely new addition starting in Reptilia as I pointed out in 1908 [4]). It arises from two sources. 1°. from the base of the brain in front of the palaeostriatum and 2° from the latero-frontal mantle joining this region, as has been pointed out by Ell. Smith (l. c.). The palaeostriatum, but chiefly the neostriatum receives its stimuli from the tweenbrain and this may be the neurobiotactic cause of its intraventricular medio-caudally directed growth.

The neostriatum together with the archistriatum (which is separated from it in Ophidia and Lacertilia by a deep fissure, the fiss. strio-archistriatica), is called *hypopallium* by Ell. Smith, on account of their character as an ingrowth of the pallium.

[1]) Vida infra.

[2]) The forebrain of Alligator missisippensis, Journ. of Comp. Neur. Vol. 27, 1917.

[3]) A preliminary note upon the morphology of the corpus striatum. Journ. of Anat. (English), Vol. LIII, 1919.

[4]) Die Phylogenese des Rhinencephalons, des Corpus Striatum und der Vorderhirncommissuren. Folia Neurobiologica Bnd. I, 1908.

Weitere Mitteilung zur Phylogenese des Vorderhirnes und des Thalamus, Anat. Anzeiger Bnd. 1908.

There is no doubt indeed that the neostriatum partly arises as such a hypopallial ingrowth in all the higher vertebrates, though its anlage is not limited to the mantle, but, also extends over the base of the brain in front of the palaeostriatum (immediately behind the anterior olfactory ventricle).

Whilst the neo-striatum is separated from the archi-striatum by the fissura *strio-archistriatica* (see my book on the Comp. Anatomy of the brain, Vol. II fig. 534), ELLIOT SMITH has rightly pointed out that the boundary between the neo-striatum and palaeostriatum is chiefly indicated by blood vessels. I may add that besides a shallow groove may indicate this boundary line (also in Reptilia), which groove I shall call *fissura neo-palaeostriatica.*

I have now studied the ontogenetic development of the different parts of the striatum complex in birds, mammals and man, and shall give here a short review of it, leaving the archi-striatum further out of discussion, since its place in brain-anatomy as the homologue of the nucleus amygdalae of mammals is since long established.

Starting than with birds I may remind that practically all anatomists have accepted the division of the forebrain of these animals as given by EDINGER in 1896.

Underneath the pallium (in which the cortex is very primitive) and continuous with it, is the *hyperstriatum*, forming the most dorsal and most lateral part of the striate complex. This hyperstriatum is in *most* birds — not in all —, easily distinguished in two divisions, by a thin medullary lamella: the *lamina medullaris hyperstriati.* These divisions I shall call *hyperstriatum superius* [1]), and *hyperstriatum inferius* [2]).

The hyperstriatum inferius in its lateral part shows a special field characterized by large cells, and richly provided with medullary fibres: the *ecto-striatum* of authors, which like the rest of the hyperstriatum is separated from the underlying *meso-striatum* (palaeostriatum augmentatum) by the *lamina medullaris dorsalis* of authors, which I prefer to call *lamina medullaris externa* since it does not only form the dorsal but also the lateral boundary of the mesostriatum. This lamina medullaris externa is very richly provided with bloodvessels as is also observed by HUNTER (Sydney) in the Kiwi.

[1]) This was called by SCHROEDER pars fronto-dorsalis hyperstriati. It consists of the areae A. C. and D. of Rose's (c.f. SCHROEDER: Der Faserverlauf in Vorderhirn des Hühnes, Journ. of Pschych. und Neur. Bnd. 18, Erg. Heft 1912, and Rose „Die zytolectonische Gliederung des Vorderhirns der Vögel". Ibidem Bnd. 21, 1914).

[2]) This corresponds with the areae G^1, G^2, G^3 of Rose's and with the striatum parichale of *Kalisher* (Comp. Kalisher: Abhandl. der Akad. der Wissensch. Berlin. 1900, 1901. 1905).

Besides the boundary of the mesostriatum and hyperstriatum in some birds is marked on the ventricular side by a slight groove, my *fissura neo-palaeostriatica.*

In caudal direction the mesostriatum, which extends to the ventricular surface becomes smaller and smaller, thus exhibiting a sort of cauda, which follows for some distance the caudal pole of the hyperstriatum inferius, and may be called *substantia palaeostriatica caudata* (see fig. 11 and 12).

In some large birds, like Pelicanus, the hyperstriatum and meso-striatum may be separated from each other — starting at the ventricular side — by an obtuse object without cutting, which probably is due to the medullary external lamella being so richly provided with bloodvessels.

In the centre of the mesostriatum (or palaeostriatum augmentatum) the so called *basal nucleus* of authors (palaeostriatum primitivum) is found, a cluster of large cells, separated in front of the augmented part of the palaeostriatum by another lamella the lamina medullaris ventralis of authors, *lamina medullaris interna* mihi.

The archistriatum or nucleus amygdalae of which I shall not speak here further is pushed backward and ventrally in birds by the enormous development of the hyperstriatum. Consequently the fissura strio-archistriatica, so conspicuous in Lacertilia and Ophidia, has become invisible in birds (as is already the case with Crocodiles).

In order to study the embryonic development of these parts in birds, I made use of haematoxyline and silverseries of the chick of 4, 5, 5½, 6, 7, 9 and 11 days of incubation and of an embryo of the ostrich some days before birth.

In a five days embryo of a chick, we find in a transverse section made on the level of the foramen Monroi, four protrusions in the ventricle (fig 4). The lower protrusion *a* is the *eminentia basimedialis* which some sections more frontally continues in the septum. This forms the basi-medial grey substance and has not to do with the striate complex.

The other three protrusions form parts of the so called striate complex.

The protrusion *b* is the primordium of the *palaeostriatum.* Its centre (less dark in fig. 4), is the basal nucleus or *palaeostriatum primitivum,* which is augmented by the surrounding darker cells, the *palaeostriatum augmentatum.*

This protrusion has only a small frontal extension (as is seen in the sagittal section, represented in fig. 5. It is chiefly confined to the level of the foramen Monroi and continues backward in the side wall of the recessus praeopticus (r. o. fig. 5). The protusion *b* is separated by a fissure (the *fissura neo-palaeo-striatica)* from the

tuberculum *c* which is less protruding but continues further frontally than *b*, bending down more or less to the base of the brain.

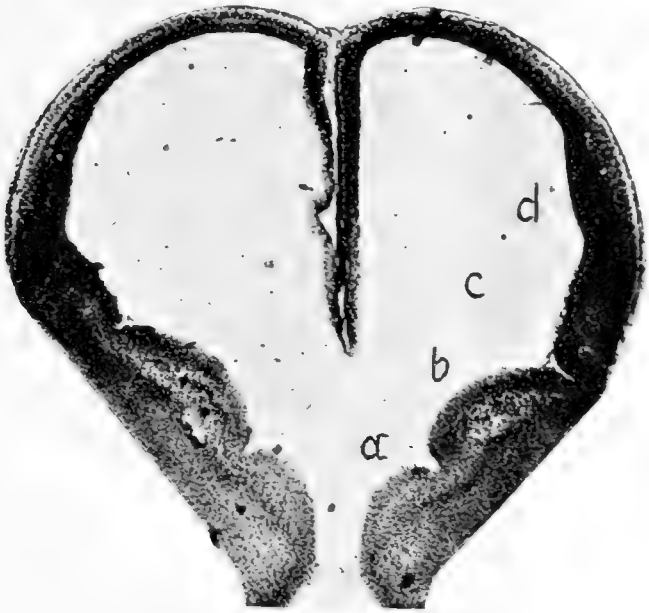

Fig. 4. Transverse section of the forebrain of a chickembryo of 5 days on the level of the foramen Monroi. *b* = primordium of the palaeostriatum, *c* = primordium of the hyperstriatum inferius, *d* = primordium of the hyperstriatum superius. For *a* see text.

This protrusion *c* appears to be the primordium of the *hyperstriatum inferius*. Caudally the groove which separates it from *b* fades away, the cells of *c* extending over *b* (comp. also fig. 6).

Dorsally from *c*, arising equally from the mantle is *d*, merely a thickening of the pallium in this stage which however appears to give rise to the *hyperstriatum superius*.

Figure 6, representing a sagittal section, is taken from an embryo of six days of incubation. The section shows the relation of the *hyperstriatum inferius c* to the palaeostriatum augmentatum *b*, which extends frontally to the triangular fissure, a part of the fissura *neo-palaeostriatica*.

It is further seen that *c*, the hyperstriatum inferius, arises on this level from the basal region in front of the palaeostriatum corresponding with the tuberculum olfactorium (t. o.). The hyperstriatum inferius thus partly has a basal origin (*partly* because more laterally it is continuous also with the mantle as we already saw in the transverse section of fig. 4).

Examining the same series on a more lateral level (fig. 7), we meet with the hyperstriatum superius *d*, and see that this arises from the mantle only, i. e. from the brainwall above the small split that indicates the communication between the lateral ventricle and the olfactory ventricle (already in fig. 6 the frontal part of the pallium shows a thickening at this place).

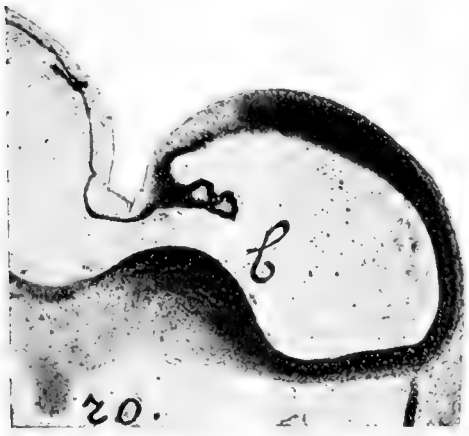

Fig. 5. Sagittal section of the forebrain of a chickembryo of $5^1/_2$ days.
b = primordium of the palaeostriatum.
r o. = wall of the recessus opticus.

In the last section of this series which I reproduce (fig. 8), all the parts of the striatum complex of birds are already visible in their mutual arrangement: the hyperstriatum superius (d.) forming the most dorsal part and extending over the rest, being continuous frontally with the pallium. Underneath it we find the hyperstriatum inferius *c* being in this

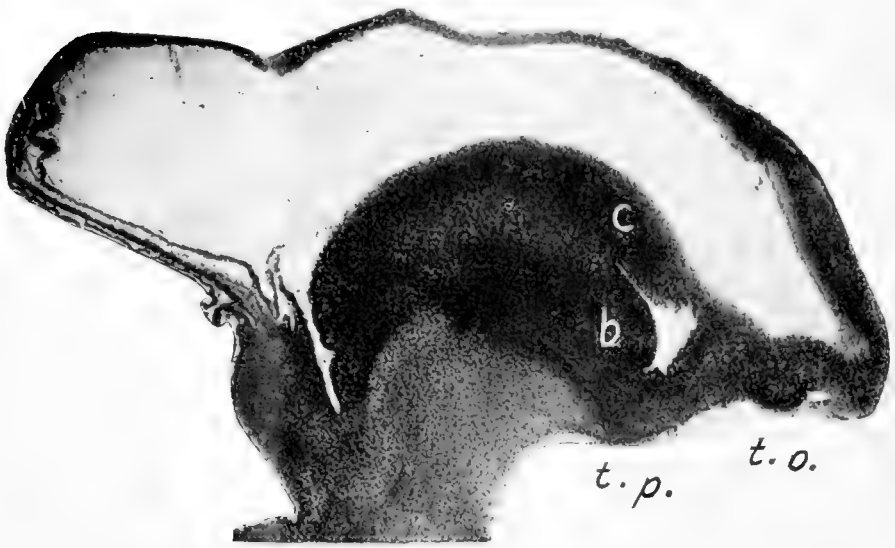

Fig. 6. Sagittal section of the forebrain of a chickembryo of 6 days on a level lateral to fig. 5
t. p. = tuberculum parolfactorium, *t. o.* = tuberc. olfact.
b = palaeostriatum augmentum, *c* = hyperstriatum inferius. (= mesostriatum).

section continuous with the most frontal part of the basis cerebri (more laterally with the mantle) and covering *b*, the meso-striatum or palaeostriatum augmentatum, in which the lighter centre (richly provided with fibres) is the primitive palaeostriatum, the basal nucleus.

If we now look at the figures of a 11 days embryo of the chick, we find that the chief alteration exhibited, is the enlargement of both parts of the hyperstriatum, which not only have increased in

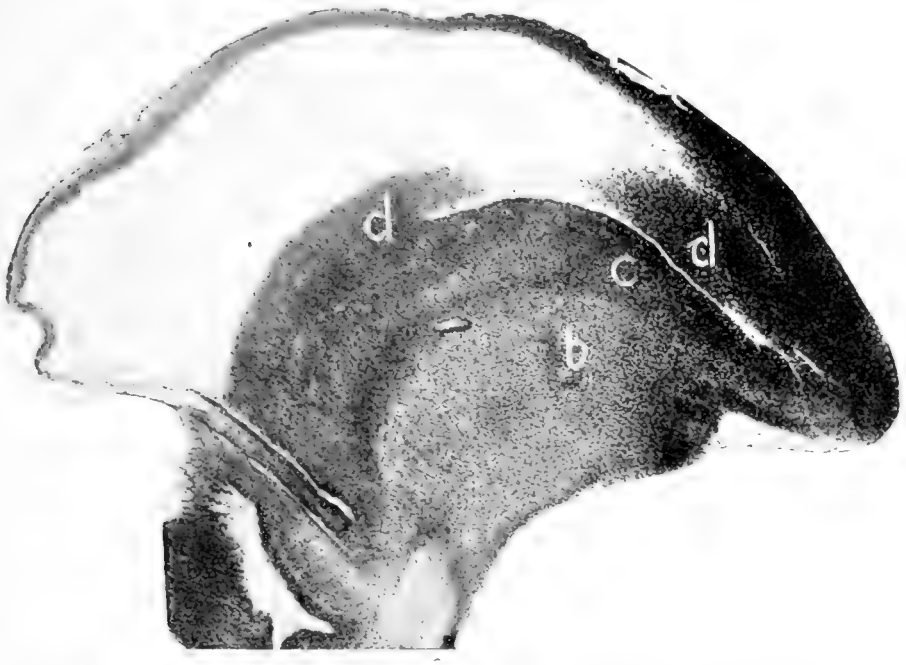

Fig. 7. Sagittal section of the forebrain of a chickembryo of 6 days.
b = palaeostriatum augmentatum (= mesostriatum).
c = hyperstriatum inferius.
d = hyperstriatum superius.

thickness (as appears from the fact that much less of the ventricle has remained free), but also has enlarged in medial direction.

The latter fact is evident from a comparison of figg. 9 and 6, which are taken on approximately corresponding levels (rather medial).

Whereas in fig. 6 on this level nothing is as yet visible of the hyperstriatum, the latter is very clearly shown in fig. 9, as a result of its growth in medial direction, further extending into the ventricle. It also shows the division in hyperstriatum superius and inferius.

In this figure we see moreover that the hyperstriatum superius is continuous only with the brainwall above the ventriculus,

being entirely derived from the mantle [1]), not from the basal part of the brain.

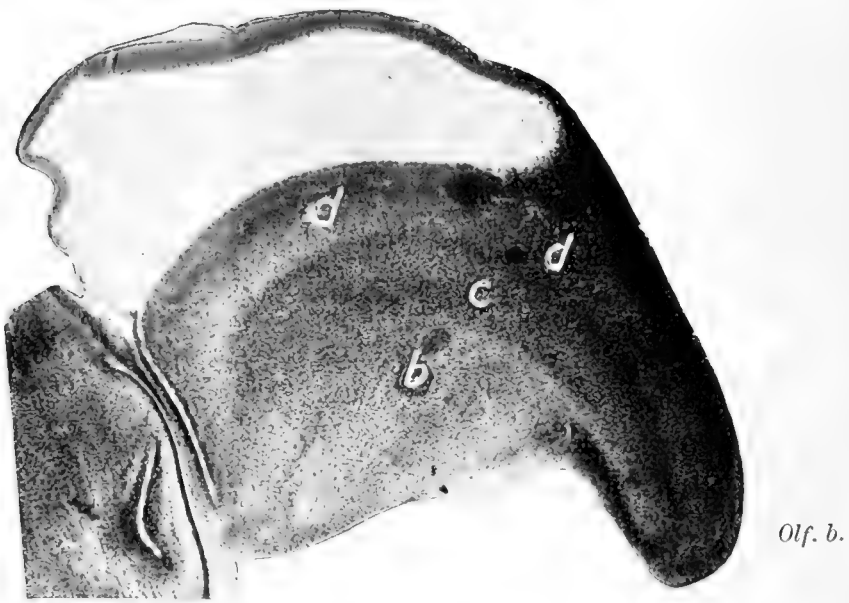

Fig. 8. Sagittal section of the forebrain of a chickembryo of 6 days (lateral to fig. 7) d = palaeostriatum augmentatum (= mesostriatum) c = hyperstriatum inferius, d = hyperstriatum superius.

In fig. 9 only a small part of the palaeostriatum (b.) is seen, viz that part which is continuous with the recessus praeopticus.

Fig. 10 is interesting to us because it shows that the hindpole of the striatum nearly only consists of hyperstriatum inferius (c), the lamina medullaris hyperstriati (in this stage of development) ending only little beyond the contact of hyperstriatum superius and pallium. In the same figure (but better in 11 and 12) is seen that the hyperstriatum inferius is continuous with the base of the brain (whilst more laterally it is continuous in the pallium).

Of the palaeostriatum besides the part that is continuous with the recessus opticus a frontal part is seen in fig. 10, seemingly separated from the hindpart by a recess of the ventricle. This is however only seemingly so, this aspect being caused by the fact

[1]) One might ask if the part called hyperstriatum superius here is not partly the "mediale Sagittal-Wulst" of the cortex with which the hyperstriatum superius in many birds (f. i. the Cacatua) coalesces. This however is not so here, though later the hyperstriatum superius continues in the medio-dorsal mantle, without showing any medullary limitation.

that the palaeostriatum following the lateral convexity of the brain is curved and not cut here in its entirely length. In fig. 11 this separation is smaller and in fig. 12 it has entirely disappeared.

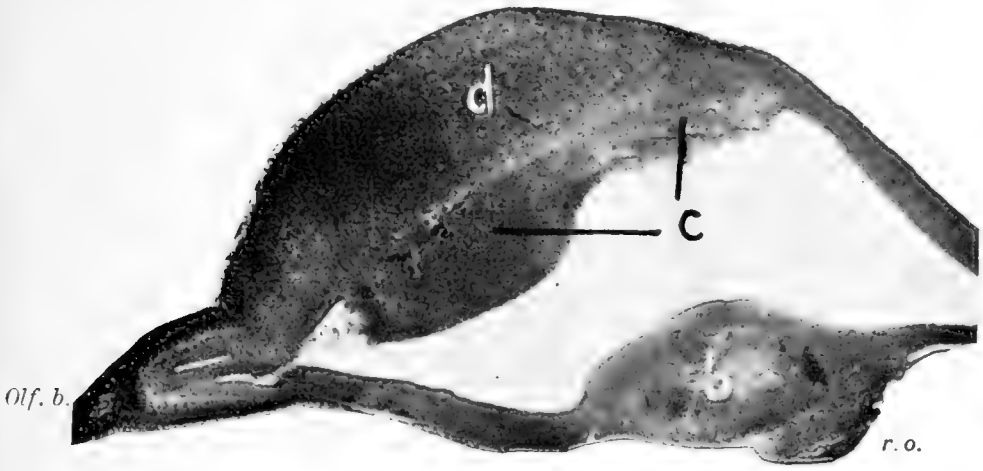

Fig. 9. Sagittal section (rather medial) of the forebrain of a chickembryo of 11 days.
r. o. = transition to the recessus opticus.
b = posterior part of the mesostriatum or palaeostriatum.
c = hyperstriatum inferius.
d = hyperstriatum superius.

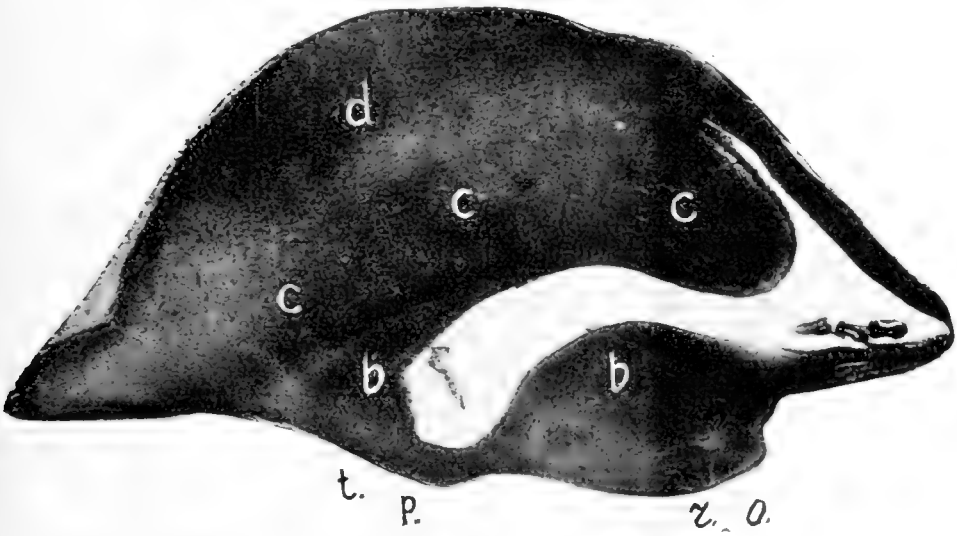

Fig. 10. Sagittal section of an 11 days chickembryo lateral to fig. 9.
d = hyperstriatum superius; *c* = hyperstriatum inferius;
b = palaeostriatum augmentatum (= mesostriatum);
t. p. = tuberculum parolfactorium; *r. o.* = recessus opticus.

The following three figures of this embryo (fig. 11, 12 and 13), show very clearly the presence of the lamina medullaris externa between the hyperstriatum inferius *c* and the *palaeostriatum b*, and also the fact that this lamina is a place of predilection for blood-vessels (v. s. = vasa sanguinea). In fig. 11 this lamina has become specially clear by the retraction of the tissue (these are silverpreparations), which retraction finds a natural place of predilection at this spot on account of the loose character of this lamina to which I already referred.

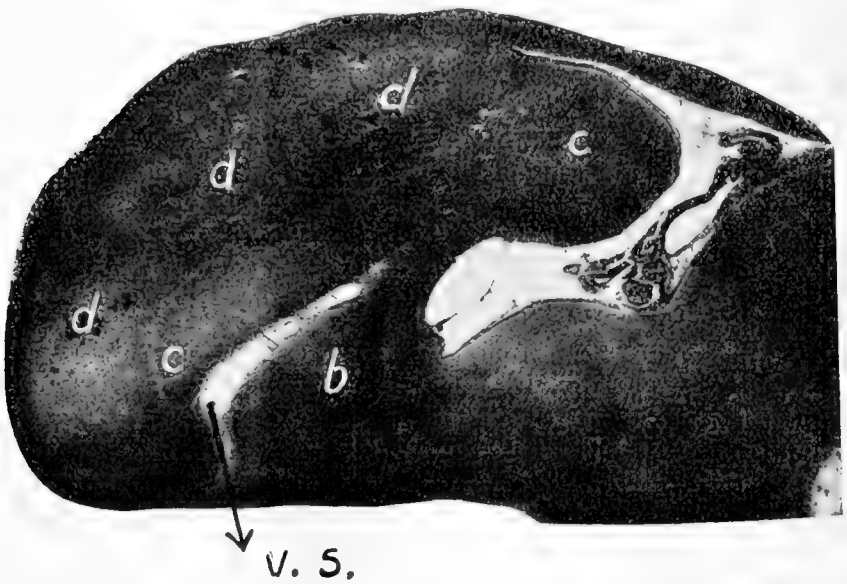

Fig. 11. Sagittal section of a chickembryo of 11 days, lateral to fig. 10. By the retraction of the tissue the vascular cavities (*v. s.*) in the lamina medullaris externa are very evident.
d = hyperstriatum superius, *c* = hyperstriatum inferius, *b* = palaeostriatum augmentatum = mesostriatum).

Figures 12 and 13 moreover show us that the hyperstriatum superius *d* diminishes in lateral direction while *c* enlarges acquiring its connection with the frontal pallium, near the ectostriatum (E. S.).

It is further of interest to note in fig. 11 and 12 that a part of the palaeostriatum *b* continues with and underneath the hyperstriatum inferius bending backward over the recess of the ventricle (above the secondary epistriatum or archistriatum E.).

The caudal enlargement of the palaeostriatum with and underneath the hyperstriatum is what I have called the *substantia palaeostriatica caudata*.

In fig. 13 we see the division of the palaeostriatum by the lamina medullaris interna, the inner segment of the palaeostriatum being the basal nucleus or palaeostriatum primitivum. In this figure also

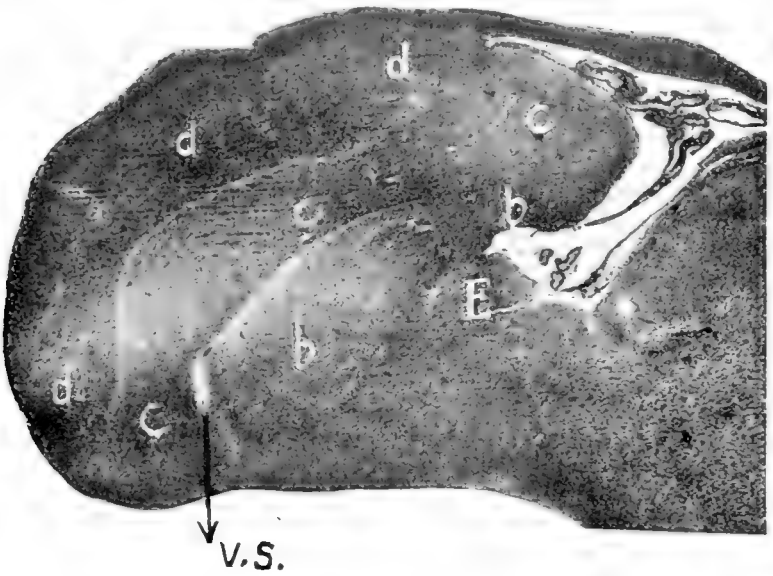

Fig. 12. Sagittal section lateral to fig. 11. Note the dorso-caudal tail of the mesostriatum (*b*) the substantia palaeostriatica caudata, underneath the caudal pole (*c*) of the neostriatum.

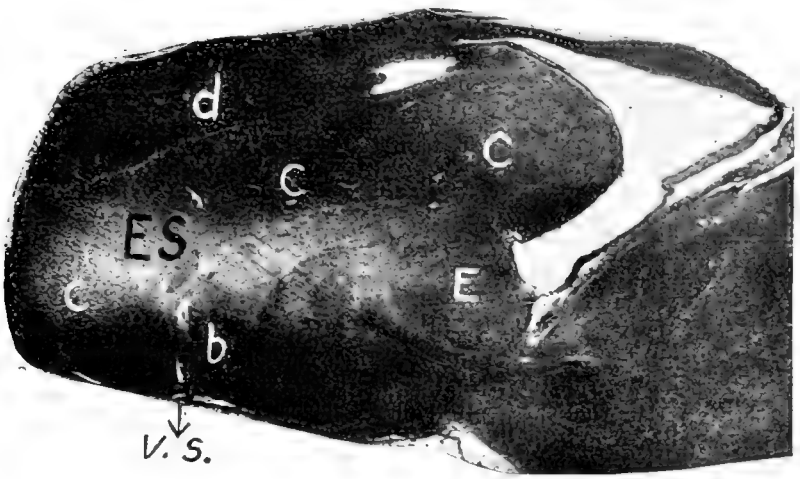

Fig. 13. Sagittal section of the forebrain of an 11 days chickembryo lateral to fig. 12. The hyperstriatum superius (*d*) is smaller, the hyperstriatum inferius (*c*) larger here than in fig. 12. The latter shows its transition in the pallium. *E S* = ectostriatum, *E* = secund. epistriatum or archistriatum; between the latter and *b* the basal nucleus.

the groove between hyperstriatum *c* and archistriatum E (the fissura strio-archistriatica) is visible (but not indicated).

Resuming my results concerning birds, I may conclude that here (apart from the archistriatum or amygdala) at least two chief divisions of the striatum may be distinguished: the *palaeostriatum*, which is enlarged to a *palaeostriatum augmentatum* (or meso-striatum) and which arises entirely from the base of the brain in front of the recessus praeopticus, and the *hyperstriatum* of which the upper part arises entirely from the mantle (hyperstriatum *superius*), while the underpart (hyperstriatum *inferius*), arises from the mantle (laterally) as well as from the base of the brain in front of the palaeostriatum. Both parts of the hyperstriatum thus show the fact, that intraventricular protrusions of striatal type may originate from the pallium as well as from the base of the brain, as I already pointed out for the primary epistriatum in bony fishes, and as was pointed out by Ell. Smith for the neostriatum of Reptiles.

Before dealing with the question whether the hyperstriatum superius of birds is included in the neostriatum of mammals (as the hyperstriatum inferius is), or if it is homologous to the claustrum, I shall shortly describe the embryonic development of the striate body in the rabbit and in man, about which already His [1]), Hochstetter [2]) and Miss Hines [3]) have given us such valuable informations.

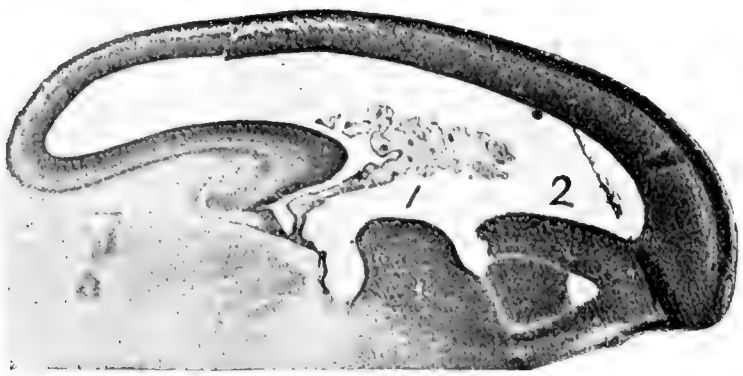

Fig. 14. Sagittal section of the forebrain of a rabbit-embryo of $2^1/_2$ c M. total length.
1 = contains ventrally the palaeostriatum, arising on the level of the foramen Monroi.
2 = the neo-striatum arising partially from the mantle

[1]) His, Die Entwicklung des menschlichen Gehirns, Leipzig 1904.

[2]) Hochstetter. Beiträge zur Entwicklung des menschlichen Gehirns. Deuticke, Wien 1920.

[3]) Hines. Studies in growth and differentiation of the telencephalon in man. Journ. of comp. Neur. Vol. 34, 1922.

In a sagittal section of the brain of a rabbit of $2\frac{1}{2}$ cM. (fig. 14), we see two proliferation centres of striatum cells. The centre of proliferation marked with 1 contains archistriatic cells covering the primordium of the *palaeostriatum*, the latter being its ventral part arising from the base of the brain about the level of the foramen Monroi, and being continuous with the wall of the preoptic recess. In front of this and arising partly from the base of the

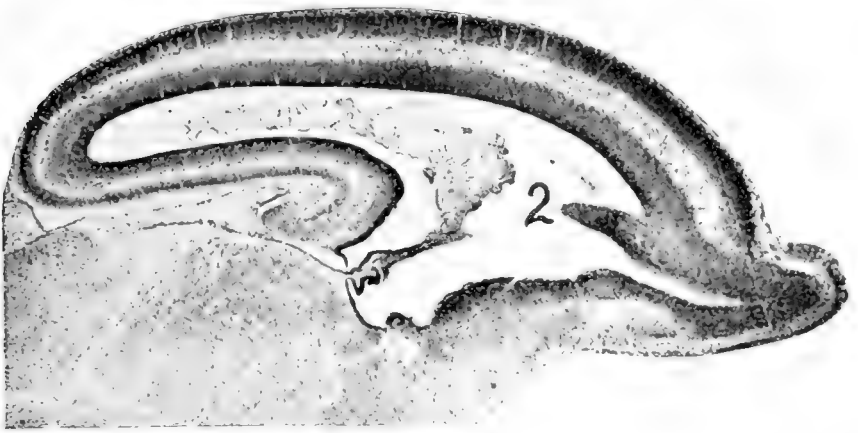

Fig. 15. Sagittal section of the forebrain of a rabbit-embryo of 4 c.M. total length. At 2 the transition of the neostriatum in the deep layer of the frontal-pallium above the olfactory ventricle is seen.

brain, partly from the mantle, we see a part of the anlage of the neostriatum, marked with 2 [1]).

Examining more lateral sections one sees that 2 enlarges backward and unites with the anlage 1.

I will not deal extensively with the mammalian ontogeny but only reproduce here still another section taken from a rabbit embryo of 4 cM., in which the continuity of the neostriatum (2) with the pallium above the olfactory ventricle is particularly evident (fig. 15).

Also in the human embryo the two parts of the striate body (I do not speak here about the amygdala) are evident, even more so than in the rabbit.

Fig. 16 shows a frontal section through the forebrain of a human embryo of 27 mM total length in front of the Foramen Monroi. At the left side of the figure the two primordia of the striatum may

[1]) The cluster of cells between 2 and the base of the brain continue medially into the septum.

be seen, which have been distinguished by His as the *crus epirhinicum* and the *crus mesorhinicum* [1]).

Both crura are separated by a fissure which until now has been named *fissura intercruralis*, but which may be called *fiss. neopalaeostriatica* since my researches have convinced me that the mesial crus is the primordium of the palaeostriatum whereas the lateral crus is the primordium of the neostriatum.

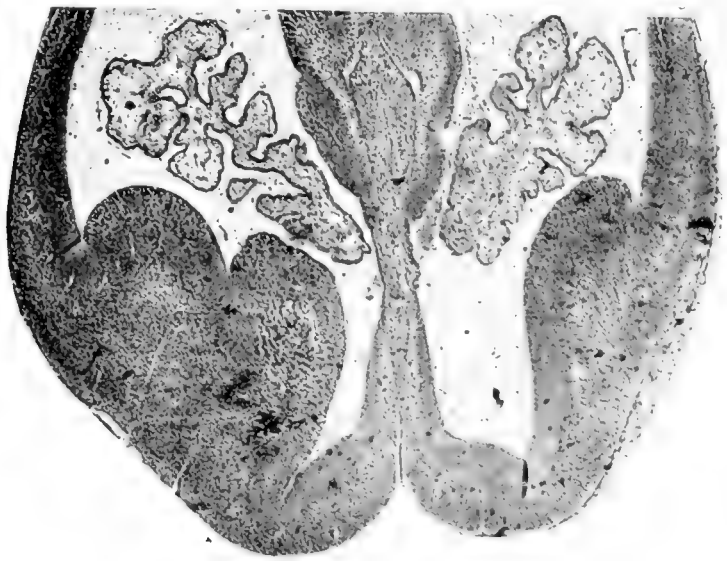

Fig. 16. Transverse section of the forebrain of a human foetus of 27 m.M. total length. This section being slightly oblique, the right side shows a more frontal level than the left one.

The mesial crus does not extend as far frontally as the lateral one, as the figure — on the right side — shows, where the mesial crus or palaeostriatum [2]) has already disappeared, the neostriatum

[1]) His called the caudo-medial edge of the latter *crus metarhinicum*, but it is better not to distinguish this as a separate part since it is merely the caudo-medial side of the mesorhinic crus. It is better to speak only of a lateral and medial primoria as also Hochstetter and Miss Hines do.

[2]) I may mention here that in this embryo of 27 mM. the transitory cavities of the corpus striatum, which Essick first described (Carnegie embryologic public. No. 222), as being constant in human embryos from 15—20 mM, and less constant up to 24 mM, were still present. They are confined in my material to the palaeostriatum. I quite agree with Essick that they may be due to insufficient drainage of the brain in that stage in which the production of metabolic solutions may surpass the possibility of drainage, the more so since phylogenetically as well as ontogenetically the dual source of production of liquor (choroid plexusses and ependyma on one hand and intra cerebral vessels on the other) is established

anlage continuing still some distance in front, being continuous not only with the base of the brain, but also with the mantle of the frontal pole, immediately above the olfactory ventricle (this is why His has called it *crus epirhinicum*).

The fissure between the neo- and palaeostriatum becomes less and less deep during further development. In an embryo of 27 centimeters, it has become very shallow, by the prepondering development of the neostriatum, which more and more overlaps the palaeostriatum, as we found it also to be the case in birds with the hyperstriatum inferius.

The fissura neo-palaeostriatica may however still be seen in the full-grown human cerebrum (f. i. about the level of the comm. anterior, fig. 17: F.N.P.S.) forming the ventro-mesial border of the caudate nucleus.

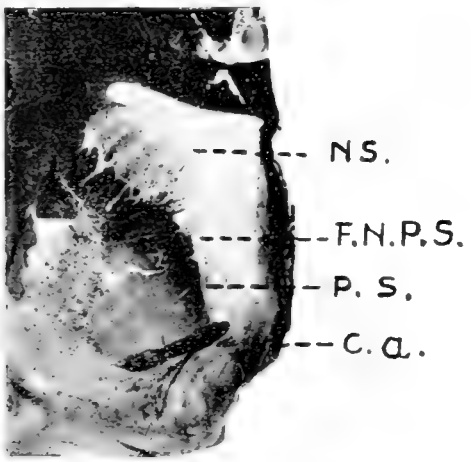

Fig. 17. Transverse section through the corpus striatum of an adult man on the level of the comm. anterior (c. a).
N.S. = Part of the Neostriatum (nu. caud.).
F.N.P.S. = Fiss. neo-palaeostriatica.
P.S. = Palaeostriatum (covered by the taenia semicircul.)

Underneath this fissure runs the *stria-semicircularis*, which covers here some small vestiges of grey substance lying on the ventricular side of the capsula interna and still belonging to the palaeostriatum,

priory to the resorptive function of both choroid plexusses and Virchow-Robin spaces (Compare also my book on Comp. Anatomy of the N. S. p. 820 and WEED Contributions to Embryolog. publ. by the Carnegie instit. Vol V. 1917).

the main mass of which lies laterally to capsula interna forming the *globus pallidus*.

Small stripes of grey substance are occasionally found between the main lateral mass of the palaeostriatum (the globus pallidus), and its mesial vestiges, chiefly between the fibres of the capsula proper and the anterior (olf.) crus of the comm. anterior.

Also other facts prove the homology of the globus pallidus and those vestiges with the palaeostriatum augmentatum (meso-striatum) of birds. So in some animals (Hypsiprymnus f. i.), we occasionally find a continuation of the lamina medullaris externa (the limiting layer between globus pallidus and putamen) in the striatal part mesial to the capsula interna, which lamella also medially may be richly provided with bloodvessels.

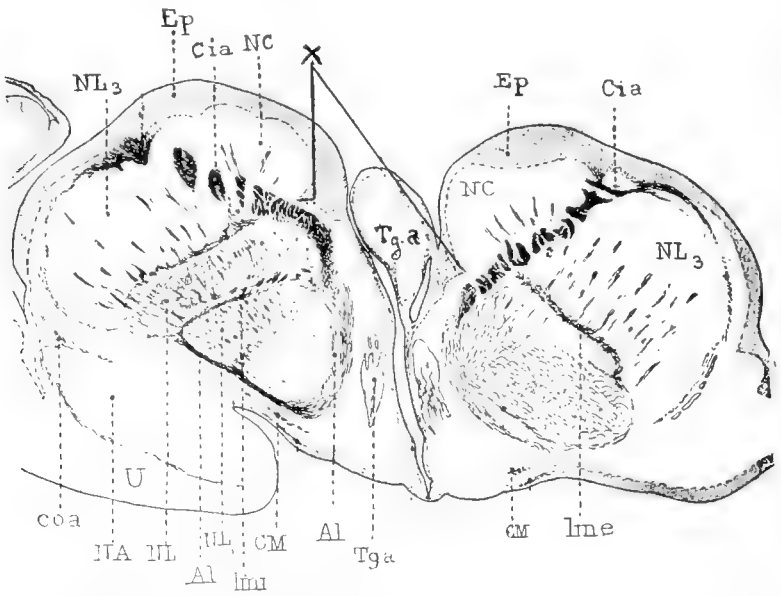

Fig. 18. DEJERINE's case "Longéry". Note the hypertrophy of the ependyma (Ep) specially in the fiss. neo-palaeostriatica at ×. — *L. m. e.* = lamina medullaris externa, *L. m. i.* = lamina medullaris interna.

Another method to reinforce this homology is the method used by SPATZ [1]), who showed that of all parts of the striatum the globus pallidus obtains the deepest blue if applying sulfur ammonium to the fresh (or formalinefixed) brainmaterial, — on account of its richness in iron.

I have applied this reaction to fresh chicken brains and found

[1]) Ueber den Eisennachweis im Gehirn, besonders in Zentren des extrapyramidal motorischen Systems, Ister Teil. Zeitschr. f. d. gesamte Neur. und Psych. Bnd. 77, 1922.

the mesostriatum (palaeostriatum augmentatum) to do the same [1]). Whereas however the dark blue colour in the striatum of mammals is confined to the part of the palaeostriatum lying laterally to the internal capsula (the globus pallidus) in birds the deep stain reaches the ventricular side of the palaeostriatum. This difference is apparently due to the accumulation of myelinated fibres in the capsula interna in mammals, myelinated fibres being insensitive to this reaction. Only in such mammals where the capsula is less dense may the blue colour penetrate in it, as SPATZ found to be the case in Ungulates [2]).

It may be mentioned here, that as in birds (fig. 11 and 12) also in mammals the mesial part of the palaeostriatum may continue some distance caudad underneath the neostriatum, viz. under the caudate nucleus. This *substantia palaeostriatica caudata* accompanies the stria semicircularis on its lateral side, and in some mammals (Elephas) is separated from the nucl. caudatus (neostriatum) by a fissure (the continuation of the fissura neo-palaeostriatica), or a medullary lamella with bloodvessels. This may be also observed sometimes in man.

I still will call attention to the fact that the neo-palaeostriatic fissure, generally best indicated on the level of the commissura anterior, may acquire a much more pronounced character in pathology. An example of this is given by the case LONGÉRY described by DEJERINE in his text book [3]).

In this case the hypertrophied ependyma (Ep. fig. 18), is especially thick at the limit between neostriatum and palaeostriatum (at X in fig. 18), filling up the neo-palaeostriatic groove.

The striatum in this case is further interesting to us, because it shows such a marked similarity with the striatum in birds, which is due to the reduction of the pallium in this case (a hydrocephalic). As a consequence of this reduction the capsular fibres are much less developed than in normal condition, which attributes to the avian aspect of this corpus striatum.

It needs not be repeated here that the division made in the mammalian neostriatum in a putamen and caudate nucleus is not an intrinsic one. The putamen

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[1]) In birds the deepest blue is shown by the "Sagittalwulst", specially its frontal part, then comes the meso-striatum, then the caudal part of the hyperstriatum inferius. In the thalamus the nucl. rotundus chiefly acquires this colour.

[2]) That the palaeostriatum is not confined to the globus pallidus alone is also proved by the figures of HOCHSTETTER's embryologic collection, where the early differentiating region is seen to penetrate into the capsula.

[3]) DEJERINE. Anatomie des centres nerveux, Tome II, fig. 202.

may enlarge medially in such animals (as Ornithorrhynchus) where the fibres that may be called capsula interna fibres, take a more medial course than usually. Moreover we know that frontally, where the capsula interna fails, the putamen and caudate nucleus fuse and that this fusion is larger the smaller the frontal extension of the capsula is. (E. DE VRIES) [1]). Such a fusion may also occur caudally (f. i. in Elephas).

Also the separation of the palaeostriatum and neostriatum by the lamina medullaris externa is very evident in man (as also the lamina medullaris interna).

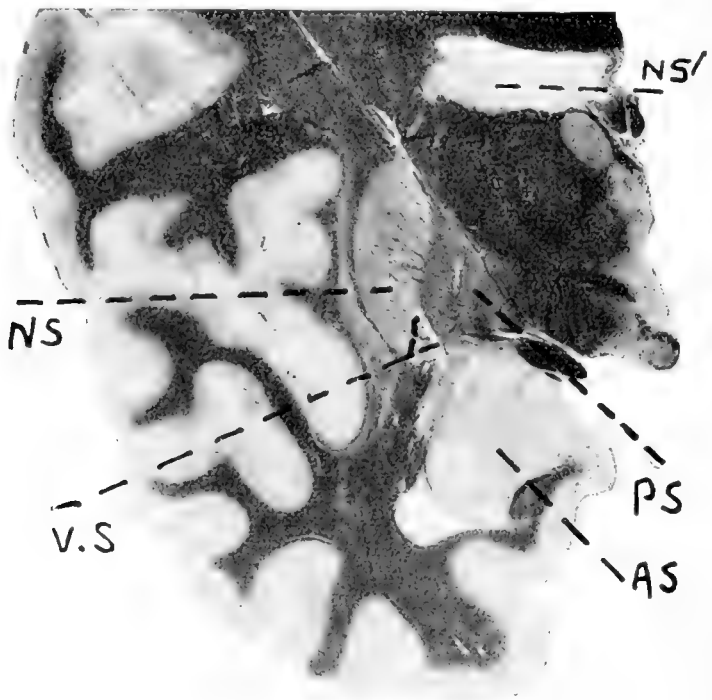

Fig. 19. Neostriatum (*NS* and *NS'*), archistriatum or amygdala (*A.S.*) and palaeostriatum (*P. S.*). *V. S.* = large bloodvessel in the lamina medull. externa. Laterally from the neostriatum (*N. S.*) the claustrum is seen.

At last I want to call the attention to the fact that in the normal human cerebrum the lam. medullaris externa — as in birds — is a place of predilection of bloodvessels more than the other (internal) lamina (vide fig. 19).

Also the connections of the avian striatum (which we know chiefly through the works of BOYCE and WARRINGTON, EDINGER, WALLENBERG, and HOLMES and

[1]) E. DE VRIES. Das Corpus Striatum der Säugetiere. Anat. Anzeiger, Bnd. 37, 1910.

SCHROEDER) and those of the mammalian striatum (which are known by the works of MONAKOW, DEJERINE, WILSON, THE VOGTS, RAMSAY HUNT, LEWY), shows in many respects a great resemblance.

In both classes it is chiefly the internal segment of the dorsal thalamus (nucl. anterior and nucl. medialis) together with the ventral thalamus and subthalamic and peduncular region, which are connected with the striate body (as is already the case in Reptiles).

So in mammalia a connection is known to exist between the nucleus anterior thalami and the caudate nucleus (a connection which I can confirm for the marsupials). It seems possible that this connection is homologous or at least analogous to the fibre tract which WALLENBERG showed to exist between the medio-dorsal part of the neo-striatum in birds and the nucl. anterior dorsalis in these animals.

The antero-lateral part of the striatum of birds, which may be homologous to the anterior part of the putamen of mammals has connections which are homologous to the mammalian, if at least the ventral peduncular nuclei of birds are homologous to the corpus subthalamicum and substantia nigra of mammals which is very likely so.

Moreover in both classes this striatal region seems to give fibres to the commissura supraoptica dorsalis of Meynert (which also contains fibres from the palaeo-striatum augmentatum). Concerning the palaeostriatum augmentatum (meso-striatum in birds), it may be that this, partly at least, has to do with trigeminal functions. The considerable enlargement which the original palaeostriatum (small as it is in Amphibia) acquires in Reptilia (chiefly in the Crocodile) and in birds, may be due to projections of the trigeminus, which acquires a very important size and function in the crocodile and is of prepondering importance in birds, the more so since smell and taste are of so little importance here, and the oral sense is so important for life, as I pointed out in 1908. (Folia Neur.).

Also the fact that in Ornithorrhynchus and in Elephas (E. DE VRIES), the palaeostriatum is so well developed may support this point of view, since the trigeminus is of prevailing importance here, (the Vth nerve is at least three times larger in Monotremes than in other animals and in the Elephant it provides the trumpets sensibility of muscles and skin). The sensibility provided by the trigeminus to facial muscles is generally of great importance as is proved by the disturbances of tonic innervation of the face, so often seen in man as a consequence of striatial lesions.

As far as concerns birds also ROGER's [1]) experiments seem to prove this conception.

Of course the connections of the striate body are not exhausted with this enumeration, so f. i. there remains to be mentioned the fibre system proceeding from the nucleus ruber to this body as demonstrated by v. MONAKOW and others. It is interesting in this respect that SCHROEDER (l. c.) even mentions — for birds — a direct continuation of the brachium anterius cerebelli, to the palaeostriatum augmentatum of birds. This sort of connections on account of their cerebellar component fall also in the range of motor coördinations. (C.f. also LEWY [2])).

The exact character of all these systems has not yet been sufficiently scrutinized,

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[1]) An experimental study on the corpus striatum in the pigeon. Journ. of comp. Neur. Vol. 35., 1922.

[2]) LEWY, Die Lehre vom Tonus und der Bewegung. Jul. Springer, Berlin, 1923.

but so much is true (comp also KINNIER WILSON's [1]) experiments and the clinical cases) that the integrating tonetic factor may have an important role in this.

That also visceral disturbances may occur (liver, bladder) in diseases of the corpus striatum and that sympathetic functions have been found to exist here is not so strange in connection with the fact that the primitive striatum developes near the frontal end of the sulcus limitans (which according to several authors, c. f. HERRICK) ends in the preoptic recess. Moreover we know from the researches of BOEKE, DUSSER DE BARENNE, AGDUHR and the BOER that also in muscle-tonus sympathetic fibres may act a part.

Since visceral and tonetic conditions act an important part in emotions, I would moreover not be astonished if the striatum would prove more and more to have to do with emotions.

At last the question remains if also the *hyperstriatum superius* of birds is included in the neostriatum of mammals and man.

This problem is not easy to solve. One might be inclined to believe that this region of the avian brain is more likely related to the mammalian *claustrum*, a supposition I already made in my textbook.

As the hyperstriatum superius, the claustrum is entirely of pallial origin. Though it may not be derived from (the sixth layer of) the cortex, yet all its cells are derived from a pallial matrix. DE VRIES [2]) has clearly shown that the claustrum in embryologic stages does not derive from the cortex, but that it derives from the lateral brainwall (which at this spot must be called pallium) between the upperedge of the neostriatum and the cortical layers, separated from the latter by fibres of the capsula extrema.

Also the figures given by LANDAU [3]) in his anatomy of the forebrain shows that the way the claustrum developes is that of an intraventricular growth of the pallium (a hypopallial growth in the sense of ELL. SMITH), though apart from the cortex. In its mode of formation the claustrum thus resembles the hyperstriatum superius.

Still in another point there is resemblance between both. Whereas the neo-striatum in mammals (like the hyperstriatum inferius in birds) developes partly from the base of the brain immediately behind the olfactory bulb, partly from the pallium lateral to the olfactory bulb, the hyperstriatum superius like the claustrum only developes from the pallium starting in front immediately above the anterior olfactory lobe of the avian brain.

The fact that the claustrum is very small in Monotremes and Marsupials (where it extends, as in many mammals, partly under-

[1]) An experimental research into the anatomy and physiology of the corpus striatum. Brain Vol. 36, 1913—1914.

[2]) Bemerkungen über Ontogenie und vergleichender Anatomie des Claustrums, Folia Neurobiol. Bnd. IV, 1910.

[3]) Die Anatomie des Grosshirns, Bircher, Bern 1923.

neath the fiss. rhinalis viz. from the region which is covered by the palaeopallium) and only in primates attains a size comparable to birds, does not necessarily contradict this homology, since there there are more respects in which the human brain resembles more the avian brain than the lower mammals do, f. i. in its oculomotor nucleus (comp. Vol. II of my book fig. 261 with figg. 294—295).

Moreover there seems to be a considerable difference in the development of hyperstriatum superius also in birds. As a matter of fact I have not been able to see it in the Kiwi (see HUNTER's work on this peculiar bird [1]) (to be published in the English Journ. of Anat. 1923—24).

In the casuary, and the ostrich it is present, but not yet in the same size as in the chick.

The bloodsupply of the hyperstriatum superius is not in contradiction with such a homology, since the hyperstriatum superius next to many branches of cortical vessels, receives a few branches of the basal arteries, as SHELLSHEAR [2]) proved to be also the case with the claustrum of mammals. Perhaps that also the function of these regions shows a certain relation in so far as degeneration of the claustrum seems to give incoordination of movements (athetosis) while also in experimental degeneration of the hyperstriatum superius disturbances of motor function occur (ROGERS). The question may certainly not be considered settled as yet, but the possibility may be kept in mind that the hyperstriatum superius is not involved in the neostriatum of mammals, but in their claustrum, though this region of the telencephalon in birds has taken a development which in this form and size may be peculiar to these animals only just as the large development of the primary epistriatum is peculiar to Teleosts and Ganoïds.

CONCLUSIONS:

Resuming we may state that also the comparative ontogeny of the striate complex in (Reptiles, Birds, Mammals, and Man proves that apart from the secondary epistriatum or archistriatum (amygdala) two chief parts may be distinguished: *palaeostriatum* and the *neostriatum*, which are separated from each other by the *lamina medullaris externa* (richly provided with bloodvessels) and the *fissura-neo-palaeostriatica*.

[1]) In connection with this it is interesting to note that in PARKER's figures on the embryology of the Kiwi brain (see Transactions of the Roy. Soc., London) Vol. 182, 183, 1892 and 1893), only two intraventricular primordia (my *b* and *c*, are drawn, and *d* the primordium of the hyperstriatum superius seems to be lacking here).

[2]) The basal arteries of the forebrain and their functional significance, Journ. of Anatomy (English), Vol. 55, 1920.

The first part contains the basal nucleus or *palaeostriatum primitivum* (its only constituent in Amphibia) and the mesostriatum which developes from the same mass as the basal nucleus including the surrounding tissue (*palaeostriatum augmentatum*).

The total palaeostriatum in man is represented by the globus pallidus and eventual vestiges of grey substance occurring in and mesially to the capsula interna (underneath the *fiss. neo-palaeostriatica*) including a vestige of grey substance which is a continuation of the latter and (as in birds) lies underneath the caudate, nucleus: the *substantia palaeostriatica caudata*, which in some animals may be separated from the caudate nucleus by a continuation of the lamina medullaris externa and of the fiss. neo-palaeostriatica. The palaeostriatum arises entirely from the base of the forebrain near the anterior wall of the preoptic recess. It consequently is entirely telencephalic in character not of diencephalic origin as SPATZ [1]) supposes, though its cells in adult animals may be continuous with the ventral and peduncular cells of the thalamus and midbrain as I already pointed out in 1908 [2]).

The *neostriatum* (putamen and caudate nucleus) in mammals, arises as well from the base of the brain in front of the palaestriatum (immediately behind the anterior olfactory ventricle) as from the adjacent pallium (ELL. SMITH). The partly pallial origin of the neostriatum (already supposed — but not proved — by WERNICKE(?), OBERSTEINER and KÖLLIKER) explains the fact that in many cortical affections of the brain frontal lobe chiefly) as general paralysis (MILLS, L. BOUMAN, FORSTER, GANS) also the neostriatum is often affected, more often than the palaeostriatum.

Whether the hyperstriatum superius of birds which arises *only* from the mantle is included in the neostriatum in mammals or not, cannot as yet be settled with certainty.

The possibility exists that it is represented in mammals and man by the *claustrum*, which also is a ventricular ingrowth of the pallium (a hypopallial product in the sense of ELL. SMITH, l.c.).

Difference must be made between a cortical ingrowth and a (hypo) pallial ingrowth. Both are formed in the mantle, but need not necessary to be related, though a pallial ingrowth may be followed by a cortical ingrowth (as f.i. is seen with the amygdala).

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[1]) Ueber Beziehungen zwischen der Substantia nigra des Mittelhirns und dem Globus pallidus des Linsenkerns. Erg. Heft zum Anat. Anzeiger Bnd. 55, 1922.

[2]) Weitere Mitteilungen ueber die Phylogenese des Corpus Striatum and des Thalamus, Anat. Anzeiger Bnd. XXX, 1908.

Geology. — *"On Tertiary Marine Deposits with fossil fishes from South Celebes"*. By Prof. H. A. BROUWER and Dr. L. F. DE BEAUFORT.

(Communicated at the meeting of January 27, 1923).

The Origin and the Age of the Deposits by Prof. H. A. BROUWER.

During the construction of a road near Patoenoeang Asoe E in South Celebes a fossil-fish, of which only the posterior part is preserved, was found at the surface of one of the detached blocks of limestone. Mr. A. HUISMAN, the engineer who supervised the road-construction, sent me this fragment some time ago, informing me that in spite of further examination of the locality, the anterior part of the fish had not been found. The block was found near Patoenoeang Asoe E, Section Maros, at the base of steep rocks, about 50 m. above sea-level.

The limestone splits easily along the plane of stratification and on further examination in my laboratory [1]), it was found to contain another fossil fish far more complete than the first. Both were studied by Dr. L. F. DE BEAUFORT. His results are given below.

The rock in wich the *Clupea* and *Lutjanus* are embedded is a whitetolight brownish compact limestone, which resembles some types of lithographic limestone from the neighbourhood of Solnhofen and Eichstätt in Franken, which contain the numerous well-preserved upper-jurassic fossils, among which numerous fishes occur. Under the microscope compact limestone proves to be free from foraminifera or other organic remains.

As regards their conditions of origin the rocks of Celebes are also very much like the lithographic limestones of the Upper-Jura in Franken. The latter rocks are found to rest in shallow-basins in coarse, unstratified or rudely stratified limestones, which are reefs on a large scale; the interjacent depressions were filled up with stratified deposits,

Near Solnhofen etc. these lithographic beds contain various, beautifully preserved organic remains. Fresh- or brackish-water fossils are lacking, but remains of tracks of land-animals are of

[1]) By Dr. P. KRUIZINGA, conservator at Delft.

frequent occurrence in the formation, from which it may be inferred that the lagoons between the coral-islands and reefs temporarily emerged above the sea-level and were exposed to the air.

Similar relations prevailed in that part of South-Celebes, where the fossil-fishes have been found. From personal experience I know the limestones near the cascade of Bantimoeroeng not far from the locality Patoenoeang Asoe E, and the numerous authors, who have described other parts of South-West Celebes, all mention these limestones, which often rise abruptly with steep walls from the surrounding plains, presenting typical reefmasses of Tertiary age.

Numerous foraminifera are found in these reefrocks; but also corals and shells are found. Rocks of oolitic structure are also recorded. As well as the limestones, which contain the fossil fishes and perhaps represent a lagoon-deposit, these oolitic limestones show the characteristics of deposits in a sea with reefs and lagoons.

Regarding the precise age of these rocks data have been published, notably by Bücking [1]) and Verbeek [2]) and afterwards by 't Hoen [3]). We now know that Eocene limestones with nummulites and discocyclines occur amongst these rocks, as well as Miocene limestones with lepidocyclines.

Up to now no account has been given of foraminiferal limestones from the immediate surroundings of the locality where the fossil fishes have been found, the nearest rocks that were examined are those near the cascade of Bantimoeroeng, east of Maros, which contain *Cycloclypeus* and *Alveolina*. The age of these rocks may be Upper-Eocene or Oligocene. We have stated already that with the Old-Tertiary rocks also Oligo-Miocene limestones occur at various places.

The age of the fossil-fishes cannot be established exactly, because they show a slight relation only to the fauna of other regions. The *Clupea* (*Sardinella*) is undoubtedly closely related to recent species, the *Lutjanus* possibly so. This might induce us to decide on a recent age of the deposits. But among the many herrings, for instance, from the Tertiary of Europe and America, none are described as showing a closer relationship to *Sardinella*. Different climatic conditions may have been of influence on the distribution of the fishes in Tertiary

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[1]) H. Bücking. Beiträge zur Geologie von Celebes. Samml. d. geol. Reichsmus. in Leiden. Ser. I. Bd. VII. I. 1902. p 118.

[2]) R. D. M. Verbeek. Molukken Verslag. Jaarb. Mijnwezen 1908. Wetensch. Ged. p. 52.

[3]) C. W. A. P. 't Hoen. Verslag over de resultaten van geol. mijnbouwk. verkenningen en opsporingen in Zuidwest-Celebes. Jaarb Mijnwezen 1915. Verhand I. p. 244.

times. VERBEEK [1]) pointed to this in connection with the fact that Eocene fish-species of the Highlands of Padang differ from the Tertiary species of Europe, whereas they bear a close relationship to the species still living in the East-Indian Archipelago, so that they seem to be Miocene rather than Eocene. From this it seems to follow that the Tertiary fishes of the tropics are not suitable to determination of age, and the species here described could be of the same age as the Eocene or Oligo-Miocene reefs.

Finally we wish to point out that after this discovery of fossil fishes, about which only very little is known as yet in the East-Indian Archipelago, it may be expected that on closer inspection of the locality more fossils will be found. The lithographic limestone of Solnhofen is poor in fossils. That the remarkable fauna of this formation has gradually become known is due to the quarry-industry and to the special attention given to the occurrence of fossils.

Description of the fossil fishes by L. F. DE BEAUFORT.

Prof. H. A. BROUWER entrusted me with the study of two fish-fossils imbedded in tertiary limestone, which have been found during the construction of a road in South Celebes.

The smaller and more complete fossil can be recognized at once as a *Clupeid*. As the anterior part of the head as well as the pectorals, ventrals and anal are missing, a further determination would have been doubtful, if the scales had not been extraordinarily well preserved. The greater part of the scales show a number of small holes in their posterior part, whereas the anterior part possesses more or less distinct transversal grooves, which are interrupted in the middle. As far as I know such perforated scales have only been found in four closely related species of herrings, which inhabit the Indo-Australian Archipelago.

These species belong to the genus *Clupea* sensu latiore. Following BLEEKER, WEBER and I (Fishes of the Indo Australian Archipelago II, 1913, p. 68) have placed these species with a number of others, which however do not show the characteristics mentioned above, in the subgenus *Harengula*. TATE REGAN (Ann. Mag. Nat. Hist. (8) **XIX**, 1917, p. 377) has raised this subgenus to the rank of a genus and has separated from it as *Sardinella* those species, which differ from *Harengula*, besides in some other characteristics, also in the structure of the scales. In *Harengula* the transversal grooves of

[1]) R. D. M. VERBEEK. Topographische en Geologische Beschrijving van een gedeelte van Sumatra's Westkust. Batavia 1883, p. 355.

the scales are uninterrupted, in *Sardinella* they are interrupted in the middle. The recent species with perforated scales mentioned above, as well as my fossil, have also interrupted transversal grooves on the scales, and TATE REGAN ranges these recent species therefore under *Sardinella*. It is clear, that my fossil belongs to the same group. TATE REGAN does not mention the small perforation of the scales in his short description and, therefore, I do not know if such perforations occur in other species, which belong to *Sardinella* and which inhabit the Atlantic, Mediterranean, Black Sea, Indic and Pacific. From the foregoing however it will be clear, that the species with scales of this structure form a natural group, and that the fossil belongs to it, which I proceed to describe now as:

Clupea (*Sardinella*) *brouweri* n. sp,

The total length of the specimen cannot be ascertained, as the praeorbital part of the head is wanting. The vertebral column is also broken at different places and some of the vertebrae have been shifted over each other, or got loose from each other. I estimate the length to be 150 mM. It is also difficult to count the number of vertebrae. I think I can distinguish forty-two of them which is somewhat less than the numbers, given by TATE REGAN l.c. for *Sardinella*. DELSMAN (Bijdragen tot de Dierkunde, Afl. **XXII**, 1922, p. 29) records forty-five vertebrae in *Clupea fimbriata*, one of the species with perforated scales.

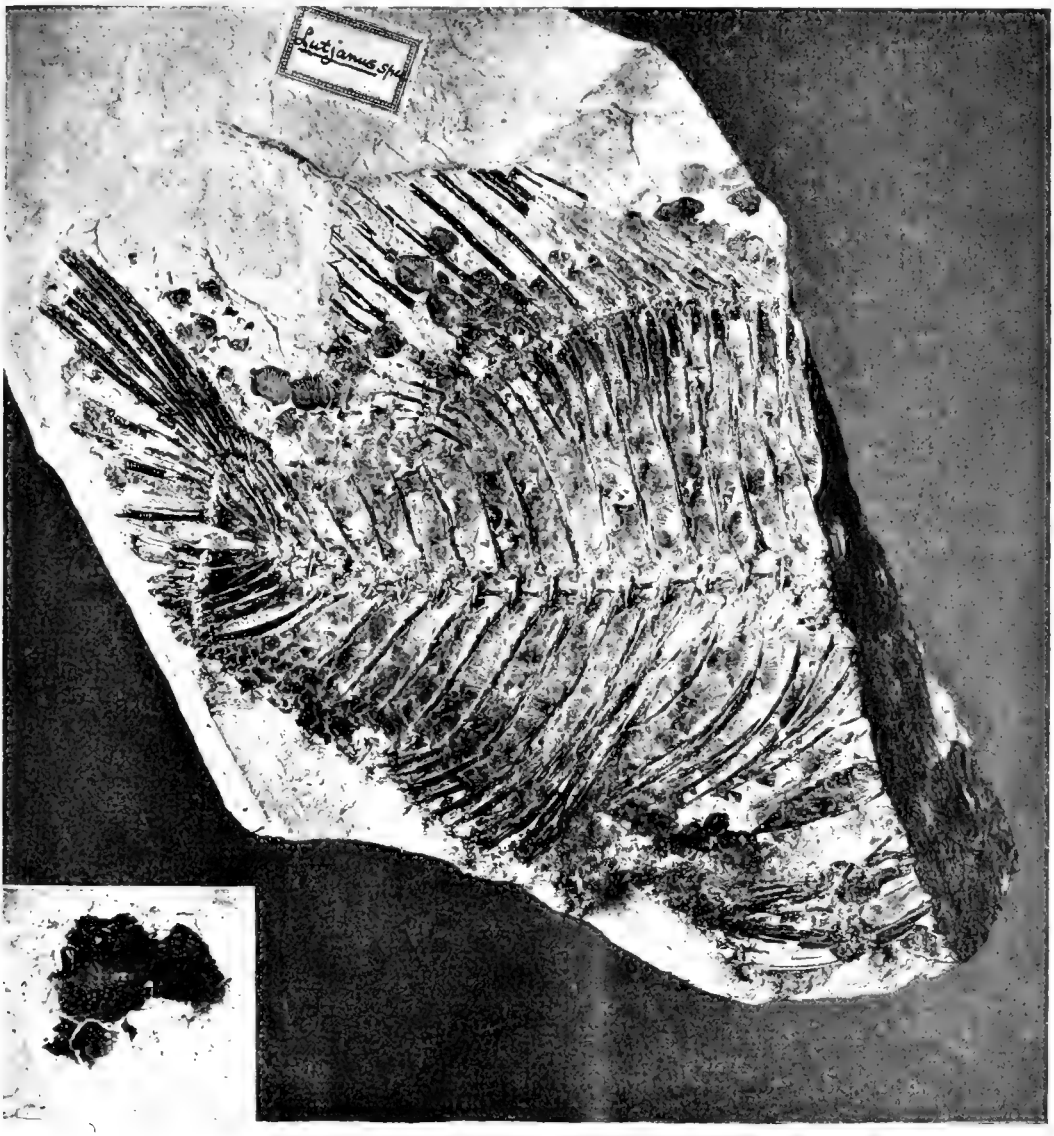
Of the head skeleton really only the opercles and a part of the orbitalia have been well preserved. The ventral part of the opercle shows delicate vertical stripes, caused by sensory-canals and which, although in a somewhat different form, are also present in *Clupea fimbriata* and *perforata*. The preaoperculum also shows some sculpture. Under favourable light fine lines, radiating from one point, may be detected, which I do not find so well developed in recent species. The operculum is not quite twice as high as long.

The dorsal rays cannot be counted accurately as part of the scales of the back have shifted on that fin. I think I can distinguish fifteen of them. Neither is it possible to ascertain the exact position of the dorsal fin, as the vertebral column has been distorted, as mentioned above. The origin of the dorsal is situated about in the middle between snout and base of caudal and is placed above the twenty-seventh vertebra, counted from the caudal. The longest D. ray is about equal to the height of the operculum.

The whole of the ventral part of the fish is severely damaged.

Nothing can therefore be said about the ventrals and anal. Of the pectorals and the pectoral-girdle only some rudiments are present.

As said above, the scales are well preserved, although most of



a — Scales of *C. (S.) brouweri* × 2.

b — *Lutjanus* spec. × 1/2.

them are dislocated and shifted in a dorsal direction. Some of them are even quite isolated, which facilitates however their examination.

In comparing the scales with those of related species, we discover the greatest likeness with the scales of *Clupea perforata* and *fim-*

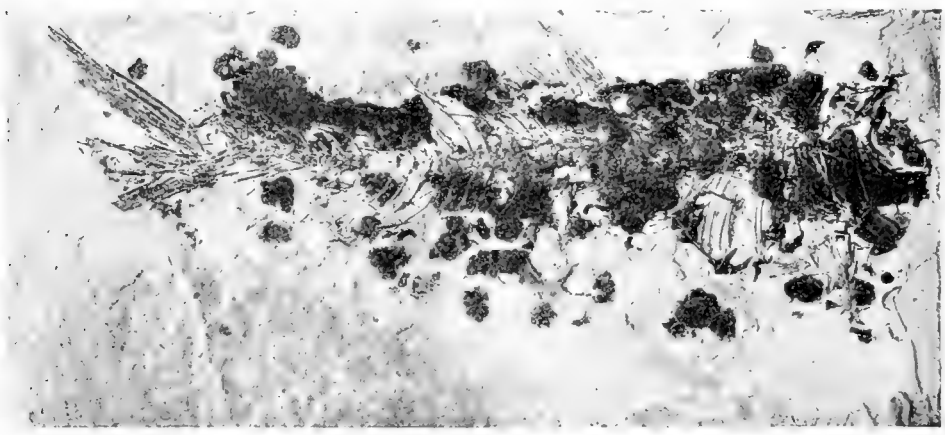
briata, which species, however, have holes of larger dimensions. Also the posterior border of the scales of these species is more ragged. The first transversal groove, which is not interrupted, is visible in all scales of my specimen, the following interrupted ones (about three in number) are specially developed on the caudal scales. For the rest the scales are practically smooth: only here and there some fine parallel stripes are visible.

As might be expected, keeled dorsal scutes are absent. The ventral ones are partly very well preserved, but all are dislocated and dispersed, so that their number cannot be made out. The dorsal prolongations of the ventral scutes or spines are beautifully conspicuous here and there. They seem to be shorter than in recent species. Possibly they are broken.

C. (*S*) *brouweri* shows the greatest resemblance to *C. fimbriata* and *perforata,* but differs in the sculpture of the opercles and in minor details of the scales.

The determination of the second fossil is less certain. It consists of the posterior part of a fish, which undoubtedly belongs to the *Perciformes.* The greater part of the caudal, all caudal vertebrae, some of the rib-bearing trunk-vertebrae, all pterygophores of the anal as well as those of the hinderpart of the dorsal are beautifully preserved in situ. The anal is broken.

The soft portion of the dorsal is intact, but only some spines of



Clupea (Sardinella) brouweri n. sp. × 1.

the precedent part are preserved. The anterior part of the fish is wanting. The ventrals and part of the pelvic have been spared. Nothing else of the shoulder-girdle remains than the caudal part of the posteleithrum. The vertebrae of the trunk bear long parapo-

physes to which the ribs are attached. There are fourteen caudal vertebrae. The caudal has seventeen rays, two of which are probably simple. The dorsal has $x + 3$ spines and 17 soft rays, the anal has 11 and the ventrals have 6 rays.

Part of the scales are extremely well preserved. They are more or less rectangular, with a convex posterior border. From the centre about ten diverging grooves run to the anterior border. A great number of crowded parallel rows of extremely small flat spinelets run to the posterior border. For the rest, the surface of the scales is covered with delicate small lines, concentrically arranged round the middle of the scale and scalloped where they cross the grooves.

With these scanty particulars a further determination had to be tried. A first indication was the great difference between the number of soft dorsal and anal rays. In consequence a number of forms, in which the soft dorsal and the anal are of nearly equal length, could be excluded. Farther on the structure of the scales put me on the right track and brought me to the recent genus *Lutjanus*. It is true that most species of this genus have fewer D-rays than my specimen, but in some of them the number is about the same, f.i. in *Lutjanus sebae*, which species I, therefore, selected for closer comparison. A skeleton of lastnamed species shows so much likeness with my fossil, even in details, that I scarcely doubt that this too belongs to *Lutjanus*.

What can these two fossils now teach us about the age of the deposits, in which they were fossilized?

As far as I know, no other tertiary Teleosts are known from the Indo-Australian Archipelago, than those from a freshwater-deposit in the Padangsche Bovenlanden formerly described by Günther (Geol. Mag. (2) III, 1876). As far as I know, forms related to our fossils are lacking too in the tertiary fish-fauna of the neighbourhood. Neither amongst the tertiary fishes from Australia (Chapman and Pilchard, Pr. Roy. Soc. Victoria (2 XX, 1907) nor amongst those of Siam (Andersson, Upsala Bull. Geol. Inst. XIII, 1916) a species of *Sardinella* or *Lutjanus* has been described. The Clupeid, recently described by Jordan (Proc. Cal. Acad. of Sciences IX, 1919) from Japan, is not related to our specimen. It is even uncertain, if it is a Clupeid at all.

Among the many herrings, described from the tertiary of Europe and America, I do not know of any species, related to *Sardinella*. Smith Woodward (Cat. Fossil Fish British Mus. IV, 1901, p. 152) gives the following description of the scales of *Clupea numidica*, from the Upper Miocene of Algeria: "Scales sometimes pitted in

their exposed portion." But "pitted" is different from "perforated". Besides *C. numidica* has 55 vertebrae.

No other fossil *Lutjanus* is known than a dubious species *Lutjanus hagari*, described by JORDAN and GILBERT (Stanford University Publications 1919) from the Miocene of California and which later on has been ranged by JORDAN in the related genus *Neomanis*.

Therefore, an opinion of the age of the two fish-fossils cannot be more than a guess. When we take in mind, that during the Miocene most of the recent genera were not yet in existence, as JORDAN has pointed out recently and when we remember, that the *Sardinella* is certainly related to recent species and the *Lutjanus* probably so, I feel on these grounds inclined to consider them not older than miocene.

Both fishes have been found in one stone, the dimensions of which are about $40 \times 20 \times 6$ cM. Moreover in the same stone some scales of other fish-species occur, which I do not venture to determine. This shows, that fish-rests are probably abundant in these layers. A further exploration would certainly be worth while, and could give us more solid information about the age and the character of these deposits.

Geology. — "*Fractures and Faults near the Surface of Moving Geanticlines*. III. *The Horizontal Movement of the Central-Atlantic Ridge*". By Prof. H. A. BROUWER.

(Communicated at the meeting of January 27, 1923).

Many explanations that have been given for tectonic structures are unsatisfactory on account of the geometrical treatment of the problems and a preference to vertical movements. The geometrical treatment draws attention to the change in position of parts of the earth's crust, while the velocity of the movement receives no further consideration. Because of the predilection for vertical movements we often explain the observed facts by vertical movements, until it is proved that faulting must have been effected in another direction.

In regions, which are not accessible to direct observation, i.e. the parts of the earth's crust covered by the sea, the existing morphology is explained by rising and by subsiding movements, while the factor time is neglected. Subsidence of continents and subsidence of "land-bridges" are common expressions in geological literature. Velocity and direction of the movement are hardly or not at all considered in these inadequate interpretations of dynamic phenomena. The reason is obvious, the forces causing the movement are unknown, and the velocity of the movement cannot be measured.

Another way of studying these problems is the comparative-tectonic method. Our object in this paper is to consider the results achieved by applying this method to the movement of a region, which is almost entirely covered by the sea, of which the morphology is known in broad outlines, and which is still moving, as we know from numerous earthquakes. It is the S-shaped ridge, of which the existence has been proved by numerous soundings and parts of which emerge from the sea, as e.g. the Azores and the islands of St. Paul and Tristan da Cunha. In previous papers [1]) we pointed to the significance of the bending-points of the horizontal projection of a geanticlinal axis for a judgment upon the horizontal movement of geanticlines. Transverse fractures, which may be more or less

[1]) These Proceedings XXIII, p. 570; XXV, p. 327.

H. A. BROUWER. The horizontal movement of geanticlines and the fractures near their surface. Journ. of Geology. 1921, XXIX, p. 560—577.

gaping are the surface expression of velocity-differences in a horizontal direction; horizontal transverse faults prevail at greater depth, while with increasing plasticity deformation takes place without fracture-movements. If these tectonic zones of different depths are all visible at the surface, they enable us to trace the movement for a considerable space of time, because then the different phases of the movement are observable. If the movements are still going on, the epicentra of earthquakes will be accumulated near the places with considerable velocity-differences and may be disposed along more or less transverse fractures. In this connection we point to the region in the neighbourhood of Sunda Strait between Java and Sumatra, to the earthquake lines near the bending-point between the Alps and the Carpathian mountains, to Cook-strait between the Northern and the Southern island of New Zealand and to many others.

Zuid Amerika = South America. Romanche diep = Romanche Deep.
Afrika = Africa Azoren = Azores

Fig. 1. 2978 etc. Depths of the sea in meters on the Central-Atlantic Ridge.

If a submarine ridge has a bending-point, the strongly curved shape of the ridge may have been developed from an originally simpler form by velocity-differences in a horizontal direction. Where the velocity-differences are greatest, the epicentra of earthquakes will be numerous, and from an accumulation of epicentra near a

bending point it may be concluded that velocity differences in a horizontal direction are a characteristic of the present movement.

In the Central Atlantic Ridge there is a distinct bending-point between the island of St. Paul and the Romanche Deep, while quite close to it there is a zone of strong seismic activity. Further application of the comparative method would lead to the conclusion that the Central Atlantic Ridge is not only moving now, but has been moving for a long time, with velocity-differences in a horizontal direction. The tectonic structure of the ridge is not accessible to observation. However, there are indications that a further application of the comparative method is possible. The morphology is still little known, but the soundings have proved the existence of very great depths, viz. in the Romanche Deep, where a depth of 7370 m. has been sounded.

This depth has been considered as a remarkable phenomenon for the Atlantic Ocean. The situation *close to the bending-point* points to an origin such as already previously suggested by us with regard to abnormally deep straits near the bending-points of rows of islands. Just as is the case in Manipa Strait between Ceram and Boeroe. The Romanche Deep can be explained by difference in velocity of horizontal movements for neighbouring parts of the ridge along the axis.

We only find the results of the *differences* in velocity in a horizontal direction, the absolute horizontal movement cannot be inferred from the surface characters with the comparative method. We do not know whether the Central Atlantic Ridge originally had a more rectilinear form. Neither do we know whether the bending of the strong curve between the Azores and the island of St. Paul is still increasing, or whether the southern portion with Ascension and Tristan da Cunha is moving with less velocity than the northern in a western, or in an eastern direction, or whether it has become stationary now.

Many widely different views have been brought forward concerning the origin of the Central Atlantic Ridge. Some authors [1]) look upon it as a rising geanticline, as a mountain range in statu nascendi. Up to now these authors never considered the horizontal movements, which as evidenced before often are much more important than the vertical movements in rising geanticlines. Another explanation [2]) has been afforded representing the ridge as the filling of an originally

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[1]) E. Haug, Traité de Géologie I, 1907, p. 164.

[2]) A. Wegener, Die Entstehung der Kontinente und Ozeane. 1922, p. 42.

narrow gaping fracture, which opened to the present Atlantic Ocean by horizontal movements of continental areas.

In either view regarding the origin of the ridge the movements can take place with velocity-differences in a horizontal direction. Other explanations, such as the ridge being of volcanic origin or the highest parts of a subsided continent (horst), do not consider horizontal movements. Vertical movements may occur and may have occurred in some places perhaps in an upward, in other places in a downward direction, and varying at different periods, because no movement of the earth's crust will have exactly a horizontal direction for a long time, just as it will never have exactly a vertical direction.

The comparative method does not enable us to trace out the movement of the Central Atlantic Ridge down from its earliest development. It proves, however, that the simple explanations by upward and downward vertical movements, which have been suggested, cannot be maintained.

Botany. — *"On stimulation in auxotonic movements"*. By Prof. J. M. JANSE. (Communicated by Prof. J. C. SCHOUTE).

(Communicated at the Meeting of January 27, 1923).

Many movements (curvatures) of very different plant-organs are caused by a change in the speed of growth on one side of the organ; collectively they are often called "auxotonic" movements. Various stimuli, among which those of gravitation and of light are by far the most important, may be the indirect cause of these movements; these stimuli are received locally and conducted to the growing zone in which the bending will afterwards take place.

The theory hitherto generally accepted was that the normal vertical longitudinal growth was a separate phenomenon, and that, for instance, the gravitation-stimulus appeared only after the plant-organ had been given a different position. In a recently published paper [1]), I expressed as my opinion that, on the contrary, the normal length-growth is also due to the gravitation-stimulus which by an increased growth of the cells equally on all sides would cause, for instance, the vertical growth of the main-axis and of the radicle. In this position there would even be the maximal stimulation corresponding to their maximal speed of growth in this position, which is experimentally demonstrated. The experiments carried out by WIESNER, MOLISCH [2]) and CZAPEK [3]) speak in favour of this theory; they showed that after the tip of the radicle had been cut off, the rate of growth diminished appreciably within the next 24 hours; this diminution would undoubtedly have been still more apparent if the observations had been recorded also during the ensuing days, because the growth during the first day must still have been influenced by the stimulus received before the amputation of the tip.

It is generally assumed that the stimulation by gravitation depends upon the pressure of the specifically heavier starch-grains (statoliths upon the outer layer of the protoplast of certain cells (statocysts);

[1]) Reizwirkung bei Rektipetalität und bei senkrechtem Wachstum: Jahrbücher für wissenschaftliche Botanik, 1922, Bd. 61, p. 590.

[2]) Berichte d. d. bot. Gesellschaft, 1883, Bd. 1. p. 362.

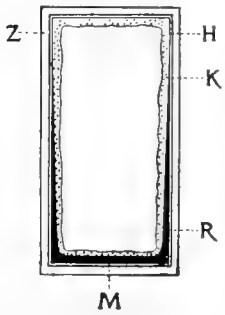
[3]) Jahrb. für wiss. Botan., 1895, Bd. 27, p. 246.

such a stimulus, however, as has already been demonstrated by Noll (Heterogene Induction, 1892), can be the cause of a movement only if the sensitiveness of this outer layer is unequal at different parts. As the vertical position, in the said organs, was regarded as the one in which no stimulation took place, it was supposed that the part adjoining the lowest transverse wall was not sensitive. If, however, also the longitudinal growth be induced by the gravitation-stimulus, as we suppose here, that part would have to be on the contrary the most sensitive.

However this may be, it is sure that there must always exist a certain connection between the position of the place of the greatest (or least) sensitiveness in the statocyst and the direction of normal growth of each organ, so that this, for instance in the cells of the vertically-growing stem, must be found at a different place to those of a horizontally-growing rhizome, etc.

This ought to imply further that when an organ of itself changes its position, this should be preceded by a shifting of the outer layer of the protoplast inside the cell. The supposition of such a shifting of the outer layer would, however, be inconsistent with the general assumption that this layer is immovable, an hypothesis, it is true, but one which for other reasons, e. g. the existence of the plasmodesms, might be called probable. This inconsistency suggests the query as to whether it is not more probable to assume that the excitable portion of the statocyst forms a separate organ of the cell, which might then lie between the outer layer and the granular protoplasm, but quite independent from the former.

This protoplasmic part, which alone should be sensitive to the pressure of the starch-grains, might be termed the "*static apparatus*" and should be capable of shifting, consequent on some influence



The accompanying diagram represents, schematically, the supposed position of the "static apparatus" in the statocyst: Z = cellwall, H = outer layer, K = granular protoplasm, R = the static apparatus of which, M = the middle-field.

The unequal thickness of the static apparatus in the drawing serves *merely* to indicate the local difference in sensitiveness of this apparatus which should be greatest in the middle-field.

from inside or outside, without the outer layer of the protoplast being involved in this movement. Moreover this apparatus should have to be most sensitive in the middle-field, while this sensitiveness

should diminish towards the edges as represented in the accompanying sketch. The apparatus need not be present in cells which are insensible to stimulation.

The normal vertical position of the main axis and radicle would seem to imply that in these organs the middle-field lies against the basal transverse wall of the statocyst. But in a horizontally-growing rhizome, for instance, it ought to ly next to the lower longitudinal wall, for then only there would be maximal stimulus, accompanied by the maximum, equal all round, speed of growth, whereby the rhizome would keep its horizontal position.

If now a certain shifting of the static apparatus is required to produce a new position of equilibrium, then inversely we might deduce from the changement in the position of equilibrium what shifting should have taken place in each separate case, but therefore it were necessary to know also in what part of the organ the static apparatus occurs. This shifting cannot be microscopically controlled, for the present at least, but if it should appear from the following lines that by assuming such a shifting we succeed in giving a simple explanation of widely different and often very complicated phenomena, this must favour our supposition of the presence of a *movable* excitable organ in the sensitive cell. It must be borne in mind, however, that it is therewith immaterial whether we think of an "static apparatus" as indicated above, or of the outer layer as a whole, provided this be but movable; in future we shall suppose the presence of a "static apparatus".

If it be possible by this means to explain why a plant-organ which has a certain position of equilibrium is able to keep this position during its growth, it does not, however explain the familiar phenomenon of an organ that is brought out of its equilibrium returning to this position, not only of its own accord, but also by shortest possible way; so a root, for instance, placed horizontally will curve downwards in a *vertical* plane until the tip points perpendicularly again. That this movement is of great advantage for the later development of the plant is of course no sufficient explanation of its cause, especially since the preparations for the movement are made long before the utility of the bending could be perceived by the plant. We should have to ask, therefore, why it is that a part of the plant makes a *useful* movement and how it comes that the new position is acquired by the shortest way.

This question which, as it seems to me, is proposed here for the first time so sharply, is connected so deeply with the more intimate life of the cell that it can not surprise that no entirely complete

answer to it can be given yet, but nevertheless we can endeavour to arrive a step nearer at its solution.

We shall confine ourselves now in the first place to the stimulus of gravitation.

We have thus supposed that the static apparatus of the statocyst lies in such a position that the middle-field, which forms its most sensitive part, adjoins the lower wall of the cell when the organ is in equilibrium, whatever this position may be. When this position be changed, if, for instance, a root be placed horizontally, the starch-grains which shift under the influence of gravitation, come into contact with the less sensitive border of the apparatus; if then, after some time, the tip bends downward, the starch-grains, again shifting, will gradually come into contact with the more and more sensitive parts of the apparatus till, when the tip stands vertical, they will have reached the most excitable place again; thus we see that the curving downward is accompanied by a continual increase of the stimulus and that the speed of this increase will be greatest when bending takes place in a vertical plane.

Could it be that this increase of the stimulus is the indirect cause of the bending and at the same time of the choice of the shortest way?

Of itself this "striving after the maximal stimulation", as we might term it, cannot be regarded in the plant as the direct cause of any movement, although it might later on be of aid in explaining it; nevertheless cases are known in which there exists a rather direct connection between this striving after an ever stronger stimulus and the movements.

So, for instance, in positive chemotaxis: if e.g. spermatozoides of ferns be placed in a weak solution of malic acid in which the concentration is unequal at different places, they will move towards the place of the strongest concentration, i. e. in the direction of the increasing concentration or stimulation.

It is known, regarding some of the senses of man and animals, such as the eye, the ear, and perhaps also of the static organ when the organism is at rest, that they adjust themselves automatically (reflectorily) to a stronger stimulus, i. e. that the same stimulus which causes the sense-perception also excites other nerves and through them certain muscles, which last thereby move the sense-organ in such a way that it receives then the strongest possible impression; thus here too we have the case of a movement with the aim of increasing the stimulation. If such a comparison with the plants were entirely justified, which could not be decided at present, we

might go further and state that, because the sense-perception is wanting in the plants, their bending might be compared with the purely reflectory movements of animals.

However, although it must be admitted that within the scope of physiology comparisons between plants and animals may be successfully drawn in many cases (as is probable especially with regard to stimulation, for the reason that in both groups of living organisms one and the same relation appears to exist between stimulus-intensity and stimulus-effect: the law of WEBER), this must be done always with the greatest caution. Bearing this in mind it nevertheless appears to me that the facts furnish us with sufficient reason to assume the striving on the part of the plant to receive the greatest possible stimulation by the quickest way as a supposition, just as we know this is the case with regard to positive chemotaxis. It must be left to later researches to reduce this striving after maximal stimulation to an actual cause of movement.

With the aid of a number of examples taken from the different groups of auxotonic movements, I now wish to demonstrate very shortly how simple the explanation of these phenomena becomes when we set forth from the assumptions mentioned above.

The different movements may be brought to certain groups according to the (supposed) position of the static apparatus and to the shifting which it should undergo.

A. STIMULATION BY GRAVITATION.

I. *Stationary position of the static apparatus.*

a. In the first place the static apparatus might lie against the lowest transverse wall of the statocyst. This should be so in the case of the vertically-growing main-root and main-axis, where the, maximal, stimulus should be the cause of the vertical growth of both by the equal lengthening of the cells allround.

If these same organs be placed in another position, e. g. horizontally, they will show positive (root) or negative (stem) geotropism. This we should now try to explain by the striving after a stronger stimulation. In the horizontal position the starch-grains press upon a part of the less sensitive border of the static apparatus; if they have to come into contact with the middle-field, the most sensitive part, the root will have to bend downwards, the stem, on the other hand, upwards. The explanation of these opposite movements requires therefore no new supposition; it follows from the circumstance that in the statocyst the middle-field in the case of the root lies against the transverse wall which is turned away from the growing-zone

(where the bending occurs) and in the case of the stem against the one that is turned towards it.

If in the centrifugal-experiment, the statolithes are moved outwards in the statocyst, then, for the same reason as given above, the stem must react by bending towards the centre, whereas the root will curve away from it.

b. Normal horizontally-growing plant-organs, such as rhizomes and some rootlets of epiphytes, can only maintain their position of equilibrium and continue growing in the same direction if the middle-field lies at the lowest longitudinal wall of the statocyst, for the same reason again that it is only in this position that the starch-grains will come in contact with this middle-field. Whether these organs also attain their quickest growth in this position has still to be investigated.

c. Besides lying against the transverse and longitudinal walls, the middle-field might also lie between the two, i. e. slanting; in such cases the organ should also exhibit a slanting position of equilibrium, the size of the angle it makes with the perpendicular depending upon the position of the apparatus with regard to the axis of the statocyst. This would explain the fixed position which the lateral branches and lateral roots of the first order always assume, and which is so different in different plants (cf. e.g. *Araucaria*, the common foliage trees, *Populus piramidalis*).

II. *Variable position of the static apparatus.*

Various organs of plants undergo a change in equilibrium during their normal development which could be ascribed now to a shifting of the apparatus at a certain moment, that is to say, if it can be demonstrated that gravitation-stimulus or longitudinal growth plays a part in the phenomenon.

The shifting may take place either at a certain moment or be continuous; moreover it may occur autonomously or as a result of some outside cause. According to this we may distinguish the following cases:

a. The position of the apparatus changes, autonomously, at a certain moment.

During the germination of the seed of a twining plant the young stem is at first vertical, but very soon the summit assumes a more or less horizontal position and at the same time the twining commences. It is possible that this transition from negative to transversal geotropism were preceded by an autonomous displacement of the

static apparatus, whereby the middle-field is shifted from the lowest transverse wall to one of the longitudinal walls; by the bending of the stem this longitudinal wall would then become the lowest of the statocyst. If the apparatus in the cell shifts over 90°, the new position of the stem-tip will become exactly horizontal; if, on the contrary, it moves less, the stem-tip will, as is often the case, assume a corresponding upward slope.

Similar changes in position, as seen in many flowers before and after flowering, may be explained in an equally simple way. The flowers of *Narcissus*, for example, when in bud stand perfectly upright, but when about to open are practically horizontal, which again would point to a preceding shifting of the apparatus from the lower transverse wall to one of the longitudinal walls. In *Agapanthus* the same movement occurs, but goes farther on, because after fertilization the ovary bends still further downward; in this case a further shifting in the same direction should have taken place, by which ultimately the middle-field arrived at the apical transverse wall.

In all these movements the bending is accompanied by a distinct growth of flower- and of fruit-stalk. Amputation of the flower-bud will prevent these movements, for which reason it is assumed that the statocysts are situated in the ovary.

Other plants again exhibit the phenomenon that the peduncle which stands upright during bloom, after fertilization increases much in length and curves downward; this is most striking with those plants which bury their young fruit in the ground, e. g. *Trifolium subterraneum*, *Arachis*, etc.; here the shifting of the apparatus from the lowest transverse wall to the highest should take place in one phase.

In all these cases the change of position of the static apparatus is clearly a result of a separate new stimulation which is either the growth of the flower or the process of fertilization.

A shifting of the apparatus in a contrary sense should take place in those cases in which the tip of the sympodial rhizome bends vertically upwards for the purpose of producing leaves and flowers, because this upward curve would have to be preceded by a displacement from the lowest longitudinal wall to the basal transverse wall.

The best known instance of a particular curvation is that of the flowerstalk of *Papaver* (to which those of the peduncles of the inflorescenses of *Tussilago Farfara* are closely connected), since there the movement has to take place before the flowering in one sense and after the fertilization in the opposite direction. VÖCHTING in 1882 succeeded in demonstrating that these movements are inti-

mately connected with the geotropic-stimulus both of the stalk and of the ovary, while the "rectipetality" should also play a part it.

VÖCHTING gave the name of "rectipetality" to the phenomenon that a plant-organ, which has curved upon irritation, begins straightening itself out again as soon as the stimulation has ceased. This he regards as a separate quality of plant-organs since it further appeared that the straightening required no new stimulation. It seems to me, however, as I set forth also in my article quoted above, that rectipetality must be regarded rather as a consequence of the original stimulation which, being gradually conducted to the opposite side of the organ, causes a contrary curving.

In *Papaver* the young flowerbud stands upright on a short and vertical peduncle; soon, however, the rapidly growing stalk makes a curve of 180°, so that the bud now hangs inverted. In this position the peduncle continues to grow which takes place at the bend, without however the curve increasing, owing to the simultaneous tendency towards rectipetality, and so it seems as if the growth is limited entirely to the part below the bend. When the flower is fully formed, the bud rises again and this upright position is also retained by the fruit.

Amputation of the ovary only (inside the bud before it is full-grown) checks the growth of the stem, which then stretches straight out as much as possible; the cessation in the growth should be regarded as a result of the cessation of the gravitation-stimulus in the ovary, the straightening of the stalk as caused by the "rectipetality" which is then the result of the stimulus received before the ovary was cut off.

The peduncle as well as the bud is negatively geotropic; the static apparatus should thus again be supposed to lie against the basal transverse wall and this position should remain unchanged in the peduncle. The reason that the growing stalk bends over at an angle of 180° should be attributed to a shifting of the static apparatus in the statocysts of the ovary from the lowest transverse wall to the uppermost, while the erecting of the full-grown bud later on should be preceded by the opposite movement in the same cells.

This example shows well how simply these seemingly complicated movements can be explained upon our assumptions.

A last group will comprise the epinastic and hyponastic movements which are so common in plagiotropic organs.

These movements depend upon temporary inequalities in the speed of growth between the upper and lower surface of the organ (especially leaves), whereby the growth predominates now on one

side and then on the other. The reason of these changes in the speed of the growth is unknown, but, while a renewed research into these movements is highly desirable, it may be taken as fairly certain that, although all apparently similar, they are not so in reality, since they are evidently not all governed by the same stimuli. The influence of gravitation, for instance, can be demonstrated in many of them, so that for this reason and also because the movements depend entirely upon longitudinal growth, there is every reason to assume that statocysts are also present in these organs. With respect to the place where they occur in leaves in general, not much is known, and it would therefore be useless to make further premises regarding the shifting of the static apparatus before sufficient data on this point have been obtained.

Some movements, however, might already be explained in a similar way as above; so, for instance, the movements of the leaves in the unfolding buds of *Aesculus*; in the bud, and also as soon as it opens, the petiole and leaflets stand vertically upright, after which the leaflets make a downward bend of 180° at the joint (shifting of the apparatus from the basal to the apical transverse wall); finally the leaflets, as well as the petioles, take up an almost horizontal position (shifting of the apparatus in both to the undermost longitudinal wall).

In connection with the above I may refer to the very important though apparently almost totally forgotten observations of HOFMEISTER [1]), from which it would seem that the lateral growth of the leaves in the bud is frequently influenced by their vertical position so that the half of the leaf pointing upwards in the bud will grow faster than that pointing downwards. If these observations be correct they would form a further indication that statocysts are also present in the leaves and would thus be able to exercise an influence upon the growth of the cells. This would agree with my view, expressed above, namely that the static apparatus also governs the normal growth in length. We shall return later to the consideration of the influence of gravitation upon the normal position of the leaves, as this also should be connected with the influence of the static apparatus (page 184).

b. The position of the static apparatus is altered by external influences.

Sometimes an external influence leads to a change in the position of organs, as, for instance, amputation of the main-axis.

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[1]) Allgemeine Morphologie der Gewächse. 1868, § 23.

If the terminal bud or a part of the main-axis be cut off, the lower lateral bud or lateral shoot will develop more strongly than it would otherwise have done, and will at the same time bend upwards until it assumes the position entirely, or almost, of the main-axis; amputation thus causes an accelerated growth as well as strong geotropic bending.

The absolute relation between the two, so striking here, is simply explained now by the circumstance that both are dependent upon stimulation of the static apparatus.

If, for instance, the almost horizontal lateral axis of *Araucaria*, after amputation of the terminal bud, gradually assumes a vertical position, this might have been preceded by shifting of the apparatus from the lowest longitudinal wall to the basal transverse wall, i. e. a shifting in the direction of the wound. TANGL [1]) and NESTLER [2]) now have demonstrated that the result of a wound is that in the neighbouring cells the protoplasm tends to accumulate in the direction of the wound; if it be that the static apparatus had a share in this shifting, this alone could be a reason for the appearance of the negative-geotropic movement.

It might be mentioned in this connection that, according to RICHTER [3]), even a plant of so much more simple structure as *Chara*, shows the same phenomenon, namely, that after amputation of the terminal bud, the adjoining lateral branch grows out more quickly and bends sharply upwards.

Amputation of the radicle has not the same effect upon the side-roots of the first order; SACHS [4]) has demonstrated that the lateral roots already present show no change in position, but that the after the amputation new formed lateral roots grow out in a more vertical direction, thus showing rather an influence upon the position of the apparatus in the newly formed cells instead of producing a shifting in those already present.

c. The static apparatus changes its position continually.

When the static apparatus is at rest in any organ, that organ assumes a certain position of equilibrium; in the case of a continual autonomic shifting, on the contrary, the organ will never arrive at a position of equilibrium and therefore never be at rest. Such ceaseless movements are known in the nutations and in the twining of plants.

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[1]) Sitzungsber. der K. Akad. der Wissensch., Wien, 1 Abt., 1884. Bd. 90, p. 25.

[2]) Ibidem, 1898, Bd. 107, p. 708.

[3]) Flora, 1894, p. 416.

[4]) Arbeiten des botan. Inst. zu Würzburg, Bd. 1, p. 622.

The nutations are now considered to be movements which take place without any stimulations, but their explanation is still wanting. As they depend, however, entirely upon longitudinal growth, they will be considered here to be induced by the stimulus of gravitation.

The least common case of nutation is seen in the peduncles of *Allium Porrum* which first hang over to one side, then straighten out and afterwards bend over to the other side, and so on. A slight displacement of the static apparatus might induce this movement; if, for instance, the middle-field lies against the basal transverse wall, the stalk, as we have seen, will assume a vertical position; should it then move slightly to one side, the stalk, in its effort to find the new equilibrium, would have to bend over to the same side; if the apparatus then moves back across the transverse wall and then shifts slightly to the opposite side the stalk would become straight and then also have to bend to that side, and so on. This autonomous shifting of the apparatus to and fro across the basal transverse wall would thus be sufficient to cause indirectly the "swinging nutation".

Much more frequent is the "rotating nutation", in which the tip of the stem moves as if over a conical surface; it may very well be imagined that this movement is brought about owing to the apparatus, as in the preceding instance, lying somewhat to the side of the transverse wall but is now pushed round in a circle, as it were, though in such a way that the middle-field remains always at the same distance from the centre of the transverse wall. The stem would then again have to follow the whole movement, always making the same angle with the perpendicular. The more the apparatus shifts, and keeps aside from the transverse wall during the nutation, the greater will be the angle at the apex of the cone described by the tip of the stem.

The twining movement was regarded by Sachs as being intimately connected with the rotating nutation, also because at that moment in both the influence of gravitation seemed to be excluded. Later on, however, it was demonstrated by Noll that in the twining the effect of this stimulus showed itself as "lateral geotropism" whereby the gravitation stimulus brings forth the lateral movement of the apex by causing a difference in growth between the two opposite lateral sides of the stem.

This lateral geotropism thus causes the apex of the stem to swing round, with the tip in a more or less horizontal position, while at the same time the tip twists round its own axis in the opposite

direction. It appears to me that these movements may also be explained by an autonomous shifting of the static apparatus. We have seen (page 6) that the tip of the young stem which at first is vertical soon afterwards assumes an almost horizontal position, after which it begins to twine; this was then explained by a shifting of the apparatus from the basal transverse wall to one of the longitudinal walls which then by the bending became the lowest. And if this apparatus were displaced now again in the statocyst, so that it goes round the cell, but always keeping at one of the longitudinal walls, this would cause the tip twisting aback and at the same time its rotating in the horizontal plane, since this twisting could not take place without a simultaneous and equally rapid rotation (one turn for each circle described in the horizontal plane). This displacement should take place in the one direction in plants which twine to the left and in the opposite direction in plants which twine to the right. If the summit of the stem is not perfectly horizontal in rotating as often occurs, the apparatus should have to lie still at the longitudinal wall but somewhat shifted towards the basal transverse wall and should be carried round in this same position in the cell.

It is worth noting in this connection that this displacement of the apparatus, and also the nature of the movement of the stem, agree largely with those described for the rotating nutation above-mentioned; for this reason, and because, in our opinion, *both* are to be regarded as dependent upon the gravitation-stimulus, the old supposition of SACHS is confirmed again, viz. that twining and rotating nutation are movements intimately connected with each other. The only difference would consist in the size of the apical angle of the cone described by the tip of the stem (which in twining plants may be as much as 180°) and thus, with regard to the static apparatus, in the distance, which exists continually during the shifting between the middle-field and the centre of the basal transverse wall.

This discussion, though necessarily too short, may however suffice to show that with the help of our theory it is possible to give even a simple explanation of the lateral geotropism.

A shifting of the apparatus back to the original position at the basal transverse wall would again lead to the negative geotropism wich causes the stem to raise itself when the twining ceases and by which the convolutions are pressed against the support.

The twisting of the stem which can frequently be observed as an accompanying phenomenon and which probably also depends upon the gravitation, cannot be discussed here.

B. Stimulation by Gravitation and Light.

Many plant-organs curve under the influence of an unequal illumination, as this causes an unequality in the longitudinal growth at different sides of the organ (heliotropism). Since this depends thus entirely on increase in length, these movements must be regarded here as being brought about both by gravitation and by light.

It is known that light can cause certain movements of protoplasm: the swarm-spores move towards light (positive phototaxis), whereby, according to the experiments of Engelmann, it is the uncoloured portion of the swarm-spore which receives the stimulus; if green cells are exposed to the light after having been kept in the dark, the chlorophyl-grains undergo a definite change of position, but resume their original place when withdrawn from the light.

These reasons would already be sufficient to assume that the position of the static apparatus also can undergo the influence of light, but such an assumption will become still more probable when it can be shown by different examples that a similar shifting of the apparatus, i. e. *towards* the light, could furnish us with a rather simple explanation of very different familiar phenomena.

a. Positive and negative heliotropism.

In the vertical position of main-axis and radicle, as was said above, the middle-field of the static apparatus should lie against the basal transverse wall of the statocyst; if these organs receive light from the side, and the apparatus, as we have just supposed, moves towards the source of light, these organs can no longer be in rest, and they can find the new equilibrium, i. e. the starch-grains will come to rest again on the most sensitive middle-field of the apparatus, only if the stem moves towards the light, and the root on the contrary from the light; thus the familiar positive and negative heliotropic curvatures.

If the plant is replaced in the dark the organs return to their vertical position, from which we should have to infer that after cessation of the light-stimulus, the apparatus of itself returns to their former place at the basal transverse wall. Consequently this is the same thing observed with the chlorophyl-grains in the above-mentioned cases, namely, that they are brought out of their position of equilibrium by light and return to it when replaced in the dark. Pfeffer [1]) considers this a matter of course.

b. It is known that certain rhizomes react to light in such a

[1]) Pflanzenphysiologie, 1904, Vol. 2, p. 780.

manner that when their tip receives the light they acquire positive-geotropism and bend downward; when the tip pierces the ground again and is thus no longer illumined, the transverse-geotropism reappears

These movements too may be explained in the simplest way from our suppositions. In the normal rhizome, as we have seen (page 6), the static apparatus should lie against the lowest longitudinal wall; if, under the influence of light, the apparatus is displaced again towards the source of the light, i. e. in the direction of the apical transverse wall, the tip will have to bend downwards in order that the starch-grains may again reach the middle-field, and this is just the movement that we see the rhizome make. When again in the dark the apparatus, and therefore the rhizome too, will resume its former position, as in the preceding case.

c. The sleep-movements of leaves, as will be known, are influenced by light to such a degree that it was long believed that light alone was the cause of them. Later, however, exhaustive researches, in particular those of PFEFFER, showed that gravitation has also a share in them. This has been most clearly demonstrated for instance in the experiment with *Phaseolus,* in which the petioles of the two first leaves were secured during the day in their normal position, so that only the leaflets could make the sleep-movement. When the plant was then turned upside down, the nyctitropic movement took place at night, but showed exactly the reverse of what in the normal position occurred, i. e. in the light the leaves now stood vertically upright, whereas in the dark they were spread out horizontally. Thus, with respect to gravitation the leaves moved in the same direction as before, with regard to light however in a manner exactly contrary to the normal way, from which it is evident that it was the gravitation in the first place which governed the nyctitropic movement and determined the equilibrium of the leaf.

As practically nothing is known regarding the position of the statocysts in the leaves (see page 179), it is still difficult to express here any opinion with respect to the eventual shifting which the static apparatus might undergo here under the influence of light, the more so because there are so many varieties of nyctitropic movement. Important in this respect for an explanation in the sense as meant here, however, is the fact that it proved the presence of the principal auxiliary, namely the static apparatus itself, in leaves which show sleep-movement.

d. What has been written concerning the sleep-movements is really also applicable to the movements which cause the leaves to

assume their natural position; apparently they are influenced only by light, but here again the experiments of PFEFFER have shown that gravitation plays an important part; f. i. many leaves when brought away from their normal position can return to it in the dark, which evidently can be effected only through the medium of the gravitation-stimulus.

Here again the lack of data regarding the position of the statocysts in the leaves prevents us from prosecuting the research as to these movements in connection with our theory.

C. STIMULATION BY LIGHT ALONE.

Auxotonic curves are seldom caused by the light-stimulus alone; the instance of this most fully investigated is that of the "transversal heliotropism", whereby certain leaves place themselves perpendicular to the incident bundle of light. HABERLANDT [1]) endeavours to explain this movement by assuming that the middle-field of that portion of the outer layer that adjoins the lower wall of the sensitive epidermical cells is more sensitive to light than its surroundings. If now the leaf seeks to reach the desired position by the shortest way, this must be accompanied by the quickest increase in the intensity of the stimulus, exactly in the same way thus as was assumed above with respect to the stimulus of gravitation.

The stimulation of an sensitive organ causes everywhere a certain sensation or movement, whereby, however, the *nature* of the sensation or of the movement, is determined *solely* by the special properties of those parts of the organism which lie *outside* the perceiving sense-organ; consequently the nature of the stimulus can never excercise any influence whatever upon the effect that the organism shows.

If this conclusion should hold good for the plant too, as is very probable from the nature of the case, and if we also bear in mind that all auxotonic movements mentioned are executed in the same way, it would follow that it is sufficient for the plant to possess only one single sensitive organ for all these movements, induced by gravitation, by light or by both.

Therefore not even for transversal-heliotropism an exception should be made, for if we consider that a static apparatus without statoliths (starch-grains) could not be stimulated by gravitation but can nevertheless remain sensitive to light, it might very well be possible

[1]) Die Lichtsinnesorgane der Laubblätter, 1905, p. 127.

that the mentioned apparatus of HABERLANDT, sensitive to light and in which the starch-grains are always lacking, might be identical with our static apparatus (provided that in this case it should be unmovable), for both exercise exactly the same influence upon the growth-phenomena in the joints, etc.

These expositions might serve to show that the hypothesis of the presence of a movable "static apparatus" in the statocyst affords such a great advantage in the consideration and the grouping of the mentioned auxotonic movements, that it is entitled to be duly regarded as a working-hypothesis of sufficient foundation and further that there is probably in plants (and in animals?) a general striving towards an increase of stimulation which might later serve to find a further explanation of how these appropiate movements be brought about.

Leyden, January 23, 1923.

Mathematics. — *"On the Points of Continuity of Functions"*. By Prof. J. WOLFF. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of February 24, 1923).

Let $f(P)$ be a function of the coordinates of a point P in a space with an arbitrary number of dimensions. The points where f is continuous, form an *inner limiting set*, i.e. the intersection of an enumerable set of open sets of points Ω_n, where we may assume that Ω_{n+1} is a part of Ω_n for any n. For the points, where the function oscillates less than $\frac{1}{n}$, form an open set Ω_n because the oscillation is an upper semi-continuous function. The set of the points of continuity is the intersection of all Ω_n, $n = 1, 2, 3, \ldots$. YOUNG [1]) has shown that to any inner limiting set E given in a linear interval, there belongs a function in that interval which is continuous in the points of E and discontinuous in any other point. We shall give here a simple proof, which is directly valid for spaces of any number of dimensions.

1. Let a set of points E be given as the intersection of an enumerable set of open sets Ω_n, where Ω_{n+1} is a part of (or coincides with) Ω_n.

We define $f(P)$ for any point of space in the following way: in the first place $f(P) = 0$ if P lies in E. Now let P be a point not lying in E, n_P the least value of n for which Ω_n does not contain the point P.

We put

$$f(P) = \frac{\psi(P)}{n_P} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\psi(P)$ is the function which in the points of space of which all the coordinates are rational, is equal to 1, in any other point of space equal to -1.

We may say that (1) holds also good for the points of E, if there we assume $n_P = \infty$.

2. Now we shall show, that $f(P)$ is continuous in the points of E and discontinuous outside them.

[1]) W. H. YOUNG. Wiener Sitzungsber., vol 112, Abt. II[a], p. 1307.

Let us first assume that P belongs to E. In this case $f(P) = 0$. If ε be an arbitrary positive number, we may choose the natural number ν in such a way that

$$\frac{1}{\nu} < \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

As P lies in Ω_ν and Ω_ν is open, there exists a region U round P which lies also in Ω_ν. For any point Q of U we have therefore $n_Q > \nu$, so that according to (1) and (2)

$$| f(Q) | < \varepsilon,$$

Hence f is continuous in any point of E.

Let us now assume P to lie in the complement of E. If P is not an limiting-point of Ω_{n_P}, it has a neighbourhood U which has no point in common with Ω_{n_P} and which lies in $\Omega_{n_P - 1}$. For any point Q of U we have in this case $n_Q = n_P$. Hence

$$| f(Q) | = | f(P) |$$

As the points where f is positive as well as the points where f is negative, lie everywhere dense on U, the oscillation of f in P is equal to $2 | f(P) |$.

If however P is an limiting-point of Ω_{n_P}, every neighbourhood U of P contains a part of Ω_{n_P}. For any point of that part $n_P > n_Q$, hence

$$| f(Q) - f(P) | \geq \frac{1}{n_P} - \frac{1}{n_P + 1} \quad . \quad . \quad . \quad (3)$$

As the points Q for which the inequality (3) holds good, have P for a limiting-point, P is a point of discontinuity of f. Herewith the theorem has been entirely proved.

Mathematics. — *"Inner Limiting Sets"*. By Prof. J. WOLFF. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of February 24, 1923).

HOBSON has been the first to prove the following theorem: [1])

An enumerable set of points which has no part that is dense in itself, is an inner limiting set, i.e. the common part of an enumerable set of open sets each of which we may assume to contain the following one.

BROUWER has given an extremely short proof, but just as HOBSON he makes use of the transfinite ordinal numbers [2]).

In the proof which follows here, no use is made of these numbers.

1. If $E_1, E_2, \ldots$ are inner limiting sets, if further each E_k is a part of an open set Ω_k, while no two Ω_k have any points in common, also the sum $E_1 + E_2 + \ldots$ is an inner limiting set..

For we may write:

$$E_k = \Omega_{k1}\,\Omega_{k2}\ldots\ , \quad k = 1, 2, \ldots .$$

which means that E_k is the set of points lying in Ω_{ki} for every i. The Ω_{ki} are open sets of which we may assume that they all lie in Ω_k. The set

$$(\Omega_{11} + \Omega_{21} + \ldots)(\Omega_{12} + \Omega_{22} + \ldots)\ldots .$$

contains $E_1 + E_2 + \ldots .$, but no point outside them, as $\Omega_{ki}\,\Omega_{lj} = 0$ for $k \neq l$. Now the auxiliary theorem has been proved.

2. We call a set of points E an inner limiting set *in a point* P if there exists an open set containing this point, so that the part of E lying in this set is an inner limiting set. This holds also good for the part of E lying in an arbitrary open set which is a part of the above mentioned one.

3. If an enumerable set E is an inner limiting set in each of its points, E is an inner limiting set.

We call the points of E: $P_1, P_2, \ldots$

[1]) Proc. London M.S. (2) 2, p. 316—323.

[2]) These Proceedings, Vol. XVIII p. 48 (1915).

Round P_k as centre we take an interval I_k (a quadrangle, a cube, etc. according to the number of dimensions of the space in which E is given), so that EI_k is an inner limiting set, taking care that the boundary of I_k contains no point of E, which is possible on account of E being enumerable.

By I_k we understand the open interval, by $\bar{I}_k$ we shall indicate the closed one, by an accent, the complement of a set. Now

$$E = EI_1 + EI_2(\bar{I}_1)' + EI_3(\bar{I}_1)'(\bar{I}_2)' + \ldots$$

From N°. 1 there follows now immediately that E is an inner limiting set.

4. Let E be enumerable and not an inner limiting set. In this case according to N°. 3 the set D of the points E in which E is not an inner limiting set, is not empty. Let P be a point of D and I an interval with P as centre. EI is according to N°. 2 not an inner limiting set, hence neither is $EI-P$; according to N°. 3, $EI-P$ contains a point Q in which $EI-P$ is not an inner limiting set, hence E is not an inner limiting set in Q, so that Q lies in D. From this there follows that D is dense in itself and from that the theorem which was to be proved.

Petrography. — *"On the Rocks of Doormantop in Central New Guinea"*. By W. F. GISOLF. (Communicated by Prof. G. A. F. MOLENGRAAFF).

(Communicated at the meeting of February 24, 1923).

During a causerie about New Guinea, delivered at Batavia, Dr. H. J. LAM of Buitenzorg, at a meeting of the "Koninklijke Natuurkundige Vereeniging", showed a sample of a rock from Doormantop, which directly engrossed my attention to such an extent that I asked him to leave it to me for examination. He readily did so. Afterwards he furnished me with more samples of the same material, for which kindness I hereby tender him my best thanks. The geologist of the Mamberamo-expedition Dr. P. F. HUBRECHT, was staying in East-Java at that time, and was not in a position, within the first ten months, to send me any material. However, when asked, he did not object to an examination of the samples nor to publication of the results. I have much pleasure in thanking him also for his kindness.

The first samples that came to hand, present a schistose structure, chiefly due to parallel bands of magnetite; they are of a dark green colour, with a thin light-brown non-detached weathered crust of a cavernous appearance, on either side a relatively considerable quantity of magnetite reveals itself in non-crystallized masses; the erosion has spared the magnetite, so that it projects $^1/_2$ — 1 centim from the rock. A blow of a hammer made the rock split along the magnetite, thus effecting the first separation between the rock and the ore.

Some slides were made of the part from which the magnetite had been removed as much as possible. Under the microscope the rock proved to consist of magnetite with fresh olivine and a colourless, lath-shaped mineral of moderate refringence and very weak birefringence. The structure is slightly varying, the olivine now encloses the colourless mineral, now it mingles with it as if they were crystallized out simultaneously; the magnetite encloses the colourless mineral and occurs xenomorphic in the aggregate olivine-unknown material. The magnetite is polarimagnetic. A little apatite presumably occurs.

The olivine, which extinguishes undulatorily, but not to such a degree as is the case in most peridotites, looks very fresh and is absolutely free from weathering. The apparent weathering in the crust appears to be merely a brown colouring; serpentinization as an effect of atmospheric influence is absent. In another slide the refractive indices were, by the immersion-method, fixed at 1.66 and 1.70, after the Canada balsam had been carefully extracted by the use of xylol. The thickness of the slide was $^1/_{35}$ mM. (measured by detaching the slide and fixing it with tallow vertically on the object glass): the highest interference-colour observed was green 3rd order, making γ-α about 0.04, which agrees with the determination of the indices. The observation perpendicular to an optical axis in convergent light revealed on rotation of the table a slightly curving beam, at which the optically positive sign and a large axial angle could be established. Presumably one has to do here with a ferromagnesium olivine with about 10 % to $12^1/_2$ % iron-olivine and 90 % to $87^1/_2$ % magnesium-olivine (See Doelter Handbuch der Mineralchemie II, I p. 16).

The colourless mineral, however, caused most trouble in its determination. Long as well as short laths occur; quadratic sections are lacking; the birefringence is low, sometimes next to zero; in one and the same lath the double refraction is not always the same, but varies, without attaining however, the so called „Pflock-structure". All the laths show straight extinction; the elongation is invariably positive; cleavage lines run lengthwise through the crystal, especially in the middle and parallel to the outline. It was very difficult to obtain an interference figure. Therefore it was surmised that the mineral might belong to the melilite group, but this surmise proved to be untenable, as it was in no way supported by further microchemical and optical testing.

For this reason I applied to Dr. Lam for more material. This additional supply enabled me to identify the mineral. The coverglass was taken from all the slides, which were rinsed repeatedly with xylol, in order to remove any trace of Canada-balsam from the margins of the slide before being examined by the immersion-method. The refractive index appeared to be 1.58.

Being treated with hydrochloric acid and washed cautiously, gelatination ensued; when moistened with fuchsin and again washed carefully, the olivine as well as the unknown mineral under consideration appeared to be gelatinized. To make sure that the silex-gel of the olivine had not spread over the unknown mineral as well and might thereby be misleading, the whole procedure was repeated

and brought to light that the mineral under consideration gelatinized sooner than the olivine. In the liquid that had been collected microchemically the presence of calcium could not be detected.

Finally each individual lath was examined conoscopically; thus I succeeded in establishing in several of them that the mineral is biaxial, and that the axial plane is always perpendicular to the longer axis, the elongation being always positive. This is possible only if the mineral is developed into flakes perpendicular to the optical A-axis; it thus became more and more probable that the mineral could be rhombic. If so we must have to do with antigorite in its primary form.

The idea of a secondary postmagmatic genesis should be dropped altogether, the antigorite laths traverse freely from one olivine-crystal into another; subsequently they form with them as it were a eutectic crystallization and ultimately become the predominant mineral (See the microphoto fig. 1 and 2). All this applies to the material rich in large magnetite masses. Other material, finer grained

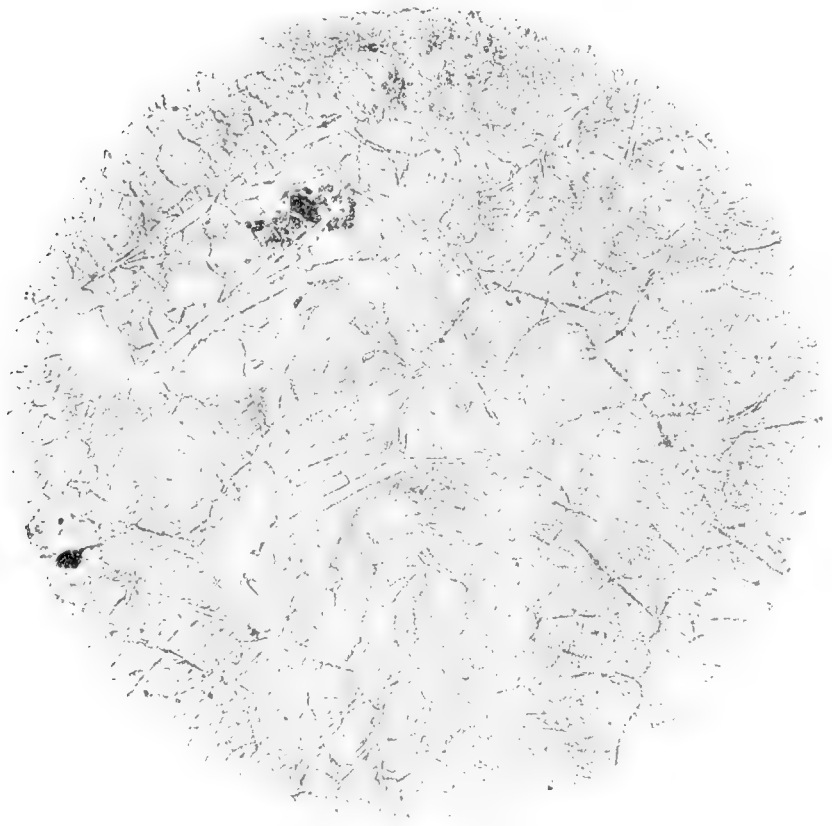

Fig. 1. Primary antigorite in olivine.

and poorer in large magnetite masses appeared to my surprise to be made up of the ordinary antigorite-serpentine with so-called olivine-

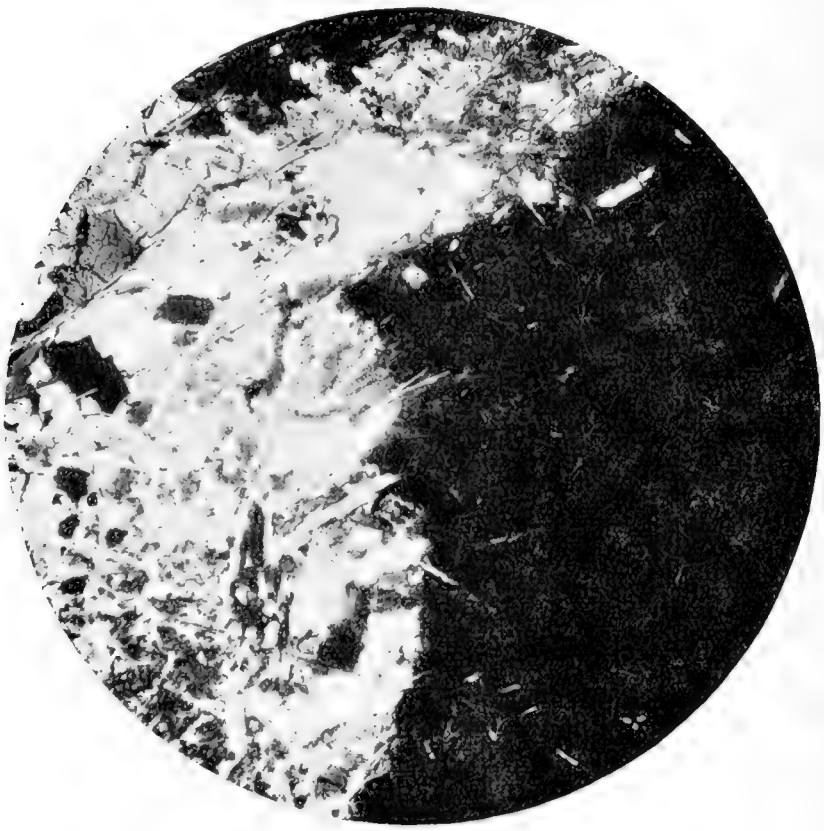

Fig. 2. Same as fig. 1, in polarized light with crossed Nicols.

rests. It appears to me that the cross-grained structure in those serpentines and the fine grain are caused by rapid crystallization. Also in these specimens the olivine is quite fresh.

The sequence of cristallization as manifested in the slides is the following:

Antigorite; antigorite-magnetite; antigorite-olivine-magnetite.

To all appearance the latter combination is a eutecticum, although it is not impossible that the magnetite is resorbed later.

Since the rock lies near the surface, the conditions for serpentinization by meteoric agencies have been favourable. However, of this the rock does not present any recognizable trace. The question, therefore, urges itself upon us whether the serpentine might perhaps be always of a magmatic origin, at all events not a product of weathering.

Now, as to the genesis of this rock we may broach the supposi-

tion (until the system H_2O—MgO—FeO—Fe_2O_3—SiO_2 shall be investigated) that the magma, from which this rock originated, crystallized under such a pressure that the gaseous components (notably watervapour) could not escape and consequently were taken up into the rock substance from the very beginning of the crystallization, thus occasioning a primary origin of serpentine. Putting it chemically [1]: the crystallization begins in the serpentine-field and terminates in a point serpentine-olivine (magnetite ?), which is perhaps located close to the connecting line olivine-serpentine (because Fe_3O_4 takes up only litlle space in an olivine-serpentine structure). In the case of eutecticum this point will be found on the same side as Fe_3O_4, and in that of resorption in the common field of serpentine-olivine.

It may be suspected that in other peridotites, in which olivine crystallized first, the said pressure was less, so that, indeed, the gases could escape at the beginning of the crystallization, but were taken up again afterwards at the final crystallization, so that in similar cases serpentinization of olivine might be considered as an apomagmatic (hydrothermal) process. Expressing it chemically: the crystallization then begins in the olivine field; on increasing pressure the stability field of the olivine is subsequently left for that of the serpentine. The consequent segregation of magnetite is self-evident after what has been said before. Magnesium is also set free for the forming of periclase or picotite or magnesite. As the gases move upwards it is obvious that serpentinization will occur chiefly in the upper zones of peridotite-masses and on rents in the solidifying and consequently shrinking peridotite-masses.

Erosion being a downward process, first the marginal portions are laid bare, so that in the field the serpentine will in many cases be found prior to the olivine, which fact, I think has lent support to the erroneous but current view that serpentine is a weathering product.

After the foregoing had been written (August 1922), the chemical analysis came to hand (Dec. 29).

Of a sample freed as much as possible from magnetite an analysis was made at the Head Office of the Mining Department by Mr. A. TER BRAAKE and Mr. G. J. WALLY. The loss of water has been

[1]) To simplify matters it has been assumed that the serpentine and olivine are very definite compounds, which is not the case, of course. For the thermal-pressure-diagram of the five-substance-system a six or seven-dimensional space would have to be used, which would not facilitate the conception.

determined at 100° and at 200°, the latter temperature was maintained for three days, viz. until the weight remained constant.

The result of the analysis is:

SiO_2	40.46
MgO	40.20
FeO	7.69 (determined as Fe_2O_3)
Al_2O_3	4.12
H_2O (100°)	6.14
H_2O (200°)	1.60
	100.21

CaO, MnO, Cr_2O_3, NiO are absent, as well as P_2O_5; no estimation could be made of K_2O and Na_2O, because in Java platinum chlorid at that time could not be obtained. In the determination of the iron-amount FeO and Fe_2O_3 were not estimated separately; it is likely, however, that they are both present.

It is evident that the chemical analysis fully confirms the microscopical examination.

Judging from the analysis also pyroxene is probably present, either separately as in so many peridotites, or in solid solution with, or as a component of the serpentine.

Presumably the latter is the case, since pyroxene has not been found in any of the slides.

It must be remembered that DAUBRÉE[1]) already succeeded in demonstrating that at a high temperature serpentine passes after melting into olivine + enstatite, while water escapes:

$$H_4Mg_3Si_2O_9 = Mg_2SiO_4 + MgSiO_3 + 2H_2O.$$

When leaving aside the watervapour, this case is merely a subdivision of the system $MgO—SiO_2$, which has been examined by ANDERSEN and BOWEN. DAUBRÉE's experience[2]) is in complete harmony with their results; so for instance from a mixture of the system $Mg_2SiO_4—MgSiO_3$ on cooling first Mg_2SiO_4 crystallizes, which at 1557° begins to react with the solution, in consequence of which $MgSiO_3$ is formed which is precipitated on the surface of the olivine; at the same time the solution becomes richer in silica, so that ultimately SiO_2 can be set free; as DAUBRÉE added magnesia he did not obtain cristobalite. In the light of later experiments his

[1]) DOELTER failed in this experiment. Still, it is worth while to peruse DAUBRÉE's carefully described experiments.

[2]) Comptes Rendus 1866, I, p. 660.

observations are correct; e. g. „Des aiguilles d'enstatite y sont fréquemment disséminées ou en recouvrent la surface" (i. e. of the olivine, obtained through smelting of serpentine with the addition of magnesia) and again in case he did not add magnesia: „le (péridot) se montre en moindre proportion que dans les fusions faites en présence de la magnésie.

Now since most peridotites (with the exception of dunites) consist of olivine and pyroxene, consequently of orthosilicate and metasilicate, we may venture to bring the primary and the secondary serpentinization into one focus. For a general theoretical treatment of the case the knowledge of the thermal pressure-diagram of the system H_2O—MgO—SiO_2 would be a first step. [1]) Needless to say, that this diagram will become very complicate owing to the great difference in volability of the components.

From the foregoing it is evident, however, that under a pressure of one atmosphere serpentine is unstable; it would be worth while to repeat the experiment of DAUBRÉE in watervapours of various tensions in order to establish the limit of stability of serpentine. Now if we are right in supposing that olivine and pyroxene are not stable at high pressure and in the presence of watervapour, but that they are transformed into serpentine, the former with liberation of MgO [2]) the latter with precipitation of silica, serpentinization may be accounted for as follows:

1. If the pressure is high enough serpentine crystallizes first from a magma, which is composed of x Mg_2SiO_4, y $MgSiO_3$, z H_2O; at a lower pressure the crystallization begins with olivine.

2. When olivine and (or) pyroxene are segregrated, the volatile components congregate in the upper zone of the batholite, which may give rise to a high tension, in case they have no opportunity to escape; thus the field of stability of the olivine and (or) the pyroxene is abandoned, and that of serpentine is attained, after which serpentinization of olivine and pyroxene commences, occasionally with a residue of MgO (Fe_3O_4) or (and) silica; while in most cases MgO is present as magnesite.

Already DAUBRÉE acknowledged: „Rien ne prouve d'ailleurs que l'hydratation qui s'est produite dans la transformation des roches de péridot en serpentine ait été opérée par les agents de la surface du globe".

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[1]) See e. g. H. E. BOEKE, Grundlagen d. phys. chem. Petrographie, p. 179.

[2]) E. g. as magnesite, because the component carbondioxyde is always present. Many serpentine deposits in fact contain magnesite and quartz.

It scarcely needs to be pointed out that, under the influence of the volatile components of a later intruded igneous rock, a peridotite mass may also be altered into serpentine.

Let it be recalled here that a résumé of the olivine-serpentine problem has been brought forward by W. N. Benson (Origin of Serpentine, American Journal of Science 46 p. 693, 1918). It is to be regretted, however, that the problem has not been dealt with from a physico-chemical point of view.

Finally I beg to use this opportunity to thank Mr. A. C. de Jongh, Director of the Research Committee of the Mining Department, for his willingness to have the analysis and the slides made in his laboratory.

From the above it may be inferred that many difficulties have stood in my way by the insufficiency of my laboratory-equipment. It is to be hoped that the Government of the Netherlands East Indies, which are so extremely rich in occurrences of beautiful rocks, may, at no distant date, take measures for the building of a well-equipped petrographic laboratory.

Weltevreden, Aug./Dec. 1922.

Palaeontology. — *"New Findings of Pliocene and Pleistocene Mammals in Noord Brabant, and their Geological Significance"*. By I. SWEMLE and Prof. L. RUTTEN. (Communicated by Prof. G. A. F. MOLENGRAAFF).

(Communicated at the meeting of February 24, 1923).

In the past year the Geological Institute of Utrecht obtained, partly through mediation of the Geological Survey, partly from the Government Bureau for Watersupply, some remains of fossil mammals originating from the southern and the western part of Noord-Brabant, one of the southern provinces of Holland, a district which up to the present has yielded very little in this respect. As we know, representatives of the young-diluvial fauna have been found in some localities of Noord Brabant, e. g. Bos primigenius Boj. near Den Bosch, Elephas primigenius Blumenb. near Acht, Rhinoceros antiquitatis Blum., in Hollandsch Diep. It is noteworthy, however, that in two places, near Oosterhout in the northwest, and near Westerhoven in the south of the province, remains have been recognized of a pliocene fauna, viz. Elephas meridionalis Nesti, and Rhinoceros etruscus Falc [1]).

Now, part of the remains, detected last year, have been derived from the zone between Oosterhout and Westerhoven. Three findings of mammals, belonging to the young diluvial fauna occurred in the vicinity of Esbeek S. S. E. of Tilburg, viz. a molar from Elephas primigenius Blum., found by Mr. SISSINGH on the premises of the clay-pit to the north of Esbeek, under a deposit of loam at a depth of three meters; three molars from Rhinoceros antiquitatis Blum, unearthed from a depth of $2\frac{1}{2}$ m. in peat-bearing layers of clay, during the construction of the lock in the Wilhelmina Canal near Diessen, when the canal was being dug, and a molar from Equus Caballus L., found during the construction of the same canal to the east of the Diessen-lock at a depth of $3\frac{1}{2}$ M. [2]).

The above fossils are not highly remarkable in themselves. The Molar from Elephas primigenius is a M III, sup. sin., on which

[1]) L. RUTTEN. Die Diluvialen Säugetiere der Niederlande, 1909.

[2]) Far more eastward, viz. near Breugel on the Dommel, a fragment of a horn of Bos Primigenius Boj. was found, with which the Utrecht Geol. Inst. was presented last year.

— $19\frac{1}{2}$ x are still visible on 215 × 100 × 160 mM.[1]). The extremely thin lamellae and the slight thickness of the enamel prove conclusively thath the tooth is to be referred to El. primigenius; it is remarkable however, that the enamel bands are finely folded which occurs only rarely in El. primigenius. The remains of Rhinoceros antiquitatis are three successive teeth, of one set of the right lower-jaw, viz. P 3, M 1, and M 2. They are but little worn down and have therefore belonged to a young animal; they must undoubtedly be referred to Rh. antiquitatis; the very thick enamel, the distinct striae of the enamel bands, the deep depressions and the trifling convexity of the teeth, all point in the same direction, while for the rest the teeth are almost quite similar to a set pictured by J. BRANDT[2]). The tooth from Equis caballus is also a M of the lower-jaw.

From Oosterhout, however, where already previously teeth and bones from Elephas meridionalis Nesti[3]) had been found in a superficial layer of loam, in a locality not precisely indicated, remains of bones and fragments of teeth were also sent to us, that belonged to this species. They were met with at a depth of 34,75 M. below Amsterdam-level in the first of five borings executed for the Water-company of Western Noord-Brabant. The wells are situated to the left of the road from Breda to Oosterhout on the Vraggel moor.

The bones from the well cannot be further determined, but a fragment of a tooth, most likely the posterior part of a M. 1 sup. sin. is distinctly indicative of Elephas meridionalis. It presents-3 x with a length of $7\frac{1}{2}$ and a breadth of 8-9 centims, while the height minus the root is about 8 cM. The fragment was not chewed down, but was sawn, in order to get an opportunity of studying its structure.

Indicative of El. meridionalis are: 1° the extraordinary thickness of the lamellae, which appears already from the lamallae-formula; 2° the extreme thickness of the enamel (up to 4 mM); 3° the large breadth and the small height of the tooth; 4° the way in which the chewing-figures originate, namely through fusion of the four annuli (see figure).

Not only do we recognize in this fragment all the characteristics of El. meridionalis, but those characteristics even become prominent in the extraordinary thickness of the lamellae and the enamel.

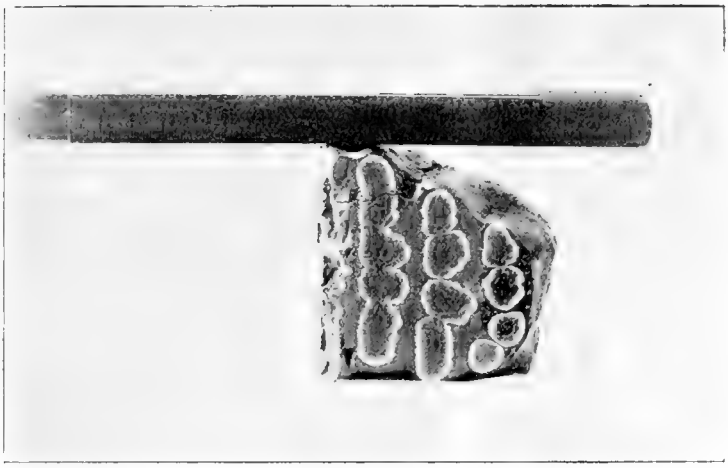
Dr. J. STEENHUIS kindly wrote us that the Geological Survey

[1]) H. POHLIG. Nova Acta Acad. Car. Leop. 53, p. 251.

[2]) J. BRANDT. Mém. Acad. St. Pétersbourg. 1849. T. XI.

[3]) L. RUTTEN. Die Diluvialen Säugetiere der Niederlande. 1909.

parallelized the part of the bore in which the tooth-fragment had been found, with the clay of Tegelen, which may be referred to the youngest pliocene or the oldest pleistocene. The tooth-fragment



corroborates this parallelism, for of late Elephas meridionalis has also been found near Tegelen [1]).

From the fact that the previous discoveries of El. meridionalis were made near Oosterhout in a loam-quarry, near the surface, it may be concluded that in this part of Noord Brabant the pliocene rises locally to the surface. In the Annexes 11 and 13 of the "Final Report of the Government Exploration of Minerals", a fault running N. 40° W is marked West of Tilburg, which, however, in Annex 11 is drawn 2 KM. farther to the east than in Annex 13. To the north-east of this fault the soil has considerably subsided, as indicated on the sketch map; to the south-east the subsidence is less marked. When mapping the finding-places of the pliocene or the old pleistocene fauna (Westerhoven and Oosterhout), it will be noted that they fall to the east of the fault, as indicated in Annex 13, while Westerhoven would also lie within the trough, when assumung the course of the fault as marked in Annex 11. It is clear, however, that the pliocene can be expected near the surface only in the least subsided region, so that it is certain that the above-named fault — marked on the map only as a "suspected" fault — must be shifted more eastward. In that case, however, the locality of El. primigenius near Esbeek falls certainly to the west

[1]) S. Richarz. Centralbl. f. Miner. Geol. u. Pal. 1921 p. 664—669; id. Stadt Gottes 1921/22. Heft III.

of the fault, and that of Rh. antiquitatis and Eg. callabus does so most probably, i. e. in the least subsided region. Two possibilities are then to be considered: in the first place near Diessen and Esbeek more recent diluvium may have overlapped the denuded pliocene and secondly the fault postulated in the above as a straight line, may proceed more irregularly, so that in reality Esbeek and Diessen come to lie east of it. — At all events it appears from the foregoing that the young fossil mammalian remains in this part of Noord Brabant, whose geology may give us still many surprises, are rather numerous and may be of use in unravelling the tectonic of this province.

The previous discoveries near Westerhoven and Oosterhout as well as the recent ones near Esbeek and Diessen were made in superficial or nearly superficial loam deposits, which but for fossil findings, would surely be referred to the "Argiles de la Campine [1]). It has already previously been pointed out that these loam-deposits may be of different ages; the palaeontological findings lend support to this hypothesis.

[1]) J. Lorié. Bull. Soc. Belge de Géol. XXI. 1907 p. 532—576.

Mathematics. — *"A Generalisation of* MERTENS' *Theorem"*. By M. J. BELINFANTE. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of February 24, 1923).

The theory of infinite series, which so far chiefly consisted of convergent series, being extended to the so-called summable and asymptotic series, it is natural to generalize as much as possible the classical results about convergent series to these classes of series.

For the well-known theorem of MERTENS this has been done by HARDY (BROMWICH, Theory of Infinite Series, p. 284), who used BOREL's method of summation. In the present paper we treat a somewhat different generalisation, whereby we are only concerned with CESARÓ's method of summation.

The product or the product-series of the series

$$a_1 + a_2 + \ldots . \quad \text{and} \quad b_1 + b_2 + \ldots .$$

is defined as the series $c_1 + c_2 + \ldots .$

where $c_i = a_1 b_i + a_2 b_{i-1} + \ldots . + a_i b_1$.

CESARÓ has proved: if two series are convergent, their product will be summable of order 1, and if two series are summable respectively of order p and q, their product will be summable of order $(p + q + 1)$ [1]).

If we call a convergent series summable of order zero, then the first part of CESARÓ's theorem is included in the second.

MERTENS' theorem, which runs as follows: "If one of two convergent series converges absolutely, their product is convergent" may now be stated thus:

The product of a series which is absolutely convergent, by a series which is summable of order p is summable of order zero.

In the first place we are led to the following generalisation:

Theorem 1: *The product of a series which is absolutely convergent, by a series which is summable of order p, is summable of order p.*

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[1]) BROMWICH. Theory of infinite series, § 125 pp. 314—316.

Further we may ask for a condition, such that the product of a convergent series by a series which is summable of order p and satisfies the condition, will also be summable of order p. This can be seen from:

Theorem 2: *The product of a convergent series by a series which is summable of order p and whose mean-values of order $p-1$ are limited, is summable of order p.*

Finally we consider the product of two series, which are summable respectively of order p and q; then we are led to:

Theorem 3: *The product of a series which is summable of order p and whose mean-values of order $p-1$ are limited, by a series which is summable of order q, is summable of order $p+q$.*

If we call a series which is summable of order p ($p \geq 1$) and whose mean-values of order $p-1$ are limited, *joinable of order p*, and if we call an absolutely converging series *joinable of order zero*, then the above theorems are included in the following:

Theorem: *The product of a series which is joinable of order p, by a series which is summable of order q, is summable of order $p+q$.*

The proofs however will be given separately. For the sake of completeness we begin with the deduction of some well-known formulas.

Let $x_1^{(1)}, x_2^{(1)}, \ldots . x_n^{(1)}, \ldots .$ be an arbitrary sequence of complex numbers; we define:

$$x_n^{(2)} = x_1^{(1)} + x_2^{(1)} + \ldots . + x_n^{(1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$x_n^{(k+1)} = x_1^{(k)} + x_2^{(k)} + \ldots . + x_n^{(k)} \quad . \quad . \quad . \quad . \quad (2)$$

We denote $x_i^{(k)}$ by $A_i^{(k)}$ if $x_i^{(1)} = 1$, whatever be i. It is easy to verify that:

$$A_n^{(k)} = \frac{(n+k-2)!}{(n-1)!\,(k-1)!} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We consider the series $a_1 + a_2 + \ldots .$ and $b_1 + b_2 + \ldots .$ with their product: $c_1 + c_2 + \ldots .$ (where $c_n = a_1 b_n + \ldots . + a_n b_1$), and we put:

$$\left.\begin{aligned} S_n^{(1)} &= s_n = a_1 + a_2 + \ldots + a_n \\ T_n^{(1)} &= t_n = b_1 + b_2 + \ldots + b_n \\ W_n^{(1)} &= w_n = c_1 + c_2 + \ldots + c_n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3a)$$

The quantities $S_n^{(k)}$, $T_n^{(k)}$ and $W_n^{(k)}$ are now also defined. The following identity will be satisfied for $p-i \geq 1$:

$$S_1^{(p)} T_n^{(q)} + S_2^{(p)} T_{n-1}^{(q)} + \ldots + S_n^{(p)} T_1^{(q)} = S_1^{(p-i)} T_n^{(q+i)} + \ldots + S_n^{(p-i)} T_1^{(q+i)} \quad (4)$$

This follows by induction:

$$S_1^{(p)} T_n^{(q)} + S_2^{(p)} T_{n-1}^{(q)} + \ldots + S_n^{(p)} T_1^{(q)} =$$

$$= S_1^{(p-1)} T_n^{(q)} + (S_1^{(p-1)} + S_2^{(p-1)}) T_{n-1}^{(q)} + \ldots + (S_1^{(p-1)} + \ldots + S_n^{(p-1)}) T_1^{(q)} =$$

$$= S_1^{(p-1)} (T_n^{(q)} + \ldots + T_1^{(q)}) + S_2^{(p-1)} (T_{n-1}^{(q)} + \ldots + T_1^{(q)}) + \ldots + S_n^{(p-1)} T_1^{(q)} =$$

$$= S_1^{(p-1)} T_n^{(q+1)} + S_2^{(p-1)} T_{n-1}^{(q+1)} + \ldots + S_n^{(p-1)} T_1^{(q+1)}.$$

We also prove with induction:

$$W_n^{(p+q)} = S_1^{(p)} T_n^{(q)} + S_2^{(p)} T_{n-1}^{(q)} + \ldots + S_n^{(p)} T_1^{(q)} \quad (5)$$

for we have by (5):

$$W_n^{(p+q+1)} = W_n^{(p+q)} + W_{n-1}^{(p+q)} + \ldots + W_1^{(p+q)}$$

$$= [S_1^{(p)} T_n^{(q)} + \ldots + S_n^{(p)} T_1^{(q)}] + [S_1^{(p)} T_{n-1}^{(q)} + \ldots + S_{n-1}^{(p)} T_1^{(q)}] + \ldots + S_1^{(p)} T_1^{(q)}$$

$$= S_1^{(p)} [T_n^{(q)} + \ldots + T_1^{(q)}] + S_2^{(p)} [T_{n-1}^{(q)} + \ldots + T_1^{(q)}] + \ldots + S_n^{(p)} T_1^{(q)} =$$

$$= S_1^{(p)} T_n^{(q+1)} + S_2^{(p)} T_{n-1}^{(q+1)} + \ldots + S_n^{(p)} T_1^{(q+1)} \quad (5a)$$

Finally we deduce from:

$$W_n^{(p+1)} = S_1^{(1)} T_n^{(p)} + S_2^{(1)} T_{n-1}^{(p)} + \ldots + S_n^{(1)} T_1^{(p)} \quad (6)$$

with the aid of $S_n^{(1)} = a_1 + a_2 + \ldots + a_n$

$$W_n^{(p+1)} = a_1 T_n^{(p)} + (a_1 + a_2) T_{n-1}^{(p)} + \ldots + (a_1 + a_2 + \ldots + a_n) T_1^{(p)}$$

$$= a_1 [T_n^{(p)} + T_{n-1}^{(p)} + \ldots + T_1^{(p)}] + a_2 [T_{n-1}^{(p)} + \ldots + T_1^{(p)}] + \ldots + a_n T_1^{(p)}$$

or

$$W_n^{(p+1)} = a_1 T_n^{(p+1)} + a_2 T_{n-1}^{(p+1)} + \ldots + a_n T_1^{(p+1)} \quad (7)$$

The n^{th} mean-value of order p of the series $a_1 + a_2 + \ldots$ respectively $b_1 + b_2 + \ldots$ is defined as

$$\frac{S_n^{(p+1)}}{A_n^{(p+1)}} \quad \text{respectively} \quad \frac{T_n^{(p+1)}}{A_n^{(p+1)}}.$$

If such a mean value of order p has a limit for $n = \infty$ we call the corresponding series summable of order p [1]). By a well-known

[1]) In our definition the first term of a series has the index 1 and not zero as is usually the case.

theorem the summability of order p (and also the convergence) of a series implies the summability of order $(p+i)$ if $i \geqq 1$ [1]).

Proof of theorem **1.**

Suppose that the series $a_1 + a_2 + \ldots$ is absolutely convergent (sum $= s$); let the series $b_1 + b_2 + \ldots$ be summable of order p, or $\lim_{n=\infty} \frac{T_n^{(p+1)}}{A_n^{(p+1)}} = t$, then we have to prove:

$$\lim_{n=\infty} \frac{W_n^{(p+1)}}{A_n^{(p+1)}} = s \,.\, t$$

Now we have:

$$W_n^{(p+1)} = a_1 \, T_n^{(p+1)} + a_2 \, T_{n-1}^{(p+1)} + \ldots + a_n \, T_1^{(p+1)} \quad . \quad . \quad (7)$$

Put $h_n = \frac{T_n^{(p+1)}}{A_n^{(p+1)}} - t$, then $\lim_{n=\infty} h_n = 0$. Substitution of

$$T_n^{(p+1)} = t A_n^{(p+1)} + h_n \, A_n^{(p+1)}$$

in (7 gives:

$$W_n^{(p+1)} =$$

$$= a_1 A_n^{(p+1)} t + a_1 A_n^{(p+1)} h_n + a_2 A_{n-1}^{(p+1)} t + a_2 A_{n-1}^{(p+1)} h_{n-1} + \ldots + a_n A_1^{(p+1)} t + a_n A_1^{(p+1)} h_1$$

$$= t \left[a_1 \, A_n^{(p+1)} + a_2 \, A_{n-1}^{(p+1)} + \ldots + a_n \, A_1^{(p+1)}\right] +$$

$$+ \left[a_1 \, A_n^{(p+1)} \, h_n + a_2 \, A_{n-1}^{(p+1)} \, h_{n-1} + \ldots + a_n \, A_1^{(p+1)} \, h_1\right] .$$

$$= P + Q \,.$$

$$P = t \left[a_1 \, A_n^{(p+1)} + a_2 \, A_{n-1}^{(p+1)} + \ldots + a_n \, A_1^{(p+1)}\right] =$$

$$= t \left[s_1 \, A_n^{(p+1)} + (s_2 - s_1) \, A_{n-1}^{(p+1)} + \ldots + (s_n - s_{n-1}) \, A_1^{(p+1)}\right] =$$

$$= t \left[s_1 \, (A_n^{(p+1)} - A_{n-1}^{(p+1)}) + s_2 \, (A_{n-1}^{(p+1)} - A_{n-2}^{(p+1)}) + \ldots + s_n \, A_1^{(p+1)}\right] =$$

$$= t \left[s_1 \, A_n^{(p)} + s_2 \, A_{n-1}^{(p)} + \ldots + s_n \, A_1^{(p)}\right]$$

$$= t \left[S_1^{(1)} \, A_n^{(p)} + S_2^{(1)} \, A_{n-1}^{(p)} + \ldots + S_n^{(1)} \, A_1^{(p)}\right]$$

$$= t \left[S_1^{(p)} \, A_n^{(1)} + S_2^{(p)} \, A_{n-1}^{(1)} + \ldots + S_n^{(p)} \, A_1^{(1)}\right]$$

$$= t \left[S_1^{(p)} + S_2^{(p)} + \ldots + S_n^{(p)}\right]$$

$$= t \; S_n^{(p+1)}.$$

Hence $\lim_{n=\infty} \frac{P}{A_n^{(p+1)}} = t \,.\, \lim_{n=\infty} \frac{S_n^{(p+1)}}{A_n^{(p+1)}} = t \,.\, s$ for $a_1 + a_2 + \ldots$ converges absolutely and is consequently summable of order p.

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[1]) Bromwich. l.c. (p. 312).

$$Q = a_1 A_n^{(p+1)} h_n + a_2 A_{n-1}^{(p+1)} h_{n-1} + \ldots + a_n A_1^{(p+1)} h_1$$

$$= a_1 h_n [A_1^{(p)} + A_2^{(p)} + \ldots + A_n^{(p)}] + a_2 h_{n-1} [A_1^{(p)} + \ldots + A_{n-1}^{(p)}] + \ldots + a_n h_1 A_1^{(p)}$$

$$= A_1^{(p)} [a_1 h_n + a_2 h_{n-1} + \ldots + a_n h_1] + A_2^{(p)} [a_1 h_n + \ldots + a_{n-1} h_2] + \ldots + A_n^{(p)} a_1 h_n$$

From $\lim_{n=\infty} h_n = 0$ it is evident that M_k may be chosen, so that $|h_{k+i}| < M_k$ if $i = 1, 2, \ldots$ and $\lim_{k=\infty} . M_k = 0$; putting $\sigma_n = |a_1| + |a_2| + \ldots + |a_n|$ and $\lim_{n=\infty} . \sigma_n = \sigma$, we have:

$$|Q| < A_1^{(p)} [|a_1| . |h_n| + |a_2| . |h_{n-1}| + \ldots + |a_n| . |h_1|] +$$

$$+ A_2^{(p)} [|a_1| . |h_n| + \ldots + |a_{n-1}| . |h_2|] + \ldots + A_n^{(p)} . |a_1| . |h_n|$$

$$< A_1^{(p)} M_1 [|a_1| + |a_2| + \ldots + |a_n|] + A_2^{(p)} M_2 [|a_1| + \ldots + |a_{n-1}|] + \ldots + A_n^{(p)} M_n |a_1|$$

$$< A_1^{(p)} M_1 \sigma_n + A_2^{(p)} M_2 \sigma_{n-1} + \ldots + A_n^{(p)} M_n \sigma_1$$

$$< \sigma [A_1^{(p)} M_1 + A_2^{(p)} M_2 + \ldots + A_n^{(p)} M_n].$$

Now whatever be $\varepsilon > 0$ it is possible to calculate k so that $M_{k+i} < \frac{\varepsilon}{\sigma}$; further let M be chosen so that $M_i < M$, then we have, if $n > k$:

$$|Q| < \sigma [A_1^{(p)} M_1 + \ldots + A_k^{(p)} M_k] + \varepsilon [A_{k+1}^{(p)} + \ldots + A_n^{(p)}]$$

$$< M \sigma [A_1^{(p)} + \ldots + A_k^{(p)}] + \varepsilon [A_n^{(p+1)} - A_k^{(p+1)}].$$

Hence:

$$\left| \frac{Q}{A_n^{(p+1)}} \right| < M \sigma \frac{A_k^{(p+1)}}{A_n^{(p+1)}} + \varepsilon \cdot \frac{A_n^{(p+1)} - A_k^{(p+1)}}{A_n^{(p+1)}}$$

$\lim_{n=\infty} . \frac{A_k^{(p)}}{A_n^{(p)}} = 0$ for $\lim_{n=\infty} . A_n^{(p)} = \infty$ because $A_n^{(p)} > A_n^{(2)} = n$.

Hence, if n is sufficiently great:

$$\left| \frac{Q}{A_n^{(p+1)}} \right| \leq \varepsilon + \varepsilon$$

and since ε is arbitrary $\lim_{n=\infty} . \frac{Q}{A_n^{(p+1)}} = 0$.

Therefore:

$$\lim_{n=\infty} . \frac{W_n^{(p+1)}}{A_n^{(p+1)}} = \lim_{n=\infty} . \frac{P+Q}{A_n^{(p+1)}} = \lim_{n=\infty} . \frac{P}{A_n^{(p+1)}} + \lim_{n=\infty} . \frac{Q}{A_n^{(p+1)}} = s . t.$$

Proof of theorem 2.

Suppose that the series $a_1 + a_2 + \ldots$ converges to s, and that the series $b_1 + b_2 + \ldots$ is summable of order p (sum $= t$); further, let the mean-values of order $(p-1)$ of the series $b_1 + b_2 + \ldots$ be limited.

We have to prove:

$$\lim_{n=\infty} \cdot \frac{W_n^{(p+1)}}{A_n^{(p+1)}} = s \cdot t$$

Put $S_n^{(1)} = s_n = s + h_n$, then $\lim_{n=\infty} \cdot h_n = 0$ and

$$\begin{aligned} W_n^{(p+1)} &= S_1^{(1)} T_n^{(p)} + S_2^{(1)} T_{n-1}^{(p)} + \ldots + S_n^{(1)} T_1^{(p)} \quad . \quad . \quad . \quad . \quad . \quad (6) \\ &= (s + h_1) T_n^{(p)} + (s + h_2) T_{n-1}^{(p)} + \ldots + (s + h_n) T_1^{(p)} \\ &= s[T_n^{(p)} + T_{n-1}^{(p)} + \ldots + T_1^{(p)}] + [h_1 T_n^{(p)} + h_2 T_{n-1}^{(p)} + \ldots + h_n T_1^{(p)}] \\ &= R + S. \end{aligned}$$

$$R = s\,[T_n^{(p)} + T_{n-1}^{(p)} + \ldots + T_1^{(p)}] = s T_n^{(p+1)}$$

$$\frac{R}{A_n^{(p+1)}} = s \cdot \frac{T_n^{(p+1)}}{A_n^{(p+1)}}.$$

Since the series $b_1 + b_2 + \ldots$ is summable of order p, we have:

$$\lim_{n=\infty} \cdot \frac{R}{A_n^{(p+1)}} = s \cdot \lim_{n=\infty} \cdot \frac{T_n^{(p+1)}}{A_n^{(p+1)}} = s \cdot t.$$

$$S = h_1 T_n^{(p)} + h_2 T_{n-1}^{(p)} + \ldots + h_n T_1^{(p)}.$$

Since the mean-values of order $(p-1)$ are limited, it is possible to find M so that:

$$\left| \frac{T_n^{(p)}}{A_n^{(p)}} \right| < M. \text{ Hence:}$$

$$|S| < M\,[|h_1|\, A_n^{(p)} + |h_2|\, A_{n-1}^{(p)} + \ldots + |h_n|\, A_1^{(p)}]. \quad . \quad . \quad (8)$$

Put $|h_n| = H_n^{(1)}$ then we have by (4)

$$\begin{aligned} |h_1| A_n^{(p)} + |h_2| A_{n-1}^{(p)} + \ldots + |h_n| A_1^{(p)} &= \\ &= H_1^{(1)} A_n^{(p)} + H_2^{(1)} A_{n-1}^{(p)} + \ldots + H_n^{(1)} A_1^{(p)} \\ &= H_1^{(p)} A_n^{(1)} + H_2^{(p)} A_{n-1}^{(1)} + \ldots + H_n^{(p)} A_1^{(1)} \\ &= H_1^{(p)} + H_2^{(p)} + \ldots + H_n^{(p)} = H_n^{(p+1)} \end{aligned}$$

The inequality (8) may now be written:

$$|S| < M \cdot H_n^{(p+1)}$$

therefore:

$$\frac{|S_n|}{A_n^{(p+1)}} < M \frac{H_n^{(p+1)}}{A_n^{(p+1)}}$$

Since $\lim_{n=\infty} . H_n^{(1)} = 0$ and hence $\lim_{n=\infty} . \frac{H_n^{(p+1)}}{A_n^{(p+1)}} = 0$, it follows that $\lim_{n=\infty} . \frac{S_n}{A_n^{(p+1)}} = 0$.

Hence :

$$\lim_{n=\infty} . \frac{W_n^{(p+1)}}{A_n^{(p+1)}} = \lim_{n=\infty} . \frac{R_n}{A_n^{(p+1)}} + \lim_{n=\infty} . \frac{S_n}{A_n^{(p+1)}} = s . t.$$

Proof of theorem 3.

Suppose that the series $a_1 + a_2 + \ldots$ is summable of order p, and that its mean-values of order $(p-1)$ are limited, then we have:

$$\lim_{n=\infty} . \frac{S_n^{(p+1)}}{A_n^{(p+1)}} = s \quad \text{and} \quad \frac{|S_n^{(p)}|}{A_n^{(p)}} < \text{a fixed number } M.$$

Let $b_1 + b_2 + \ldots$ be summable of order q, or

$$\lim_{n=\infty} . \frac{T_n^{(q+1)}}{A_n^{(q+1)}} = t.$$

We have to prove: $\lim_{n=\infty} . \frac{W_n^{(p+q+1)}}{A_n^{(p+q+1)}} = s . t$

$$W_n^{(p+q+1)} = S_1^{(p)} T_n^{(q+1)} + S_2^{(p)} T_{n-1}^{(q+1)} + \ldots + S_n^{(p)} T_1^{(q+1)} \quad . \quad (5a)$$

Put $\frac{T_n^{(q+1)}}{A_n^{(q+1)}} = t + h_n$, then $\lim_{n=\infty} . h_n = 0$

$$W_n^{(p+q+1)} = S_1^{(p)} [t A_n^{(q+1)} + h_n A_n^{(q+1)}] + \ldots + S_n^{(p)} [t A_1^{(q+1)} + h_1 A_1^{(q+1)}]$$

$$= t [S_1^{(p)} A_n^{(q+1)} + \ldots + S_n^{(p)} A_1^{(q+1)}] + [S_1^{(p)} h_n A_n^{(q+1)} + \ldots + S_n^{(p)} h_1 A_1^{(q+1)}]$$

$$= U + V.$$

$$U = t [S_1^{(p)} A_n^{(q+1)} + S_2^{(p)} A_{n-1}^{(q+1)} + \ldots + S_n^{(p)} A_1^{(q+1)}]$$

$$= t [S_1^{(p+q)} A_n^{(1)} + S_2^{(p+q)} A_{n-1}^{(1)} + \ldots + S_n^{(p+q)} A_1^{(1)}]$$

$$= t [S_1^{(p+q)} + S_2^{(p+q)} + \ldots + S_n^{(p+q)}] = t . S_n^{(p+q+1)}$$

$$\frac{U}{A_n^{(p+q+1)}} = t \frac{S_n^{(p+q+1)}}{A_n^{(p+q+1)}}$$

Since $\lim_{n=\infty} . \frac{S_n^{(p+1)}}{A_n^{(p+1)}} = s$, it follows that

$$\lim_{n=\infty} . \frac{U}{A_n^{(p+q+1)}} = s \,.\, t$$

$$V = S_1^{(p)} h_n A_n^{(q+1)} + S_2^{(p)} h_{n-1} A_{n-1}^{(q+1)} + \ldots + S_n^{(p)} h_1 A_1^{(q+1}$$

From $\frac{|S_i^{(p)}|}{A_i^{(p)}} < M$ whatever be i, we deduce:

$$|V| < M\,[A_1^{(p)} A_n^{(q+1)} |h_n| + A_2^{(p)} A_{n-1}^{(q+1)} |h_{n-1}| + \ldots + A_n^{(p)} A_1^{(q+1)} |h_1|]$$

hence if $1 < k < n$

$$V| < M\,[A_1^{(p)} A_n^{(q+1)} |h_n| + \ldots + A_{n-k}^{(p)} A_{k+1}^{(q+1)} |h_{k+1}|] +$$
$$+ M\,[A_{n-k+1}^{(p)} A_k^{(q+1)} |h_k| + \ldots + A_n^{(p)} A_1^{(q+1)} |h_1|].$$

Now whatever be $\varepsilon > 0$, we can calculate an integer k so that $|h_{k+i}| < \frac{\varepsilon}{M}$; further we can find μ so that $|h_i| < \mu$.

Then, if $n > k$ we have:

$$|V| < \varepsilon [A_1^{(p)} A_n^{(q+1} + \ldots + A_{n-k}^{(p)} A_{k+1}^{(q+1)}] + M\mu [A_{n-k+1}^{(p)} A_k^{(q+1)} + \ldots + A_n^{(p)} A_1^{(q+1)}]$$

or, since

$$A_1^{(p)} A_n^{(q+1)} + A_2^{(p)} A_{n-1}^{(q+1)} + \ldots + A_n^{(p)} A_1^{(q+1)} = A_n^{(p+q+1)} \text{ and } A_i^{(j)} \geqq A_{i-1}^{(j)}$$

$$|V| < \varepsilon\, A_n^{(p+q+1)} + M\mu A_k^{(q+1)} [A_{n-k+1}^{(p)} + \ldots + A_n^{(p)}]$$

$$< \varepsilon\, A_n^{(p+q+1)} + M\mu A_k^{(q+1)}\, A_n^{(p+1)} .$$

hence:

$$\frac{|V|}{A_n^{(p+q+1)}} < \varepsilon + M\mu A_k^{(q+1)} . \frac{A_n^{(p+1)}}{A_n^{(p+q+1)}}.$$

If $q \geq 1$, $\lim_{n=\infty} . \frac{A_n^{(p+1)}}{A_n^{(p+q+1)}} = 0$ for:

$$\frac{A_n^{(p+1)}}{A_n^{(p+q+1)}} = \frac{(n+p-1)\,!}{(n-1)\,!\,p\,!} \times \frac{(n-1)\,!\,(p+q)\,!}{(n+p+q-1)\,!}$$

$$= \frac{(p+q)\,!}{p\,!} \times \frac{(n+p-1)\,!}{(n+p+q-1)\,!}$$

$$\leqq \frac{(p+q)\,!}{p\,!} \times \frac{(n+p-1)\,!}{(n+p+1-1)!} \text{ or } \leqq \frac{(p+q)!}{p!} \times \frac{1}{(n+p)}.$$

Hence we have if n is sufficiently great:

$$\frac{|V|}{A_n^{(p+q+1)}} < \varepsilon + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we have:

$$\lim_{n=\infty} . \frac{V}{A_n^{(p+q+1)}} = 0.$$

Hence:

$$\lim_{n=\infty} . \frac{W_n^{(p+q+1)}}{A_n^{(p+q+1)}} = \lim_{n=\infty} \frac{U}{A_n^{(p+q+1)}} + \lim_{n=\infty} \frac{V}{A_n^{(p+q+1)}} = s . t$$

Remark 1.

K. KNOPP [1]) ad S. CHAPMAN [2]) have limited the order of summability of the product of two series, which are summable of order p and q, by considering non-integral orders of summability. It may happen that the theorems proved above give more result, as is seen by the following example:

The series $1 - 1 + 1 - 1 + \ldots$ is summable of order 1 and its mean-values of order 0 are limited. Hence, applying theorem 3, we see that the product of this series by a series which is summable of order p, is summable of order $(p + 1)$. Now the so-called index of summability of the series $1 - 1 + \ldots$ is zero (see CHAPMAN, l. c.); the index of a series which is summable of order p, cannot exceed p: hence the index of the product cannot exceed $p + 1$, and therefore we can only infer by CHAPMAN's theory that the product is summable of order $p + 2$.

Remark 2.

HARDY [3]) has also given the following extension of MERTENS' theorem which is totally different from the generalisations mentioned above, and which contains MERTENS' theorem as a special case:

If Σa_n is absolutely convergent and Σb_n is a finitely oscillating series whose n^{th} term tends to zero, then their product is a finitely oscillating series, and if the limits of oscillation of Σb_n are β_1 and β_2 those of the product are $s . \beta_1$ and $s . \beta_2$.

[1]) Sitzungsberichte der Berliner Math. Gesellschaft 1907 (p. 1—12)

[2]) Proceedings of the London Mathematical Society, Ser. 2 Vol. 9 (p. 369—409).

[3]) Proceedings of the London Mathematical Society, Ser. 2 Vol. 6 (p. 410—423).

Evidently the terms a_i and b_i are supposed to be real: therefore MERTENS' theorem is only a special case of this theorem when the terms of the series are real. It is however easy to see that HARDY's proof is also valid for the following extension to series with complex terms: [1])

Theorem 4: *If* Σa_n *converges absolutely to* s, *if* $b_1 + b_2 + \ldots + b_n$ *is limited and* $\lim_{n=\infty} . b_n = 0$, *then the product of the series* Σa_i *and* Σb_i *oscillates for* $n = \infty$ *about the same region as the series* $s.\ \Sigma b_n$.

The functions $\varphi(n)$ and $\psi(n)$ are said to oscillate about the same region if n tends to ∞, if the following condition is satisfied:

whatever be $\varepsilon > 0$ we can find two numbers μ and a so that it is possible to calculate whatever be $n > \mu$ a number m which satisfies the conditions:

$$|\varphi(n) - \psi(n)| < \varepsilon \qquad |n - m| < a$$

and that is also possible so calculate whatever be $m > \mu$ a number n which satisfies the same conditions.

Finally we prove the following theorem which is analogous to theorem 4 and which contains theorem 1 as a special case:

Theorem 5: *If* Σa_n *converges absolutely to* s, *if the mean-values of order* p *of* Σb_n *are limited and the mean-values of order* $(p-1)$ (*which we denote by* $U_i^{(p)}$) *satisfy the condition*:

$$\lim_{n=\infty} \frac{U_n^{(p)}}{n} = 0,$$

then the mean-values of order p *of the product-series oscillates about the same region as* $s.\ U_n^{(p+1)}$ *as* n *tends to* ∞.

Proof of theorem 4

Substituting $p = 1$ in formula (7), we have:

$$w_n = a_1 t_n + a_2 t_{n-1} + \ldots + a_n t_1$$

Hence, if $1 < k < n$:

$$w_n = [a_1 t_n + \ldots + a_k t_{n-k+1}] + [a_{k+1} t_{n-k} + \ldots + a_n t_1] = P + Q.$$

Suppose $|t_i| < t$ and $|s_i| < \sigma$ whatever be i.

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[1]) It is not clear from HARDY's article how far the author also considers series with complex terms; in the preceding pages he considers series with real terms, and his statement, as far as I am aware, is also made for real terms; yet his proof applies as well to series with complex terms.

We can find, whatever be $\varepsilon > 0$, an integer k so that:

$$|a_{k+1}| + \ldots + |a_{k+p}| < \frac{\varepsilon}{3t};$$

then we have also $|s-s_k| < \frac{\varepsilon}{3t}$ and $|Q| = |a_{k+1}\, t_{n-k} + \ldots + a_n\, t_1| < \frac{\varepsilon}{3}$

$$P = a_1\, t_n + \ldots + a_k\, t_{n-k+1} = a_1\,(b_1 + \ldots + b_n) + \ldots + a_k\,(b_1 + \ldots + b_{n-k+1})$$

$$= (b_1 + b_2 + \ldots + b_{n-k}) \,.\, (a_1 + a_2 + \ldots + a_k) + b_n s_1 + b_{n-1}\, s_2 + \ldots + b_{n-k+1}\, s_k \,.$$

$$= t_{n-k}\, s_k + R \text{ als } R = b_n\, s_1 + \ldots + b_{n-k+1}\, s_k.$$

We can find μ so that $|b_{n-k+i}| < \frac{\varepsilon}{3ko}$ if $n > \mu$; then we have also $|R| < \frac{\varepsilon}{3}$ if $n > \mu$.

Now $s_k\, t_{n-k} = s t_{n-k} - (s - s_k) \,.\, t_{n-k}$.

Since $|(s-s_k) \,.\, t_{n-k}| < \frac{\varepsilon}{3}$ (see above), we have if $n > \mu$:

$$|P - s \,.\, t_{n-k}| < \frac{2\varepsilon}{3} \quad \text{and, since} \quad |Q| < \frac{\varepsilon}{3} \text{ and } w_n = P + Q:$$

$$|w_n - s \,.\, t_{n-k}| < \varepsilon.$$

Hence we see that it is possible to calculate, whatever be $\varepsilon > 0$, an integer μ which satisfies the conditions.

Proof of theorem 5.

We have:

$$W_n^{(p+1)} = a_1\, T_n^{(p+1)} + a_2\, T_{n-1}^{(p+1)} + \ldots + a_n\, T_1^{(p+1)} \quad . \quad . \quad . \quad (7)$$

Hence, if $1 < k < n$:

$$W_n^{(p+1)} = [a_1\, T_n^{(p+1)} + \ldots + a_k\, T_{n-k+1}^{(p+1)}] + [a_{k+1}\, T_{n-k}^{(p+1)} + \ldots + a_n\, T_1^{(p+1)}]$$

$$= P + Q.$$

Let $\frac{|T_i^{(p+1)}|}{A_i^{(p+1)}} < t$ and $|s_i| < \sigma$.

Whatever be $\varepsilon > 0$ we can find an integer k so that:

$$|a_{k+1}| + \ldots + |a_{k+p}| < \frac{\varepsilon}{3t}.$$

Then $|s - s_k| < \frac{\varepsilon t}{3}$ and a fortiori:

$$\frac{|Q|}{A_n^{(p+1)}} < t \cdot [|a_{k+1}| + \ldots + |a_n|] < \frac{\varepsilon}{3}$$

$$P = [a_1 T_n^{(p+1)} + \ldots + a_k T_{n-k+1}^{(p+1)}]$$

$$= s_k T_{n-k}^{(p+1)} + [T_n^{(p)} s_1 + T_{n-1}^{(p)} s_2 + \ldots + T_{n-k+1}^{(p)} s_k]$$

Hence it follows from

$$T_{n-k}^{(p+1)} = T_n^{(p+1)} - [T_n^{(p)} + T_{n-1}^{(p)} + \ldots + T_{n-k+1}^{(p)}]$$

that

$$\frac{P}{A_n^{(p+1)}} = s^k \cdot \frac{T_n^{(p+1)}}{A_n^{(p+1)}} + \frac{T_n^{(p)}(s_1 - s_k) + T_{n-1}^{(p)}(s_2 - s_k) + \ldots + T_{n\ k+1}^{(p)}(s_k - s_k)}{A_n^{(p+1)}}$$

$$= R + S.$$

It is evident that the absolute value of S is less than

$$2\sigma \frac{|T_n^{(p)}| + |T_{n-1}^{(p)}| + \ldots + |T_{n-k+1}^{(p)}|}{A_n^{(p+1)}}$$

We now prove that we can find $\mu > k$ so that if $r > \mu$: $\frac{|T_{r-k}^{(p)}|}{A_{r-k}^{(p+1)}} < \frac{\varepsilon}{3k\sigma}$ then it follows that $|S| < \frac{2\varepsilon}{3}$ if $n > \mu$.

For we have by hypothesis $\lim_{i=\infty} \cdot \frac{T_i^{(p)}}{i A_i^{(p)}} = 0$.

Since $\frac{A_i^{(p+1)}}{A_i^{(p)}} = \frac{i+p-1}{p}$ we have $\lim_{i=\infty} \cdot \frac{T_i^{(p)}}{A_i^{(p+1)}} = 0$.

If $n > \mu$ we have:

$$\left| \frac{W_n^{(p+1)}}{A_n^{(p+1)}} - s_k \frac{T_n^{(p+1)}}{A_n^{(p+1)}} \right| < \varepsilon$$

and since $|s - s_k| < \frac{\varepsilon}{3t}$ our theorem is proved.

Remark 3.

A. Rosenblatt (Bulletin International de l'Academie des Sciences de Cracovie", ser. A 1913 p. 612—620 [1])) has proved the following theorem:

[1]) Rosenblatt's memoir not being accessible to me, the reference above is taken from an article of G. Doetsch, Mathematische Zeitschrift Bd. 11, p. 161—175.

If Σa_n is summable of order $p+1$ and its mean-values of order p are limited, if Σb_n is summable of order $q+1$ and its mean-values of order q are limited, then the product-series is summable of order $p+q+2$.

This theorem is an extension of CAUCHY's theorem that the product of two absolutely convergent series is convergent, analogous to the extension of MERTENS' theorem given in theorem 3, and, like MERTENS' theorem implies that of CAUCHY, so theorem 3 implies that of ROSENBLATT.

Mathematics. — *"On a Generalisation of* TAUBER's *Theorem concerning Power Series"*. By M. J. BELINFANTE. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of March 24, 1923).

Introduction.

In this paper we consider power series with complex coefficients, but for real values of the variable. We suppose them to converge if $|x| < 1$, and we denote by $x \to 1$ that x approaches 1 by real values from below.

TAUBER has proved the following theorem [1]):

If $\lim_{n=\infty} na_n = 0$ and $\lim_{x \to 1} \Sigma a_n x^n = s$, then Σa_n converges to s.

LITTLEWOOD [2]) has shown that the usual proof of this theorem proves more than is actually stated, and that the same proof applies to the theorem:

If $\sum_{0}^{\infty} a_n x^n$ oscillates finitely as $x \to 1$, then the limits of oscillation as $n \to \infty$ of $\sum_{0}^{n} a_i$ are the same as the limits of oscillation of $\sum_{0}^{\infty} a_n x^n$.

In the present paper we give extensions of both theorems to the so-called mean-values of HÖLDER.

§ 1 contains the proof mentioned above and a definition of the expression "oscillate about the same region"; in § 2 the definition of HÖLDER's mean-values and some necessary formula's will be treated, while § 3 contains the generalizations of TAUBER's theorem.

§ 1.

Definition [3]). We say that $f(x)$ oscillates for $x \to x_0$ about the same region as $g(y)$ for $y \to y_0$, when the following conditions are satisfied:

[1]) Monatshefte für Math. u. Phys., 1897 Bd. 8, p. 273.

[2]) Proc. of the Lond. Math. Soc., 1911 Vol. 9, p. 436.

[3]) We always suppose that x resp. y approaches x_0 resp. y_0 by real values from below.

primo: it is possible whatever be $\varepsilon > 0$ and $\xi < x_0$, to calculate a number $\eta(\varepsilon,\xi)$ so that whatever be y_1 between η and y_0, we can calculate a number x_1 between ξ and x_0 which satisfies the condition

$$| f(x_1) - g(y_1) | < \varepsilon,$$

secundo: it is possible whatever be $\varepsilon > 0$ aad $\eta < y_0$ to calculate a number $\xi(\varepsilon,\eta)$ so that whatever be x_1 between ξ and x_1 we can calculate a number y_1 between η and y_0 which satisfies the condition

$$| f(x_1) - g(y_1) | < \varepsilon.$$

Theorem 1. If $\lim_{n=\infty} na_n = 0$, $\sum_0^\infty a_n x^n$ *oscillates for* $x \to 1$ *about the same region as* $\sum_0^n a_i$ *for* $n \to \infty$.

Proof: We have by a well-known theorem [1]) that $\lim_{\nu=\infty} u_\nu = u$ implies $\lim_{\nu=\infty} \frac{1}{\nu} \sum_0^{\nu-1} u_i = u$. Hence, since $\lim_{n=\infty} na_n = 0$ implies $\lim_{n=\infty} n\,|a_n| = 0$,

$$\lim_{\nu=\infty} \frac{1}{\nu} \sum_0^{\nu-1} n\,|a_n| = 0.$$

Therefore, whatever be $\varepsilon > 0$, we can calculate an integer μ so that if $\nu > \mu$:

$$\nu\,|a_\nu| < \frac{\varepsilon}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{1}{\nu} \sum_0^{\nu-1} n\,|a_n| < \frac{\varepsilon}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, if $0 < x < 1$, we have:

$$\left| \sum_0^{\nu-1} a_n - \sum_0^\infty a_n x^n \right| \leq \left| \sum_0^{\nu-1} a_n - \sum_0^{\nu-1} a_n x^n \right| + \left| \sum_\nu^\infty a_n x^n \right|$$

$$\leq \sum_0^{\nu-1} a_n (1-x^n) \Big| + \Big| \sum_\nu^\infty a_n x^n \Big| \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\left| \sum_0^{\nu-1} a_n (1-x^n) \right| \leq \sum_0^{\nu-1} |a_n| \,.\, (1-x^n) < (1-x) \,.\, \sum_0^{\nu-1} n\,|a_n| \quad . \quad . \quad (4)$$

Substitution of $x_\nu = 1 - \frac{1}{\nu}$ in (4) gives:

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[1]) Bromwich, Theory of Infinite Series, p. 383.

$$|\sum_0^{\nu-1} a_n (1-x_\nu^n)| < \frac{1}{\nu} \sum_0^{\nu-1} n\,|a_\nu|$$

Hence by (2) if $\nu > \mu$:

$$|\sum_0^{\nu-1} a_n (1-x_\nu^n)| < \frac{\varepsilon}{2} \quad \ldots\ldots\ldots \quad (5)$$

Substitution of (1) in $\sum_\nu^\infty a_n x^n$ gives:

$$|\sum_\nu^\infty a_n x^n| < \sum_\nu^\infty \frac{\varepsilon}{2n} \cdot x^n ,$$

or a fortiori:

$$< \frac{1}{\nu} \sum_\nu^\infty \frac{\varepsilon}{2} x^n$$

or

$$< \frac{\varepsilon}{2\nu} \cdot \frac{x^\nu}{1-x} .$$

Substituting $x_\nu = 1 - \frac{1}{\nu}$ in the last inequality we have:

$$|\sum_\nu^\infty a_n x_\nu^n| < \frac{\varepsilon}{2} \quad \ldots\ldots\ldots \quad (6)$$

From (3), (5) and (6) we deduce:

$$|\sum_0^{\nu-1} a_n - \sum_0^\infty a_n x_\nu^n| < \varepsilon$$

if $\nu > \mu$ and $x_\nu = 1 - \frac{1}{\nu}$, and it follows easily that both conditions of our definition are satisfied.

§ 2.

If $t_1, t_2 \ldots$ is an arbitrary sequence of quantities, we define the so-called Hölder mean-values as follows [1]):

$$H_n^{(1)}(t) = \frac{t_1 + t_2 \ldots + t_n}{n} \quad \ldots\ldots\ldots \quad (7)$$

$$H_n^{(k)}(t) = \frac{H_1^{(k-1)}(t) + H_2^{(k-1)}(t) + \ldots + H_n^{(k-1)}(t)}{n} \quad \ldots \quad (8)$$

$$H_0^{(k)}(t) = H_{-1}^{(k)}(t) = 0 \quad \ldots\ldots\ldots \quad (8a)$$

The following relations are easy to verify:

$$H_n^{(p)}[H^{(q)}(t)] = H_n^{(p+q)}(t) \text{ if } p \geq 1,\ q \geq 1. \quad \ldots\ldots \quad (9)$$

and

[1]) This definition differs slightly from the usual one, as the latter is given for a series $u_1 + u_2 + \ldots$ and not for a sequence.

$$n \cdot H_n^{(k)}(t) - (n-1) \cdot H_{n-1}^{(k)}(t) = H_n^{(k-1)}(t) \quad . \quad . \quad . \quad (10)$$

Let

$$s_n = a_1 + a_2 + \ldots + a_n \; ; \quad s_{-1} = s_0 = 0 \; ; \quad . \quad . \quad . \quad (11)$$

then we define:

$$H_n^{(k)}(s) = A_n^{(k)} \text{ if } k \geq 1, \quad n \geq -1 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$s_n = A_n^{(0)} \text{ if } n \geq -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12a)$$

$$\sigma_n^{(k)} = n \cdot [A_n^{(k)} - A_{n-1}^{(k)}] \text{ if } k > 0, \quad n \geq 0 \quad . \quad . \quad . \quad . \quad (13)$$

From (10), (12) and (13) we deduce:

$$\sigma_n^{(k)} = A_n^{(k-1)} - A_{n-1}^{(k)} \text{ if } k \geq 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Finally we define:

$$\varphi_k(x) = \sum_1^\infty [A_n^{(k)} - A_{n-1}^{(k)}] \cdot x^n \text{ if } k \geq 0 \quad . \quad . \quad . \quad . \quad (15)$$

thus

$$\varphi_0(x) = \sum_1^\infty a_n x^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15a)$$

We prove the following identities: [1])

$$H_n^{(1)}[\sigma^{(k)}] = \sigma_n^{(k+1)} + \frac{A_{n-1}^{(k+1)}}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\varphi_k(x) + (1-x) \cdot \varphi_k'(x) = \frac{1}{x} \cdot \varphi_{k-1}(x) \quad . \quad . \quad . \quad . \quad (17)$$

$$(1-x) \cdot \varphi_k'(x) = \sum_0^\infty [\sigma_{n+1}^{(k)} - \sigma_n^{(k)}] \cdot x^n \quad . \quad . \quad . \quad . \quad (18)$$

Proof of (16).

By (14) we have:

$$\begin{aligned}
&\sigma_n^{(k)} + \sigma_{n-1}^{(k)} + \ldots + \sigma_1^{(k)} = \\
&\quad = [A_n^{(k-1)} - A_{n-1}^{(k)}] + \ldots + [A_2^{(k-1)} - A_1^{(k)}] + [A^{(k-1)}] \\
&\quad = [A_1^{(k-1)} + A_2^{(k-1)} + \ldots + A^{(k-1)}] - [A_1^{(k)} + A^{(k)} + \ldots + A_{n-1}^{(k)}] \\
&\quad = n \cdot A_n^{(k)} - (n-1) \cdot A_{n-1}^{(k+1)} \\
&\quad = n \cdot A_n^{(k)} - n \cdot A_{n-1}^{(k+1)} + A_n^{(k+1)}
\end{aligned}$$

hence:

[1]) We tacitly assume that the power series φ_k and φ_{k-1} are convergent if $-1 < x < +1$; in our applications this will be the case.

$$H_n^{(1)}[\sigma^{(k)}] = \frac{\sigma_1^{(k)} + \sigma_2^{(k)} + \ldots + \sigma_n^{(k)}}{n}$$

$$= A_n^{(k)} - A_{n-1}^{(k+1)} + \frac{A_{n-1}^{(k+1)}}{n}$$

$$= \sigma_n^{(k+1)} + \frac{A_{n-1}^{(k+1)}}{n}.$$

Proof of (17).

$$\varphi_k(x) + (1-x)\cdot\varphi_k'(x) =$$

$$= \sum_0^\infty x^n \left\{ A_n^{(k)} - A_{n-1}^{(k)} + (n+1)\cdot[A_{n+1}^{(k)} - A_n^{(k)}] - n\cdot[A_n^{(k)} - A_{n-1}^{(k)}] \right\}$$

$$= \sum_0^\infty x^n \left\{ [(n+1)\cdot A_{n+1}^{(k)} - n\cdot A_n^{(k)}] - [n\cdot A_n^{(k)} - (n-1)\cdot A_{n-1}^{(k)} \right\}$$

$$= \sum_0^\infty x^n [A_{n+1}^{(k-1)} - A_n^{(k-1)}] = \frac{1}{x}\cdot\sum_0^\infty x^{n+1}\cdot[A_{n+1}^{(k-1)} - A_n^{(k-1)}]$$

$$= \frac{1}{x}:\sum_1^\infty x^n\cdot[A_n^{(k-1)} - A_{n-1}^{(k-1)}]$$

$$= \frac{1}{x}\cdot\varphi_{k-1}(x).$$

Proof of (18).

$$(1-x)\cdot\varphi_k'(x) = \sum_0^\infty x^n\cdot\left\{(n+1)\cdot[A_{n+1}^{(k)} - A_n^{(k)}] - n\cdot[A_n^{(k)} - A_{n-1}^{(k)}]\right\}$$

$$= \sum_0^\infty x^n\cdot[\sigma_{n+1}^{(k)} - \sigma_n^{(k)}].$$

§ 3.

We prove the following extensions of TAUBER's theorem:

Theorem 2. *If* $\lim_{n=\infty} n\cdot[A_n^{(p)} - A_{n-1}^{(p)}] = 0$, *and* $|s_n| < c$ *whatever be* n, *then* $\sum_1^\infty a_n x^n$ [1]) *oscillates as* $x \to 1$ *about the same region as* $A_n^{(p)}$ *if* $n \to \infty$.

Theorem 3. *If* $\lim_{n=\infty} n\cdot[A_n^{(p)} - A_{n-1}^{(p)}] = 0$ *and* $\lim_{x\to 1}\sum_1^\infty a_n x^n = s$ [1]), *then we have also:* $\lim_{n=\infty} A_n^{(p-1)} = s$.

Proof of theorem 2.

From the fact that s_n is limited it is easy to deduce that $A_n^{(i)}$ is

[1]) See remark 2 at the end of the article.

also limited, and therefore:

$$\lim. \frac{A_{n-1}^{(k+1)}}{n} = 0.$$

Hence by (16) we conclude:

$$\lim_{n=\infty} [H_n^{(1)}(\sigma^{(k)}) - \sigma_n^{(k+1)}] = 0 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Now it is a well-known theorem that $\lim_{n=\infty} H_n^{(k)}(t) = s$ implies $\lim_{n=\infty} H_n^{(k+1)}(t) = s$;[1]) hence we deduce from (19) with the aid of (9):

$$\lim_{n=\infty} [H_n^{(i+1)}(\sigma^{(k)}) - H_n^{(i)}(\sigma^{(k+1)})] = 0$$

from which we conclude:

If $\lim_{n=\infty} H_n^{(i)}(\sigma^{(k)}) = 0$, we have also $\lim_{n=\infty} H_n^{(i+1)}(\sigma^{(k-1)}) = 0$. (20)

By hypothesis we have: $\lim_{n=\infty} n \cdot A[{}_n^{(p)} - A_{n-1}^{(p)}] = 0$ or by (13)

$$\lim_{n=\infty} \sigma_n^{(p)} = 0.$$

Hence by (19)

$$\lim_{n=\infty} H_n^{(1)}(\sigma^{(p-1)}) = 0$$

and applying (20) we get successively:

$$\lim_{n=\infty} H_n^{(2)}[\sigma^{(p-2)}] = 0$$

$$\lim_{n=\infty} H_n^{(3)}[\sigma^{(p-3)}] = 0$$

. . . . . .

$$\lim_{n=\infty} H_n^{(i)}[\sigma^{(p-i)}] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Hölder has proved[2]) that if $\lim_{n=\infty} H_n^{(i)}(t) = h$, then we have also:

$$\lim_{x \to 1} \sum_0^\infty (t_{n+1} - t_n) x^n = h.$$

In virtue of this theorem we have by (21):

$$\lim_{x \to 1} \sum_0^\infty [\sigma_{n+1}^{(p-i)} - \sigma_n^{(p-i)}] x^n = 0$$

or by (18):

$$\lim_{x \to 1} (1-x) \cdot \varphi'_{p-i}(x) = 0$$

thus by (17):

[1]) See Bromwich, Theory of Infinite Series, p. 383.

[2]) Mathematische Annalen, Bd. 20 (1882), p. 535.

$$\lim_{x\to 1}[x\,.\,\varphi_{p-i}(x)-\varphi_{p-i-1}(x)]=0.$$

Hence we infer that $\varphi_{(p-i)}(x)$ and $\varphi_{p-i-1}(x)$ oscillate about the same region as $x\to 1$. Repeating the argument for $i=1,2,..p$, we see that

$$\varphi_0(x)=\sum_1^\infty a_n x^n \quad \text{and} \quad \varphi_p(x)=\sum_1^\infty [A_n^{(p)}-A_{n-1}^{(p)}]\,x^n$$

oscillate about the same region as $x\to 1$.

By hypothesis we have $\lim_{n=\infty} n\,[A_n^{(p)}-A_{n-1}^{(p)}]=0$; with the aid of theorem 1 we deduce that $\varphi_p(x)=\sum_1^\infty [A_n^{(p)}-A_{n-1}^{(p)}]\,x^n$ oscillates as $x\to 1$ about the same region as $A_m^{(p)}=\sum_1^m [A_n^{(p)}-A_{n-1}^{(p)}]$ if $m\to\infty$.

Combining these results we see that $\sum_1^\infty a_m x^m$ oscillates as $x\to 1$ about the same region as $A_m{}^{(p)}$ as $m\to\infty$.

Proof of theorem 3.

Lemma: *If* $\lim_{x\to 1}\varphi_{k-1}(x)=s$ *and* $\varphi_k(x)+(1-x)\,\varphi_k'(x)=\frac{1}{x}\varphi_{k-1}(x)$ *then* $\lim_{x\to 1}\varphi_k(x)=s$.

Proof of the lemma: If we solve the differential equation we become:

$$\varphi_k(x)=(1-x)\int_0^x \frac{\varphi_{k-1}(x)}{x\,(1-x)^2}\,dx \;+\; C\,(1-x)$$

Since $\lim_{x\to 1}\varphi_{k-1}(x)=s$, it is possible whatever be $\varepsilon>0$ to calculate a number $\xi_1<1$ so that $\xi_1<x<1$ implies:

$$|\varphi_{k-1}(x)-s|<\varepsilon$$

$$(1-x)\,.\int_0^1 \frac{\varphi_{k-1}(x)}{x\,(1-x)^2}\,dx=(1-x)\int_0^{\xi_1}\frac{\varphi_{k-1}(x)}{x\,(1-x)^2}\,dx+(1-x)\int_{\xi_1}^x \frac{\varphi_{k-1}(x)}{x\,(1-x)^2}\,dx$$

$$=(1-x)\int_0^{\xi_1}\frac{\varphi_{k-1}(x)}{x\,(1-x)^2}\,dx+(1-x)\int_{\xi_1}^x\frac{s}{x\,(1-x)^2}\,dx+(1-x)\int_{\xi_1}^x\frac{\varphi_{k-1}(x)-s}{x\,(1-x)^2}\,dx$$

$$=\qquad I\qquad +\qquad II\qquad +\qquad III$$

$\lim_{x\to 1} I=0$; therefore we can calculate a number $\xi_2>\xi_1$ so

that $|I| < \varepsilon$ if $\xi_2 < x < 1$. Further it is possible to calculate $\xi_3 > \xi_2$ so that $|II - s| < \varepsilon$ if $\xi_3 < x < 1$, for we have:

$$II = (1-x)\int_{\xi_1}^{x} \frac{s\,.\,dx}{x(1-x)^2} = (1-x)\,.\,s\left[\log\frac{x}{1-x} + \frac{1}{1-x} - \log\frac{\xi_1}{1-\xi_1} - \frac{1}{1-\xi_1}\right]$$

$$= s + \left[(1-x)\log\frac{x}{1-x} - (1-x)\log\frac{\xi_1}{1-\xi_1} - \frac{1-x}{1-\xi_1}\right] s$$

and the expression between brackets tends to zero as $x \to 1$.

In like manner we can calculate $\xi > \xi_3$ so that $|III| < 2\varepsilon$ if $\xi < x < 1$. Combining these results we have if $\xi < x < 1$:

$$|I| < \varepsilon,\ |II - s| < \varepsilon \text{ and } |III| < 2\varepsilon,$$

therefore:

$$|I + II + III - s| < 4\varepsilon.$$

Since ε is arbitrary and $\lim_{x \to 1}. C(1-x) = 0$ we infer:

$$\lim_{x \to 1}. \varphi_k(x) = s.$$

We now prove theorem III as follows: by hypothesis we have $\lim_{x \to 1}. \varphi_0(x) = \lim_{x \to 1}. \sum_1^\infty a_n x^n = s$; applying the lemma we get:

$$\lim_{x \to 1}. \varphi_1(x) = s \quad ; \quad \lim_{x \to 1}. \varphi_2(x) = s \quad ; \quad \ldots\ldots \lim_{x \to 1}. \varphi_p(x) = s \ ;$$

or:

$$\lim_{x \to 1}. \sum_1^\infty [A_n^{(p)} - A_{n-1}^{(p)}]\, x^n = s$$

Moreover we have by hypothesis:

$$\lim_{n = \infty}. n\,[A_n^{(p)} - A_{n-1}^{(p)}] = 0$$

and therefore by TAUBER's original theorem [1]):

$$\lim_{n = \infty}. A_n^{(p)} = s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

From $\lim_{n = \infty}. n\,[A_n^{(p)} - A_{n-1}^{(p)}] = 0$, (13) and (14) we deduce:

$$\lim_{n = \infty}. [A_n^{(p-1)} - A_{n-1}^{(p)}] = 0$$

Hence by (22)

$$\lim_{n = \infty}. A_n^{(p-1)} = s,$$

Remark 1.

It is not difficult to see that the following statement is an immediate consequence of theorem 3:

---

[1]) $\lim_{n = \infty}. na_n = 0$ and $\lim_{x \to 1}. \sum_1^\infty a_n x^n = s$ imply $\lim_{n = \infty}. \sum_1^n a_i = s$.

Theorem A: *The conditions* $\lim_{x\to 1} \sum_1^\infty a_n x^n = s$ *and* $\lim_{n=\infty} n[A_n^{(p)} - A_{n-1}^{(p)}] = 0$ *are each necessary for the existence of*:

$$\lim_{n=\infty} A_n^{(p-1)} = s,$$

and taken together they are sufficient.

Indeed the necessity of the condition $\lim_{x\to 1} \sum_1^\infty a_n x^n = s$ follows from HÖLDER's theorem mentioned above, and the necessity of $\lim_{n=\infty} n[A_n^{(p)} - A_{n-1}^{(p)}] = 0$ is seen by writing it $\lim_{n=\infty} [A_n^{(p-1)} - A_{n-1}^{(p)}] = 0$ and by observing that $\lim_{n=\infty} A_n^{(p-1)} = s$ implies $\lim_{n=\infty} A_n^{(p)} = s$.

The following particular case of this theorem has been proved by TAUBER [1]):

Theorem B. The conditions $\lim_{x\to 1} \sum_1^\infty a_n x^n = s$ *and*

$$\lim_{n=\infty} \frac{1}{n}(a_1 + 2a_2 + \ldots + na_n) = 0$$

are both necessary for the convergence of $\sum_1^\infty a_n$, *and taken together they are sufficient.*

This may be seen by substituting $p = 1$ in theorem *A*, for:

$$A_n^{(0)} = s_n$$

$$n[A_n^{(1)} - A_{n-1}^{(1)}] = A_n^{(0)} - A_{n-1}^{(1)} = s_n - \frac{s_1 + s_2 + \ldots + s_{n-1}}{n-1} =$$

$$= \frac{1}{n-1}[(n-1)s_n - (s_1 + s_2 + \ldots + s_{n-1})] = \frac{1}{n-1}[(s_n - s_1) + (s_n - s_2) + \ldots + (s_n - s_{n-1})]$$

$$= \frac{1}{n-1}[(a_2 + a_3 + \ldots + a_n) + (a_3 + \ldots + a_n) + \ldots + a_n]$$

$$= \frac{1}{n-1}[(n-1)a_n + (n-2)a_{n-1} + \ldots + a_2]$$

$$= \frac{1}{n-1}\sum_1^n (p-1)a_p,$$

and we may infer the equivalence of the conditions

$\lim_{n=\infty} \frac{1}{n-1}[a_2 + 2a_3 + \ldots + (n-1)a_n] = 0$ and $\lim_{n=\infty} \frac{1}{n}(a_1 + 2a_2 + \ldots + na_n) = 0$

from the equations:

$U(x) = a_1 x + a_2 x^2 + \ldots$; $V(x) = a_2 x + a_3 x^2 + \ldots$; $U(x) = a_1 x + xV(x)$

[1]) BROMWICH, op. cit., p. 251.

A somewhat different generalization of theorem B has been given by A. KIENAST [1]). KIENAST defines:

$$s_n = \sum_1^n a_k \qquad r_n^{(1)} = \sum_1^n k\, a_k$$

$$s_n^{(1)} = \frac{1}{n} \sum_1^{n-1} s_k$$

. . . .

$$s_n^{(\lambda+1)} = \frac{1}{n} \sum_{\lambda+1}^{n-1} s_k^{(\lambda)} \qquad r_n^{(\lambda+1)} = \sum_{\lambda}^{n-1} \frac{1}{k} r_k^{(\lambda)}$$

and proves the following theorem:

Theorem C: The conditions $\lim_{n=\infty} \frac{1}{n} r_n^{(\lambda+1)} = 0$ and $\lim_{x\to 1} \sum_1^\infty a_n x^n = s$ *are each necessary for the existence of* $\lim_{n=\infty} s_n^{(\lambda)} = s$, *and taken together they are sufficient.*

The mean-values $s_n^{(\lambda)}$ differ from CESARÒ's or HÖLDER's mean-values, but in a second paper [2]) KIENAST has shown the equivalence of his mean-values with those of CESARÒ-HÖLDER.

Remark 2.

We have tacitly assumed that $\sum_1^\infty a_n x^n$ converges if $-1 < x < 1$. This is however superfluous for our purpose as the condition $\lim_{n=\infty} n[A_n^{(p)} - A_{n-1}^{(p)}] = 0$ implies the convergence of $\sum_1^\infty a_n x^n$ provided $|x| < 1$.

Indeed from $\lim_{n=\infty} n\,[A_n^{(p)} - A_{n-1}^{(p)}] = 0$ we infer the absolute convergence of $\varphi_p(x) = \sum_1^\infty [A_n^{(p)} - A_{n-1}^{(p)}] x^n$ provided $|x| < 1$.

Further we have by (17):

$$\varphi_{k-1}(x) = x \,.\, \varphi_k(x) + x\,(1-x) \,.\, \varphi_k'(x);$$

therefore the absolute convergence of $\varphi_k(x)$, which implies the absolute convergence of $\varphi_k'(x)$, implies also the absolute convergence of $\varphi_{k-1}(x)$. Repeating the argument we infer the absolute convergence of $\varphi_0(x) = \sum_1^\infty a_n x^n$ provided $|x| < 1$.

[1]) Proceedings of the Cambridge Phil. Soc., vol. 19 (1918), p. 129.

[2]) Proceedings of the Cambridge Phil. Soc., vol. 20 (1920), p. 74.

Chemistry. — "*Hydrogenation of Paraffin by the* BERGIUS' *Method*". By Prof. H. I. WATERMAN and J. N. J. PERQUIN. (Communicated by Prof. J. BÖESEKEN).

(Communicated at the meeting of February 24, 1923).

In a previous communication on the hydrogenation by BERGIUS' method of mineral oils or allied products, different experiments were discussed, which were carried out with heavy Borneo-asphalt-oil, distillation residue (pitch) of this oil, and with asphalt obtained by distillation of Mexican crude oil [1]).

The experiments in question, comprising both cracking- and berginisation experiments, were executed in a vertical immovable autoclave.

That we have now chosen another material, technically perhaps of less importance for this purpose, is owing to the peculiar advantages which commercial paraffin offers for such experiments over other materials, as asphalt. Paraffin is much more easily analysed than asphalt, and this holds also for the products prepared out of paraffin, when they are compared with the corresponding substances formed in the treatment of asphalt. Thus paraffin yields products that are less strongly coloured than Mexican asphalt. For these experiments we had an autoclave at our disposal which could be shaken continuously [2]).

The way of procedure was for the rest quite analogous to the earlier experiments; the arrangement of the apparatus is represented in fig. 1. The capacity of the autoclave was about 2500 cm.^{3}, the heating took place by means of gas, in such way that the temperature could be regulated accurately to a few degrees.

The paraffin had a Sp. Gr. ($15°/15°$) of 0,913, the solidifying point (SHUKOFF method) was 50,6°, the bromine-value, (addition) determined by Mc. ILHINEY's method [3]), was 0,5.

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[1]) Congrès international des combustibles liquides, Paris, 9—15 Octobre 1922; Chimie et Industrie, numéro spécial, Mai 1923, p. 200.

[2]) Apparatus supplied by ANDREAS HOFER, chief instrument-maker at the laboratory of Prof. Dr. FRANZ FISCHER, Kaiser Wilhelm Institut für Kohlenforschung, Mülheim—Ruhr.

[3]) Journ. Am. Chem. Soc. **16**, 275 (1894), **21**, 1084 (1899), Journ. Soc. Chem. Ind. **19**, 320 (1900); H. BECKURTS, Die Methoden der Massanalyse, Braunschweig 1913, p. 480.

Practically the bromine value of the paraffin may, therefore, be neglected. The bromine-value determined according to Mc. Ilhiney's method, is obtained by subtracting the substituted bromine from the

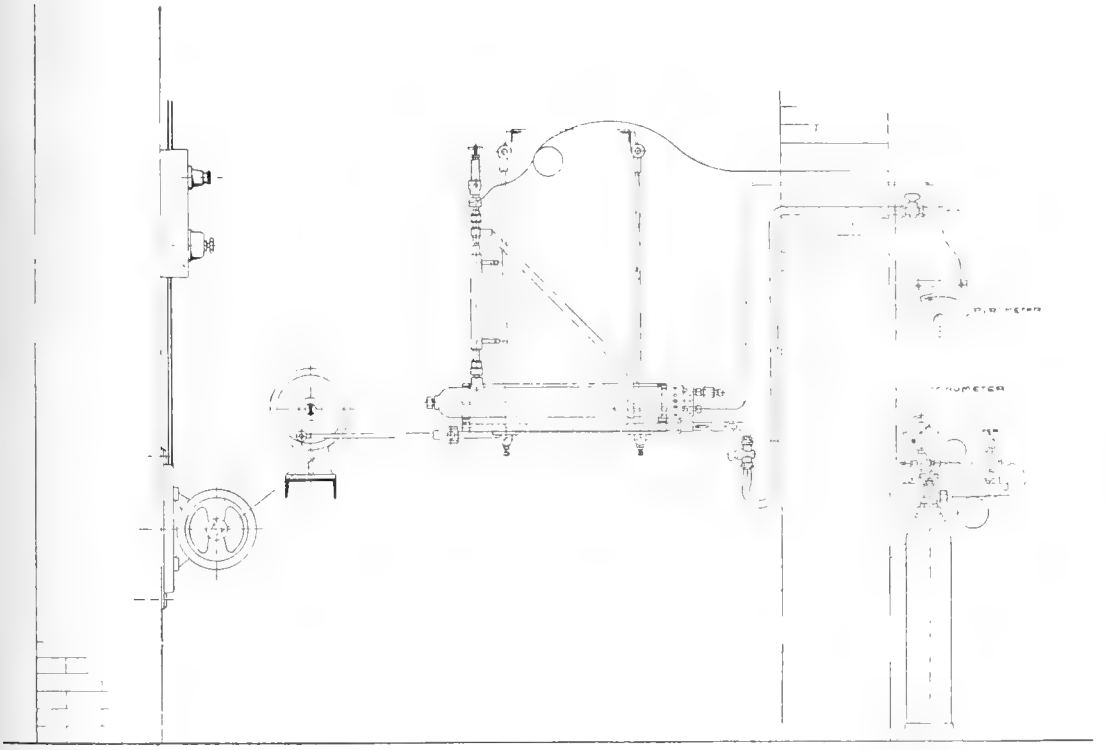
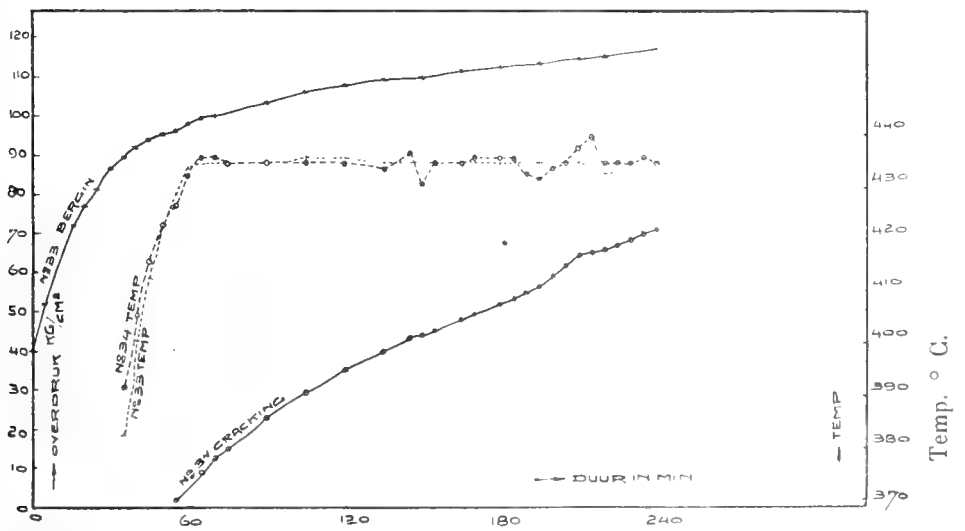


Fig. 1.



Overdruk = Pressure Duur in min. = Time in minutes

Fig. 2.

total amount of the absorbed bromine. The remaining quantity gives a measure of the degree of unsaturation, and is expressed in percentages of weight of the weighed quantity.

In every experiment 300 gr. of paraffin was taken, an equal weight of stones being put in the autoclave to promote a thorough mixing; the temperature was always 435°. Some of the results obtained are recorded in the table, and in fig. 2 an illustration is given of the variation of the pressure in the course of experiments 33 and 34. Though in experiment 33 the typical pressure curve according to BERGIUS given in our preceding communication is not obtained, probably on account of the high temperature, the difference from the cracking-pressure curves is nevertheless very striking. In all the other experiments recorded in the table the pressure curves obtained are analogous to those of 33 and 34. The oils obtained by the BERGIUS' process were coloured from yellow to red, and perfectly transparent, a small quantity of "carbon" was deposited on the bottom. The oils obtained in cracking were very dark of colour and pretty well opaque. Here too separation of some carbon is found. The small quantity of carbon which is deposited on the bottom, when the weight of carbon which had already been deposited on the stones is added, is so small, both in the cracking and in the BERGIUS' method, that practically *the paraffin may be assumed to have been entirely converted into oil and gas in both processes.*

In this we leave out of consideration experiments 35, 37, and 40, where the duration of the processes was still so short that the reaction product had remained partially solid. Hence the product obtained had to be melted out in these experiments.

It appears from the experiments made that,

1. observations can be obtained which can be perfectly reproduced (compare 35 and 37, and 46 and 48).

2. if the duration of the experiments is long enough, the paraffin is practically quite converted into liquid oil and gas, both in the cracking and in the BERGIUS' process.

3. the yield of gasoline does not differ much in the two processes.

4. there is a great difference in the nature of the residues left in the distillation of the oil obtained according to ENGLER. Its specific gravity is always smaller in the Berginisation experiments than in the corresponding cracking experiments, which is a confirmation of corresponding experiments made by BERGIUS.

5. It appears from the final pressure, also in connection with the gas analysis (percentage of hydrogen), that actually considerable quantities of hydrogen are absorbed in the berginisation.

COMPARISON OF BERGINISATION AND CRACKING AT ± 435° C.

N°. of the experiment		Time required to reach the temperature in minutes	Duration of the proper experiment in minutes	Initial pressure in atmospheres	Maximum pressure during the experiment	Final pressure after complete cooling	Obtained oils in grammes	Bromine content (addition)	Distillation of the obtained oil according to ENGLER					Gases.		
									Weight % of the distilled oil.				Sp. G. residue 15°/15°.	Quantity in Litres	Sp. Gr. compared with air	Hydrogen percentage.
									to 220°	to 300°	residue >300°	Loss [2]				
35	Berginisation	60	60	40	110.5	37.5	260 [1]	16.4	24.9	41.0	56.3	2.7	0.846	—	0.24	85.8
37	Berginisation	60	60	40	108.5	37.5	272 [1]	16 1	22.8	37 9	59.0	3.1	0.854	·	0.20	89.5
36	Berginisation	60	120	40	107	31	272	19.0	36.6	56.6	38.7	4.7	0.838	—	0.37	74.6
33	Berginisation	60	180	40	117	30	256	20.8	51.7	72.1	22.4	5.5	0.852	—	0.56	56.9
46	Berginisation	75	240	40	118	28	250	21.0	58.9	79.6	14.8	5.6	0.836	63	0.63	47.5
48	Berginisation	75	240	40	120	28	249	20.7	59.7	79.2	13.8	7.0	0.838	62.5	0.63	46.5
40	Cracking.	80	60	0	30	± 2	271 [1]	21.6	23.0	39.0	60.1	0.9	0.854	—	0.99	—
45	Cracking.	70	120	0	51.5	< 4	270	27.6	41.9	63.7	32.0	4.3	0.855		0.80	—
34	Cracking.	60	180	0	71	5	246	26.1	56.1	76.5	17.5	6.0	0.900	—	1.10	2.3
49	Cracking.	75	240	0	72	7.5	238	23.9	56.8	76.9	16.2	6.9	0.902	29	0.94	3.7

Atmospheric pressure

[1]) The product obtained was still solid and had to be melted out, which gave rise to extra losses of weight.
[2]) Belongs to the lowest fraction.

6. The bromine value caused by addition of the oils obtained by berginisation is lower than that of the corresponding cracking experiments. It is, however, very risky to draw general conclusions from this bromine value, for dissolved unsaturated gases can have a great influence on the halogen value.

The example given here proves convincingly that a determination of the yield of oil and gas from a solid substance does not suffice to enable us to form a correct opinion on the process of BERGIUS. A comparative cracking experiment is required for this. Possible results refer only to the procedure followed, in this case to the periodic process, the temperature at the experimenting etc.

It is self-evident that in practice processes that proceed continuously, will be preferred. It may, however, be considered to be an established fact that when BERGIUS' method of procedure is followed, important quantities of hydrogen added from the outside, are chemically bound. After the scientific researches of SABATIER c.s. concerning the hydrogenation of hydro carbonic vapours with catalyst and the technical hardening of fatty oils (NORMANN and others), this fact, combined with the absence of express addition of catalyst, may be considered as the third great discovery in the region of hydrogenation.

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Palaeontology. — "*Contributions to our Knowledge of the Palaeontology of the Netherlands. I. Otoliths of Teleostei from the Oligocene and the Miocene of the Peel-district and of Winterswijk.*" By O. POSTHUMUS. (Communicated by Prof. J. C. SCHOUTE).

(Communicated at the meeting of February 24, 1923).

As regards the fish-fauna of the tertiary deposits in the Netherlands the occurrence has been reported of a number of Selachii in the Oligocene of South-Limburg [1]), and of the Miocene of East-Gelderland [2]) and Overijssel [3]). No remains had as yet been found of Teleostei.

We are in a position to form an idea of the fish-fauna in the North Sea of Miocene time, from a number of otoliths occurring in material obtained from borings, undertaken by the Government (Institute for the Geol. Exploration of the Netherlands) on the Southern Peelhurst, notably from boring 20 (Helden) of the Middle-Miocene (75.4—80.4 m.), from boring 21 (Swalmen) of the Upper-Oligocene (100—160 m.), and of the Middle-Miocene (75—100 m.); likewise in material originating from boring 22 (Liessel) also of Middle-Miocene date (100—190 m.).

Moreover the test-boring U near Winterswijk, placed at my disposition some otoliths from the Septarian clay, and from the Middle-Miocene, laid bare in the bed of Slingerbeek near Winterswijk.

The following specimens have been found [4]):

Oligocene.

Middle-Oligocene (Septaria clay), Winterswijk.

Otolithus (Scopelus) pulcher, Prochazka.

[1]) W. C. H. STARING De bodem van Nederland, 2e deel, Haarlem, 1860, p. 282.

[2]) Ibid, p. 209, 210.

[3]) T. C. WINKLER. Catalogue systématique du Musée Teyler, 6me livr. 1867, p. 624.

[4]) They will before long be figured and described in a more detailed memoir.

Upper-Oligocene, Swalmen.

Otolithus (Dentex) nobilis, Koken.
„ (Percidarum) limburgensis, nov. spec.
„ (Trachinus) mutabilis, Koken.
„ (Trigla) Schuberti, nov. spec.
„ (Scopelus) austriacus, Koken.
„ (Scopelus) pulcher, Prochazka.
„ (Gonostoma?) parvulus, Koken.
„ (Gonostoma?) angustus, nov. spec.
„ (Fierasfer) muntius, Koken.
„ (Gadus) elegans, Koken.
„ (Merlangus) cognatus, Koken.

Miocene.

Middle-Miocene, Swalmen.

Otolithus (Percidarum) frequens, Koken.
„ (Trachinus) mutabilis, Koken.
„ (Trigla) rhombicus, Schubert.
„ (Gobius) aff. elegans, Prochazka.
„ (Ophidiidarum) semiglobosus, nov. spec.
„ (Ophidiidarum) swalmensis, nov. spec.
„ (Gonostoma?) parvulus, Koken.
„ (Solea) approximatus, Koken.
„ (Rhombus) rhenanus, Koken.
„ (incertae sedis) peelensis, nov. spec.

Middle-Miocene, Helden.

Otolithus (Serranus) Noetlingi, Koken.
„ (Centropristis) integer, Schubert.
„ (Dentex) nobilis, Koken.
„ (Percidarum) acuminatus, nov. spec.
„ (Trigla) Schuberti, nov. spec.
„ (Sciaenidarum) Staringi, nov. spec.
„ (Gonostoma) aff. gracilis, Prochazka.
„ (Clupea) testis, Koken.
„ (Clupea) Priemi, nov. spec.
„ (Gadus) elegans, Koken.
„ (Phycis) elongatus, nov. spec.
„ (incertae sedis) Mariae, Schubert.
„ (incertae sedis) peelensis, nov. spec.

Middle-Miocene, Liessel.

Otolithus (Dentex) nobilis, Koken.
,, (Percidarum) frequens, Koken.
,, (Percidarum) Liesselensis, nov. spec.
,, (Scopelus) austriacus, Koken.
,, (Scopelus) pulcher, Prochazka.
,, (Gonostoma?) parvulus, Koken.
,, (Clupea) testis, Koken.
,, (Fierasfer) nuntius, Koken.
,, (Gadus) elegans, Koken.
,, (Merluccius) emarginatus, Koken.
,, (Phycis) elongatus, nov. spec.
,, (Hymenocephalus) globosus, nov. spec.
,, (Hymenocephalus) medius, nov. spec.
,, (Hymenocephalus) ovalis, nov. spec.
,, (Hymenocephalus) Brinki, nov. spec.
,, (Hymenocephalus) dubius, nov. spec.
,, (Macrurus) pusillus, nov. spec.
,, (Macrurus) ellipticus, Schubert.
,, (Macrurus) debilis, nov. spec.

Middle-Miocene, Winterswijk.

Otolithus (Gadus) elegans, Koken.

The fauna of the Upper-Oligocene ot Swalmen is characterised by the absence of littoral forms; the fishes that occur, inhabit the deeper and more open parts of the sea, as e.g. Dentex, especially in the upper water-layers, or the Scopelidae, especially at greater depth. The depth may have been somewhere about 400 m. at a moderate distance from the shore. This tallies with the known data, as the Upper-Oligocene is represented in erosion-rests as far as the line Liege—Aachen—Cologne.

From Middle-Miocene data are known from localities on the Southern Peelhurst, lying in one line, that is about straight and runs about S.E.—N.W. In the South-most of these three localities, near Swalmen, the genera Rhombus, Solea and Gobius are conspicuous. They are all littoral forms, and not met with in the material of Helden, about 20 km. farther, where, however, Clupea, Serranus, and Dentex occur; these fishes we also find naar Liessel, about 18 km. farther in Noord-Brabant, where, however, Macruridae and Scopelideae predominate in the material. Judging from the remains of fishes Swalmen is not far from the ancient coastline; in the vicinity of

Helden the fauna resembles closely that of a moderately deep sea, while the remains of Macruridae, occurring in the material of Liessen, originate from deep-sea forms, so that here we have to assume a greater depth of about 1000 m. This conclusion is in accordance with the results of the inquiries of the Government Institute for the Geological Exploration of the Netherlands: the boundary-line between the continental and the marine Miocene runs about via Swalmen; the lignite formation occurs near Melick-Herkenbosch and Vlodrop, while in the profile of boring 21 the lowermost layers of the Miocene are marine, and the upper layers display a limnic facies. It seems to me that a closer inspection of material from the Groote Slenk, southwest of the Peelhurst, would be very interesting.

The tertiary fauna of this region differs from the recent fauna of the North-Sea: on the one side forms occur that inhabit greater depths than those living in the North Sea at the present day, such as Scopelidae and Macruridae, which occasionally occur at high latitudes in the Atlantic Ocean; on the other side the tertiary fauna comprises genera such as Dentex, Centropristis and Serranus, now living at lower latitudes. In my judgment the occurrence of the latter points to a change of environment, which is to be ascribed either directly to a change of climate, or to other conditions, e.g. an altered direction of the oceanic currents.

In conclusion I wish tot express my warm thanks to Prof. Dr. J. H. Bonnema for kindly placing at my disposal the material in the Geological-Mineralogical Institute of the State University of Groningen.

Palaeontology. — "*Contributions to our Knowledge of the Palaeontology of the Netherlands*". II. "*On the Fauna of the Phosphatic Deposits in Twente. (Lower Oligocene)*" By O. POSTHUMUS. (Communicated by Prof. J. F. VAN BEMMELEN).

(Communicated at the meeting of March 24, 1923).

In examining a collection of fossils, derived from the phosphatic-nodulus-bearing deposits of the localities Ootmarsum and Rossum (between Oldenzaal and Denekamp) I came upon the following formations:

Coeloma balticum SCHLÜTER, Zeitschrift der deutschen Geol. Ges. Bd. 31, 1879, p. 604, Pl. XVIII; one specimen.

Myliobates toliapicus L. AGASSIZ, Recherches sur des Poissons fossiles, vol. 3, 1843, p. 321, tab. 47, fig. 15—20; loose toothplates.

Carcharodon angustidens L. AGASSIZ, Recherches etc., vol. 3, 1843, p. 255, tab. fig. 20—25, tab. 30, fig. 3: teeth.

Notidanus primigenius L. AGASSIZ, Recherches etc., vol. 3, 1843, p. 218, tab. 27, fig. 4—8, 13—17; teeth.

Oxyrhina Desori (L. AGASSIZ) SISMONDA, Memoria della Reale. Accademia delle Science de Torino, 2d series, t. X, 1849, p. 44, tab. II, fig. 7—16; teeth.

Oxyrhina Desori L. SISMONDA mut. flandrica, M. LERICHE, Mémoires du Musée Royal d'histoire naturelle de Belgique, T. 5, p. 280, fig. 87; vertebrae.

Odontaspis cuspidata L. AGASSIZ, Recherches etc., vol. 3, 1843, p. 294, tab. 37, fig. 43—49; teeth.

Otodus obliquus L. AGASSIZ, Recherches etc., vol. 3, 1843, p. 267, tab. 31, tab. 36, fig. 22—27; teeth.

Lamna spec., vertebrae.

Phyllodus polyodus L. AGASSIZ, Recherches etc., vol. 2, 1843, p. 240, tab. 69a, fig. 6, 7;

And in addition some fragments of bone, presumably from Cetacea.

The phosphatic deposits are disposed in the profile as follows [1]):

"Underlying the Middle-Oligocene Septarian clay are ..... pale-green, very fine glauconite sands, probably referable to Lower-Oligocene, but seeming to belong to the Middle-Oligocene. At the basis of these sands a very typical conglomerate layer of loosened phosphorite nodules and shark's teeth appears, as may be found e. g. in the eocene quarries at the southern base of Lonnekerberg in the neighbourhood of Rossum, between Oldenzaal and Denekamp, and in the hills north of Ootmarsum". The phosphatic deposits

[1]) Eindverslag van de Rijksopsporing van Delfstoffen. Amsterdam, 1918, p. 114.

therefore may be estimated to be of Lower-Oligocene date; at all events they must have been formed at the commencement of the Oligocene transgression.

These formations are best compared with the Oligocene phosphatic deposits of the North-German Plain, of which those from Helmstedt have become familiar to us through the researches of Von Koenen and H. B. Geinitz [1]). It appears that all the fossils found in Twente, except Oxyrphina Desori, are also to be found near Helmstedt, which proves that the two deposits are equivalent.

This induces me to put forward some remarks about the formation of phosphatic nodules. Most authors advocate the view that the more or less rounded shape of these bodies is to be attributed to transportation, which view is adhered to by recent observers, as shown by the "Eindrapport" from which we just now quoted a passage. We contend that the nodules, in many cases, are not rounded, but more or less irregular, nay, as Staring [2]) observes, they often seem to be made up of two or more rounded nodules. The shark's teeth are in many cases enclosed in an approximately rounded phosphatic nodule: the portion that is sticking out, however, is not worn off at all, which fact clashes with the presumable genesis. H. B. Geinitz assumed the transport of the nodules to have taken place in the Recent Tertiary and based this view on the fact of their presence in the layers of Myliobates and of Lamna cuspidata, which he had examined, and which up to that time had been recognized only in the Pliocene. Now, this cannot apply to the Overijsel phosphatic deposits, in which these remains have also been met with, because the younger deposits of the Oligocene also occur here. The palaeontological argument that the rounded shape is attributable to rolling cannot be sustained. We are bound to assume that after the formations of the phosphate-concretions, the position of the deposits remained unaltered, which conception has been supported already by Dr. W. P. A. Jonker [3]) on other grounds.

I wish to conclude by gratefully acknowledging my indebtedness to Mr. J. Bernink, Director of the Museum "Natura Docet" at Denekamp, for granting me access to the fossils collected by him.

[1]) H. B. Geinitz, Die sogenannten Koproliethenlager von Helmstedt, Büddenstedt und Schleweke bei Harzburg. Abhandlungen der Naturwiss. Geselschaft „Isis" in Dresden. 1883, p. 3–14.

H. B. Geinitz, Ueber neue Funde in den Phosphatlagern von Helmstedt, Büddenstedt und Schleweke. Isis, 1883, p. 37—46.

[2]) W. H. C. Staring, De bodem van Nederland. 2e deel. Haarlem, 1860, p. 195.

[3]) W. P. A. Jonker, Het ontstaan van phosphorieten. Handelingen van het 17e Natuur- en Geneeskundig Congres, 1920, p. 94—96.

Mathematics. — "*An application of the theory of integral equations on the determination of the elastic curve of a beam, elastically supported on its whole length*". By Prof. C. B. BIEZENO. (Communicated by Prof. J. C. KLUIJVER).

(Communicated at the meeting of March 24, 1923).

In his well-known treatise „Vorlesungen über Technische Mechanik" (Vol. III, § 48) FÖPPL describes a construction, by which the elastic curve of a beam, elastically supported on his whole length, might been approximated.

If in the differential equation of this elastic curve

$$E\,I\,y'''' + k\,y = q$$

($E\,I$ = coefficient of stiffness of the beam, k = coefficient of stiffness of the supporting ground, q = specific continuous loading) the function y where known, it would be possible to refind this function by integrating four times the expression $\frac{q-ky}{E\,I}$.

This integration would graphically correspond to the construction of the elastic curve of a beam, which carries only well-known forces.

It is obvious, therefore, first to make a supposition about the elastic curve — in such a way, of course, that the reaction-forces of the supporting ground will be in equilibrium with the external forces of the beam —, then to integrate graphically the expression $\frac{q-ky}{EI}$, and finally to controll, if the before-mentioned accordance takes place.

„Im allgemeinen — such is the opinion of FÖPPL —wird man zunächst eine starke Abweichung in der Gestalt beider Kurven finden. Dann ändert man die zuerst gezeichnete Belastungsfläche so ab, dasz sich die Lastverteilung jetzt der Gestalt der gefundenen elastischen Linie nähert und wiederholt das Verfahren für diese zweite Annahme. Die Uebereinstimmung zwischen Belastungsfläche und zugehöriger elastischen Linie wird jetzt besser werden und nach mehrmaliger Wiederholung findet man mit hinreichender Genauigkeit die wirkliche Druckverteilung."

Certainly it will be possible, — under favourable conditions — to find in this way technical sufficient accordance between the supposed curve and the one, derivated from it; but generally the convergency of the described process is uncertain.

In the following paper a convergent process will be given.

2. The equation

$$E\,I\,y'''' + k\,y = q$$

is transformed in

$$y'''' + k'\,y = q'$$

if $\frac{k}{EI} = k'$, $\frac{q}{EI} = q'$.

Putting $y'''' = \varphi(x)$ it becomes:

$$\varphi(x) + k' \int_0^x \varphi(x)\,dx^4 = q' + Ax^3 + Bx^2 + Cx + D$$

or, using the well-known relation

$$\int_0^x \varphi(x)\,dx^4 = \int_0^x \frac{(x-s)^3}{3!}\,\varphi(s)\,ds$$

$$\varphi(x) + k' \int_0^x \frac{(x-s)^3}{3!}\,\varphi(s)\,ds = q' + Ax^3 + Bx^2 + Cx + D.$$

A, B, C and D are constants of integration, which enable us to satisfy the following conditions:

1°. $y'' = 0$, $y''' = 0$ at $x = 0$.

2°. $y'' = 0$, $y''' = 0$ at $x = l$.

The former conditions imply, as is seen from the relation

$$y = \int_0^x \varphi(x)\,dx^4 - \frac{Ax^3 + Bx^2 + Cx + D}{k'}$$

that the coefficients A and B are zero. The coefficients C and D are determinated by the latter conditions.

3. According to VOLTERRA the solution of the integralequation

$$\varphi(x) + k' \int_0^x \frac{(x-s)^3}{3!}\,\varphi(s)\,ds = q' + Cx + D$$

may be written as:

$$\varphi(x) = \varphi_0(x) + k'\varphi_1(x) + k'^2\varphi_2(x) + k'^3\varphi_3(x) + \ldots .$$

where

$$\varphi_0(x) = q' + Cx + D$$

$$\varphi_1(x) = -\int_0^x \frac{(x-s)^3}{3!}\varphi_0(s)\,ds$$

$$\varphi_2(x) = -\int_0^x \frac{(x-s)^3}{3!}\varphi_1(s)\,ds$$

$$\vdots$$

$$\varphi_n(x) = -\int_0^x \frac{(x-s)^3}{3!}\varphi_{n-1}(s)\,ds.$$

This solution however can only graphically be used, if the coefficients C and D are known. Nevertheless this coefficients depend on the second and first integral of $\varphi(x)$ in a point which is different from zero. Therefore we cannot find them a priori.

4. To meet this difficulty, we introduce the function

$$\chi_0(x) = q' + C_0 x + D_0;$$

C_0 and D_0 being two constants, determinated by:

$$\int_0^l \chi_0(x)\,dx = 0$$

$$\int_0^l \chi_0(x) \cdot x\,dx = 0.$$

By choosing C_0 and D_0 in this manner, we reach that 1°. C_0 and D_0 can easily be graphically found, and 2°. that the function

$$\overline{\varphi_1}(x) = -\int_0^x \frac{(x-s)^3}{3!}\chi_0(s)\,ds$$

satisfies at the point $x = l$ the conditions

$$\overline{\varphi_1}''' = 0, \quad \overline{\varphi_1}'' = 0,$$

or the conditions

$$\int_0^l \chi_0(x)\,dx = 0, \quad \int_0^l dx \int_0^x \chi_0(x)\,dx = 0$$

For:

$$\overline{\varphi''}_1(x)_{x=l} = -\int_0^l dx \int_0^x \chi_0(x)\,dx = \Big\} - x\int_0^x \chi_0(x)dx \Big\{_0^l + \int_0^l x\,\chi_0(x)\,dx = 0.$$

If we should deduce the function $\overline{\varphi_2}(x)$ from $\overline{\varphi_1}(x)$, in the manner which VOLTERRA indicates, the second and third derivates of $\overline{\varphi_2}(x)$ would not be zero at the point $x = l$. Therefore we define the function

$$\chi_1(x) = -\left[\int_0^x \frac{(x-s)^3}{3!}\chi_0(s)\,ds + C_1 x + D_1\right]$$

C_1 and D_1 being constants determinated by

$$\int_0^l \chi_1(x)\,dx = 0$$

$$\int_0^l \chi_1(x)\,.\,x\,dx = 0.$$

In this way, the second and third derivates of $\chi_1(x)$ take at the points $x = 0$ and $x = l$ the prescribed values; on the other hand fore-fold integration of $\chi_1(x)$ gives rise to a function, the second and third derivates of which are at the point $x = l$ also equal to zero.

This being stated, we are lead to define the series of functions

$$\chi_0(x) = q' + C_0 x + D_0$$

$$\chi_1(x) = -\left[\int_0^x \frac{(x-s)^3}{3!}\chi_0(s)\,ds + C_1 x + D_1\right]$$

$$\chi_2(x) = -\left[\int_0^x \frac{(x-s)^3}{3!}\chi_1(s)\,ds + C_2 x + D_2\right]$$

$$\vdots$$

$$\chi_n(x) = -\left[\int_0^x \frac{(x-s)^3}{3!}\chi_{n-1}(s)\,ds + C_n x + D_n\right]$$

where the coefficients C_i and D_i are bound by the conditions

$$\int_0^l \chi_i(x)\,dx = 0$$

$$\int_0^l \chi_i(x)\,.\,x\,dx = 0$$

and to put

$$\varphi = \chi_0(x) + k'\chi_1(x) + k'^2\chi_2(x) + \dots .$$

This function satisfies formally the equation

$$\varphi(x) + k' \int_0^x \frac{(x-s)^3}{3!}\varphi(s)\,ds = q' + Cx + D$$

and the expression y, which follows from it:

$$y = \frac{q' - \varphi}{k'} = \frac{q' - (q' + C_0 x + D_0) - k'\chi_1(x) - k'^2\chi_2(x) - \dots}{k'} =$$

$$= -\frac{C_0 x + D_0}{k'} - \chi_1(x) - k'\chi_2(x) - k'^2\chi_3(x) \dots$$

satisfies formally the conditions, imposed at the ends $x = 0$ and $x = l$.

For, substituting the expression φ in the integral equation we obtain — provided that it be allowed to integrate term by term the series, which occurs under the sign of integration:

$$C_0 x + D_0 - k'(C_1 x + D_1) - k'^2(C_2 x + D_2) - \dots = Cx + D.$$

If the series, which appears in the first member of this equation, converges, there can be disposed of the constants C and D in such a manner, that the equation becomes an identity.

Of course it would now be necessary to examine the convergency of the described process of iteration.

For this investigation however we refer to the paper of Mr. J. Droste, which follows this. We will state here only, that convergency is sure, if $\frac{kl^4}{EI} < 500$, and go on to demonstrate in which manner the process can be graphically performed.

5. At the first place the system of forces, which loads the beam, is substituted by another load, changing linearly, ($q_0 = \alpha x + \beta$), and which is statically equivalent with the first.

This substitute load causes a sinking down of the beam, determinated by

$$y_0 = \frac{\alpha x + \beta}{k}.$$

This y_0 can be considered as the first approximation of the required y, and can be brought in relation with the expression $C_0 x + D_0$, which is defined in N°. 3.

Indeed, $\alpha x + \beta$ satisfies the equations

$$\int_0^l (\alpha x + \beta)\, dx = \int_0^l q\, dx$$

$$\int_0^l (\alpha x + \beta) \,.\, x\, dx = \int_0^l q \,.\, x\, dx$$

on the contrary $C_0 x + D_0$ is defined by

$$\int_0^l (C_0 x + D_0)\, dx = -\int_0^l q' dx = -\int_0^l \frac{q}{EI} dx$$

$$\int_0^l (C_0 x + D_0) \,.\, x\, dx = -\int_0^l q' \,.\, x\, dx = -\int_0^l \frac{q}{EI} \,.\, x\, dx.$$

It follows, that $\alpha x + \beta \equiv - EI (C_0 x + D_0)$, so that:

$$y_0 = \frac{\alpha x + \beta}{k} = -\frac{C_0 x + D_0}{k}.$$

The load which really charges the beam differs from the substitute load by:

$$q_1 = q - q_0 = q - (\alpha x + \beta) = EI (q' + C_0 x + D_0) = EI \chi_0 (x).$$

By adding this load (which is in equilibrium) to the load q_0, we would regain the real conditions of loading.

However, if we add the load q, the beam gets a deflexion y_1, determinated by:

$$E I y_1'''' = E I \chi_0 (x)$$

Hence:

$$y_1 = \int_0^x \chi_0 (x)\, dx^4 = \int_0^x \frac{(x-s)^3}{3!} \chi_0 (s)\, ds + A_1 x^3 + B_1 x^2 + C_1 x + D_1.$$

The second and third derivates of y_1 being zero for $x = 0$, it follows that $A_1 = 0$, $B_1 = 0$.

Choosing C_1 and D_1 so that:

$$\int_0^l y_1\, dx = 0$$

$$\int_0^l y_1 \,.\, x\, dx = 0.$$

we identify y_1 and $-\chi_1 (x)$.

At the same time, the forces, defined by ky_1, are in equilibrium.

If the elastic ground were loaded with ky_1, it would obtain the the deflexion y_1. In this case the beam and the ground would have the same shape. However the load on the ground can only arise from the beam. The deflexion y on the ground therefore involves necessarily a reaction-load $-ky_1$ on the beam.

This latter load gives rise to another deflexion y_2 of the beam, defined by:

$$E\,I\,y_2'''' = -\,k\,y_1 = k\,\chi_1\,(x)$$

Hence

$$y_2 = k'\left\{\int_0^x \frac{(x-s)^3}{3!}\chi_1\,(s)\,ds + C_2\,x + D_2\right\}.$$

If we require again that the load ky_2, which follows from y_2, is in equilibrium, we find that:

$$y_2 = -\,k'\,\chi_2\,(x).$$

From this, we deduce $y_3 = -\,k'^2\,\chi_3(x)$ and so on. Therefore, the terms of the series:

$$y = -\,\frac{C_0\,x + D_0}{k'} - \chi_1\,(x) - k'\,\chi_2\,(x) - k'^2\,\chi_3\,(x)\ldots$$

represent elastic curves of a beam, which is loaded in a well-defined manner.

6. Fig. 1 illustrates the described construction in the case: $l = 200$ cm., b = breath of the beam $= 25$ cm., $I = 5000$ cm^4, $E = 100000$ kg/cm^2; $EI = 5 \times 10^8$ kg.cm^2, $\bar{k} = 5$ kg/cm^3, $k = b\bar{k} = 125$ kg/cm^2. The load diagram has a parabolic form; the specific load at the ends of the beam is $^1/_4$ of its value at the middle. The total load is 15000 kg. The scale of length in horizontal direction is $n = 5$ (1 cm $\longleftrightarrow$ means 5 cm $\longleftrightarrow$).

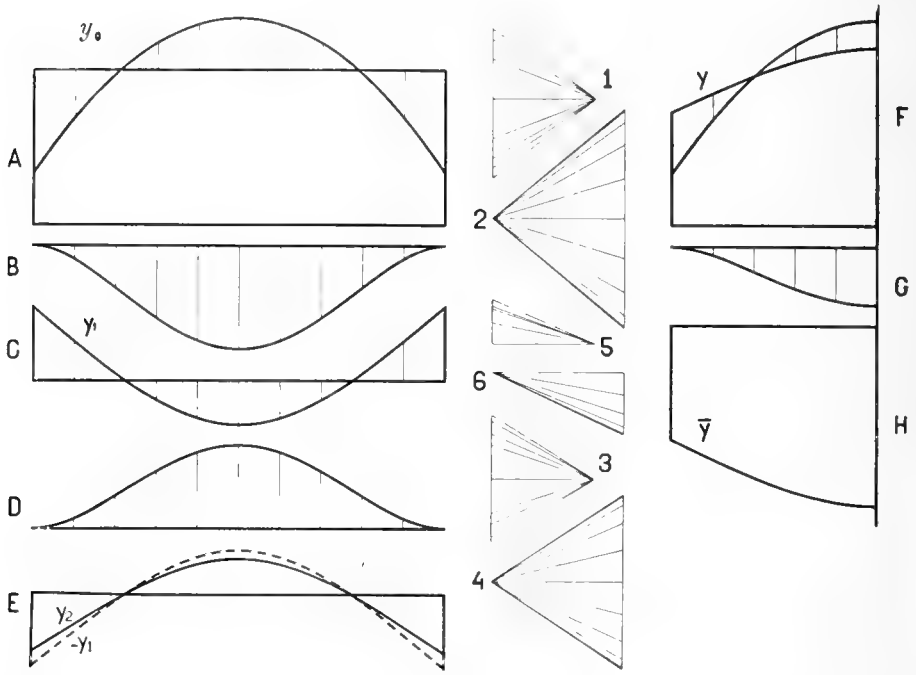
The deflexion are 25 times enlarged; 1 cm. $\updownarrow$ represents $^1/_{25}$ cm $\updownarrow$.

The linear load q_0, which is statically equivalent to the given load q, will give a sinking down to the beam, which is:

$$y_0 = \frac{15000\ \text{KG}}{125\ \text{KG/cm}^2 \times 200\ \text{cm}} = 0{,}6\ \text{cm}.$$

This sinking down is represented in figure 1a by 25 · 0,6 cm. = 15 cm.; and gives rise to the straight line y_0. This line also

represents, when the scale is altered, the load q_0; in this case 1 cm. $\updownarrow$ must be interpreted as $\frac{15000 \text{ kg.}}{200 \times 15 \text{ cm}} = 5$ kg/cm (say m_1 kg/cm).



On this scale the parabolic load q has been drawn in fig. 1*a*, so that the load $q—q_0$, — which determines the elastic curve y_1 — is represented in fig. 1*a* by the hatched area.

In the well-known manner the elastic curve y_1, which corresponds to the load $q—q_0$, is constructed (see figures 1*b* and 1*c* with the corresponding pole figures 1 and 2).

To determine the situation of the pole in the second pole figure, we make the following remarks.

In figure 1*a* 1 cm. $\longleftrightarrow$ represents n cm. $\longleftrightarrow$; 1 cm. $\updownarrow$ represents m_1 kg/cm. Therefore 1 cm² of fig. 1*a* represents nm_1 kg.

Assuming now that in the first pole figure 1 cm. (whether $\longleftrightarrow$ or $\updownarrow$) will represent m_2 cm² of figure 1*a* (in the drawing m_2 is supposed to be 5) and that the first pole distance has a length of H_1 cm (in the drawing 10 cm), we see that H_1 represents $m_1 m_2 n H_1$ kg.

Hence 1 cm. $\updownarrow$ in fig. 1*b* represents $m_1 m_2 n^2 H_1$ kg. cm. Consequently

the unity of area in fig. 1*b* means in the next integration $\frac{m_1 m_2 n^3 H_1}{EI}$ units.

The second pole distance H_2 therefore represents $\frac{m_1 m_2 m_3 n^3 H_1 H_2}{EI}$ units, if we suppose that 1 cm. of this distance represents m_3 cm² (in the drawing 10 cm²) of the area in fig. 1*b*.

From all this it follows finally that 1 cm. ↕ in fig. 1*c* represents

$$\frac{m_1 m_2 m_3 n^4 H_1 H_2}{EI} \text{ cm.}$$

Now the elastic curves y_1 and y_0 must been drawn on the same scale; hence:

$$\frac{m_1 m_2 m_3 n^4 H_1 H_2}{EI} = {}^1/_{25}$$

$$H_2 = \frac{1}{25} \frac{EI}{m_1 m_2 m_3 n^4 H_1} = 12{,}8 \text{ cm.}$$

The elastic curve y_1 once found, the drawing process is to be repeated so many times, that the last approximations may be neglected. By adding the different curves $y_0, y_1, y_2 \ldots$ we obtain the elastic curve y. The final result can be controlled as follows. We load the beam at the one side by the well-known external forces, at the other side by the continuous load ky, which follows from the elastic curve y. Then we construct the elastic curve $\bar{y}$. If the result y were exact, the curves y and $\bar{y}$ must be identical. Fig. 1*f, g, h* shows, that a difference between the curves y and $\bar{y}$ cannot be observed.

7. Considering fig. 1, it appears that the ordinates of the curves y_2 and y_1 are proportional. If the factor of proportionality is called $-\mu$, so that $y_2 = -\mu y_1$, it is easily seen that the ordinates of the curve y_3 can be written as $-\mu y_2$ and so on.

The ordinates $y_1, y_2, \ldots y_n$ at any point can therefore been looked upon as terms of a geometrical series and the curve y can be obtained by adding y_0 to the sum of all the following approximations.

Not only when the factor of proportionality μ is < 1, but also when $\mu > 1$, it may occur that the described drawing process is useful to find the elastic curve.

Supposing that the load $-ky_n$ gives rise to the deflexion $-\mu y_n$ there can be found a factor r, such that the function $r y_n$

satisfies the equation $EI\,y'''' + ky = -ky_n$. Using the relation $-EI\,\mu y_n'''' = -ky_n$, we find the condition:

$$\nu\frac{ky_n}{\mu} + k\nu y_n = -ky_n$$

whence:

$$\nu = \frac{-\mu}{\mu+1}.$$

We therefore can obtain the deflexion y of the beam by adding $\frac{-\mu}{\mu+1}y_n$ to the sum of the curves $y_0, y_1 \ldots y_n$, or by adding $\left(1+\frac{-\mu}{1+\mu}\right)y_n = \frac{1}{1+\mu}y_n$ to the sum $y_0 + y_1 + \ldots + y_{n-1}$.

Thus we can stop the drawing of curves, as soon as two consecutive ones y_n and y_{n+1} are found, the ordinates of which are proportional.

Though — generally — the above mentioned proportionality only appears exactly after an infinite number of iterations, it nevertheless will be approximately observed tolerably soon. Neglecting in such a case that part of the last found loading diagram which troubles the proportionality between its ordinates and those of the foregoing diagram, we can use the preceding remark, provided that 1° the neglected load diagram be insignificant, and 2° it gives no rise to following load diagrams which grow larger and larger.

The second condition is satisfied when $\frac{kl^4}{EI} < 14600$.

The justification of this latter statement can be given most naturally by the aid of the deductions, given by Mr. Droste. We therefore refer to his paper.

Mathematics. — "*An application of the theory of integral equations on the determination of the elastic curve of a beam, elastically supported on its whole length*". By Dr. J. DROSTE. (Communicated by Prof. J. C. KLUIJVER).

(Communicated at the meeting of March 24, 1923).

1. Under the same title and at the same time a paper [1]) of Mr. BIEZENO appears in these Proceedings. The question, suggested, in N°. 4 of that paper as to the validity of the process of iteration used in it, will be answered here.

For that purpose we observe that the function of x, satisfying the differential equation

$$\frac{d^4y}{dx^4} + \lambda y = q'(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the conditions at the ends of the interval, is a meromorphic function of λ. We might find it by means of the method of the variation of constants and then expand it in ascending powers of λ; the radius of convergence R of the power series that stands after the first term (containing λ^{-1} as a factor) might easily be calculated then. After this it will be necessary to investigate wether it agrees or not for $\lambda = k'$ with the series of paper I; it is only in the first case that the latter series will be valid for $k' < R$. For the sake of this investigation, however, and also in order to get an idea of the proportionality of the functions $\chi_n(x)$ (vid. I, 7), we prefer to use the method based upon the theory of the integral equation of FREDHOLM.

2. We construct a function of x, satisfying in the interval $(0, l)$ both the equation

$$\frac{d^4y}{dx^4} + \lambda y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the conditions $y'' = y''' = 0$ at the ends, and being continuous as well as its first three derivates everywhere in $(0, l)$ with the only exception of a saltus of the third derivate at the point ξ:

$$\left.\frac{d^3y}{dx^3}\right|_{\xi-0}^{\xi+0} = 0.$$

[1]) Referred to in the sequel as "paper I".

This function we call $K(x, \xi, \lambda)$; it represents the deflexion of the beam, loaded by a load 1, which is concentrated at the point ξ.

Putting $\lambda = -\varrho^4$ the function

$$\mp \frac{1}{4\varrho^3}\{\sinh \varrho\,(x-\xi) - \sin \varrho\,(x-\xi)\}$$

(the upper sign for $x \leqq \xi$, the lower for $x \geqq \xi$) will satisfy all conditions excepted those at the ends.

Assuming

$$K(x, \xi, \lambda) = \mp \frac{1}{4\varrho^3}\{\sinh \varrho\,(x-\xi) - \sin \varrho\,(x-\xi)\} +$$
$$+ A\cosh \varrho\,(x-\tfrac{1}{2}) + B\sinh \varrho\,(x-\tfrac{1}{2}\,l) + C\cos \varrho\,(x-\tfrac{1}{2}\,l) + D\sin \varrho\,(x-\tfrac{1}{2}\,l),$$

we may determine A, B, C and D in such a way that $K(x, \xi, \lambda)$ satisfies the conditions at the ends. This gives

$$-A\cosh \tfrac{1}{2}\varrho l + B\sinh \tfrac{1}{2}\varrho l + C\cos \tfrac{1}{2}\varrho l - D\sin \tfrac{1}{2}\varrho l = \frac{1}{4\varrho^3}\{\sinh \varrho\xi + \sin \varrho\xi\},$$

$$-A\sinh \tfrac{1}{2}\varrho l + B\cosh \tfrac{1}{2}\varrho l - C\sin \tfrac{1}{2}\varrho l - D\cos \tfrac{1}{2}\varrho l = \frac{1}{4\varrho^3}\{\cosh \varrho\xi + \cos \varrho\xi\},$$

$$-A\cosh \tfrac{1}{2}\varrho l - B\sinh \tfrac{1}{2}\varrho l + C\cos \tfrac{1}{2}\varrho l + D\sin \tfrac{1}{2}\varrho l =$$
$$= \frac{1}{4\varrho^3}\{\sinh \varrho\,(l-\xi) + \sin \varrho\,(l-\xi)\},$$

$$-A\sinh \tfrac{1}{2}\varrho l - B\cosh \tfrac{1}{2}\varrho l - C\sin \tfrac{1}{2}\varrho l + D\cos \tfrac{1}{2}\varrho l =$$
$$= \frac{1}{4\varrho^3}\{\cosh \varrho\,(l-\xi) + \cos \varrho\,(l-\xi)\}.$$

Adding the first and the third of these equations and also the second and the fourth we get two equations containing only A and C. Subtracting the third from the first and the fourth from the second we get two equations containing only B and D. In this way we obtain

$$-A\cosh \tfrac{1}{2}\varrho l + C\cos \tfrac{1}{2}\varrho l = \frac{1}{4\varrho^3}\{\sinh \tfrac{1}{2}\varrho l\cosh \varrho(\xi-\tfrac{1}{2}\varrho l) + \sin \tfrac{1}{2}\varrho l\cos \varrho(\xi-\tfrac{1}{2}l)\},$$

$$-A\sinh \tfrac{1}{2}\varrho l - C\sin \tfrac{1}{2}\varrho l = \frac{1}{4\varrho^3}\{\cosh \tfrac{1}{2}\varrho l\cosh \varrho\,(\xi-\tfrac{1}{2}\,l) + \cos \tfrac{1}{2}\varrho l\cos \varrho\,(\xi-\tfrac{1}{2}l)\},$$

$$B\sinh \tfrac{1}{2}\varrho l - D\sin \tfrac{1}{2}\varrho l = \frac{1}{4\varrho^3}\{\cosh \tfrac{1}{2}\varrho l\sinh \varrho\,(\xi-\tfrac{1}{2}\,l) + \cos \tfrac{1}{2}\varrho l\sin \varrho\,(\xi-\tfrac{1}{2}\,l)\},$$

$$B\cosh \tfrac{1}{2}\varrho l - D\sin \tfrac{1}{2}\varrho l = \frac{1}{4\varrho^3}\{\sinh \tfrac{1}{2}\varrho l\sinh \varrho\,(\xi-\tfrac{1}{2}\,l) - \sin \tfrac{1}{2}\varrho l\sin \varrho(\xi-\tfrac{1}{2}l)\},$$

From these equations A, B, C and D are easily solved; putting

$$\Delta_1(\varrho) = \cosh \tfrac{1}{2}\varrho l \sin \tfrac{1}{2}\varrho l + \sinh \tfrac{1}{2}\varrho l \cos \tfrac{1}{2}\varrho l,$$

$$\Delta_2(\varrho) = \cosh \tfrac{1}{2}\varrho l \sin \tfrac{1}{2}\varrho l - \sinh \tfrac{1}{2}\varrho l \cos \tfrac{1}{2}\varrho l,$$

we get

$$-4\varrho^3 \Delta_1(\varrho)\{A \cosh \varrho(x-\tfrac{1}{2}l) + C \cos \varrho(x-\tfrac{1}{2}l)\} =$$
$$= (\cosh \tfrac{1}{2}\varrho l \cos \tfrac{1}{2}\varrho l + \sinh \tfrac{1}{2}\varrho l \sin \tfrac{1}{2}\varrho l) \cosh \varrho(x-\tfrac{1}{2}l) \cosh \varrho(\xi-\tfrac{1}{2}l)$$
$$+ \cosh \varrho(x-\tfrac{1}{2}l) \cos \varrho(\xi-\tfrac{1}{2}l) + \cos \varrho(x-\tfrac{1}{2}l) \cosh(\xi-\tfrac{1}{2}l)$$
$$+ (\cosh \tfrac{1}{2}\varrho l \cos \tfrac{1}{2}\varrho l - \sinh \tfrac{1}{2}\varrho l \sin \tfrac{1}{2}\varrho l) \cos \varrho(x-\tfrac{1}{2}l) \cos \varrho(\xi-\tfrac{1}{2}l),$$

$$-4\varrho^3 \Delta_2(\varrho)\{B \sinh \varrho(x-\tfrac{1}{2}l) + D \sin \varrho(x-\tfrac{1}{2}l)\} =$$
$$= (\cosh \tfrac{1}{2}\varrho l \cos \tfrac{1}{2}\varrho l - \sinh \tfrac{1}{2}\varrho l \sin \tfrac{1}{2}\varrho l) \sinh \varrho(x-\tfrac{1}{2}l) \sinh \varrho(\xi-\tfrac{1}{2}l)$$
$$+ \sinh \varrho(x-\tfrac{1}{2}l) \sin \varrho(\xi-\tfrac{1}{2}l) + \sin \varrho(x-\tfrac{1}{2}l) \sinh \varrho(\xi-\tfrac{1}{2}l)$$
$$+ (\cosh \tfrac{1}{2}\varrho l \cos \tfrac{1}{2}\varrho l + \sinh \tfrac{1}{2}\varrho l \sin \tfrac{1}{2}\varrho l) \sin \varrho(x-\tfrac{1}{2}l) \sin \varrho(\xi-\tfrac{1}{2}l).$$

We now have calculated the function $K(x, \xi, \lambda)$; it appears to be a function with the denominator $4\varrho^3 \Delta_1(\varrho) \Delta_2(\varrho)$. The values of λ equating to zero this denominator are the characteristic numbers of the problem; as $K(x, \xi, \lambda)$ is symmetrical with respect to x and ξ that numbers will be all real. From this it follows that the corresponding values of ϱ have an argument that is a multiple of $\frac{1}{4}\pi$; it is easily proved to be an even multiple so that the values of ϱ wil be real or purely imaginary and the corresponding values of λ negative or zero. For that purpose we first write $1 - \cosh \varrho l \cos \varrho l$ for $2\Delta_1(\varrho)\Delta_2(\varrho)$ and then substitute in it $\varrho l = \alpha + i\beta$; equating the real part to zero we get

$$\cosh \alpha \cosh \beta \cos \alpha \cos \beta + \sinh \alpha \sinh \beta \sin \alpha \sin \beta = 1,$$

which is not satisfied by $\beta = \pm \alpha \neq 0$, for substituting $\beta = \pm \alpha$ in it we get $\sinh^2 \alpha = \sin^2 \alpha$, which is impossible for $\alpha \neq 0$. Therefore the values of ϱ are real or purely imaginary and the characteristic numbers are *negative*, except one which is zero.

If ϱ be a root of $\Delta_1(\varrho) = 0$, also $i\varrho$ will be a root (and consequently $-\varrho$ and $-i\varrho$); the same is true with respect to the roots of $\Delta_2(\varrho) = 0$. We now call the positive roots of the equation

$$\operatorname{tgh} p = -\operatorname{tg} p,$$

in the order of their magnitude $p_1, p_2, \ldots$ and the positive roots of the equation

$$\operatorname{tgh} p = \operatorname{tg} p,$$

ordered in the same way $q_1, q_2, \ldots$ Then the characteristic numbers will be

$$0, -\left(\frac{2p_n}{l}\right)^4, -\left(\frac{2q_n}{l}\right)^4 \ldots (n = 1, 2, \ldots)$$

3. We will also calculate the characteristic functions. If p represents one of the numbers p_n and q one of the numbers q_n we have to calculate the following limits:

$$\lim_{\rho\to 0} \varrho^4 K(x,\xi,\lambda),\ \lim_{\rho\to 2p/l}\left\{\varrho^4-\left(\frac{2p}{l}\right)^4\right\}K(x,\xi,\lambda),\ \lim_{\rho\to 2q/l}\left\{\varrho^4-\left(\frac{2q}{l}\right)^4\right\}K(x,\xi,\lambda).$$

To none of the limits the term $\mp\frac{1}{4\varrho^3}\{\sinh\varrho(x-\xi)-\sin\varrho(x-\xi)\}$ contributes.

For the first of the limits we find immediately

$$\lim_{\rho\to 0}\varrho^4 K(x,\xi,\lambda) = -\frac{1}{l}-\frac{12}{l^3}(x-\tfrac{1}{2}l)(\xi-\tfrac{1}{2}l).$$

To the second only the term $A\cosh\varrho(x-\frac{1}{2}l)+C\cos\varrho(x-\frac{1}{2}l)$ contributes. First we have

$$\lim_{\rho\to 2p/l}\frac{\varrho^4-(2p/l)^4}{-4\varrho^3\Delta_1(\varrho)}=\frac{1}{-l\cosh p\cos p}$$

and the numerator of the fraction we have found for

$$A\cosh\varrho(x-\tfrac{1}{2}l)+C\cos\varrho(x-\tfrac{1}{2}l)$$

changes for $\varrho=2p/l$ into

$$\cosh 2p\left(\frac{x}{l}-\tfrac{1}{2}\right)\left\{(\cosh p\cos p+\sinh p\sin p)\cosh 2p\left(\frac{\xi}{l}-\tfrac{1}{2}\right)+\cos 2p\left(\frac{\xi}{l}-\tfrac{1}{2}\right)\right\}$$

$$+\cos 2p\left(\frac{x}{l}-\tfrac{1}{2}\right)\left\{\cosh 2p\left(\frac{\xi}{l}-\tfrac{1}{2}\right)+(\cosh p\cos p-\sinh p\sin p)\cos 2p\left(\frac{\xi}{l}-\tfrac{1}{2}\right)\right\}$$

From $\cosh p\sin p+\sinh p\cos p=0$ we have

$$\cosh p\cos p-\sinh p\sin p=\frac{\cosh p}{\cos p},$$

$$\cosh p\cos p+\sinh p\sin p=\frac{\cos p}{\cosh p},$$

and consequently the numerator becomes

$$\left\{\cosh 2p\left(\frac{x}{l}-\tfrac{1}{2}\right)+\frac{\cosh p}{\cos p}\cos 2p\left(\frac{x}{l}-\tfrac{1}{2}\right)\right\}\left\{\frac{\cos p}{\cosh p}\cosh 2p\left(\frac{\xi}{l}-\tfrac{1}{2}\right)+\cos 2p\left(\frac{\xi}{l}-\tfrac{1}{2}\right)\right\}$$

In this way we find

$$\lim_{\rho\to 2p/l}\left\{\varrho^4-\left(\frac{2p}{l}\right)^4\right\}K(x,\xi,\lambda)=$$

$$-\frac{1}{l}\left\{\frac{\cosh 2p\left(\frac{x}{l}-\frac{1}{2}\right)}{\cosh p}+\frac{\cos 2p\left(\frac{x}{l}-\frac{1}{2}\right)}{\cos p}\right\}\left\{\frac{\cosh 2p\left(\frac{\xi}{l}-\frac{1}{2}\right)}{\cosh p}+\frac{\cos 2p\left(\frac{\xi}{l}-\frac{1}{2}\right)}{\cos p}\right\}.$$

In the same way

$$\lim_{\rho\to 2q/l}\left\{\rho^4-\left(\frac{2q}{l}\right)^4\right\}K(x,\xi,\lambda)=$$

$$-\frac{1}{l}\left\{\frac{\sinh 2q\left(\frac{x}{l}-\frac{1}{2}\right)}{\sinh q}+\frac{\sin 2q\left(\frac{x}{l}-\frac{1}{2}\right)}{\sin q}\right\}\left\{\frac{\sinh 2q\left(\frac{\xi}{l}-\frac{1}{2}\right)}{\sinh q}+\frac{\sin 2q\left(\frac{\xi}{l}-\frac{1}{2}\right)}{\sin q}\right\}.$$

Putting

$$\varphi_0(x)=\frac{1}{\sqrt{l}},\quad \varphi_n(x)=\frac{1}{\sqrt{l}}\left\{\frac{\cosh 2p_n\left(\frac{x}{l}-\frac{1}{2}\right)}{\cosh p_n}+\frac{\cos 2p_n\left(\frac{x}{l}-\frac{1}{2}\right)}{\cos p_n}\right\},$$

$$\psi_0(x)=\frac{2\sqrt{3}}{l\sqrt{l}}(x-\tfrac{1}{2}),\quad \psi_n(x)=\frac{1}{\sqrt{l}}\left\{\frac{\sinh 2q_n\left(\frac{x}{l}-\frac{1}{2}\right)}{\sinh q_n}+\frac{\sin 2q_n\left(\frac{x}{l}-\frac{1}{2}\right)}{\sin q_n}\right\},$$

$$(n=1,2,\ldots)$$

the functions $\varphi_n(x)$, $\psi_n(x)$ $(n=0,1,2\ldots)$ will be the orthogonal and normal characteristic numbers; they satisfy equation (2), λ being replaced by the corresponding characteristic number.

Now drawing graphs of the functions $y=tgx$, $y=tghx$ and $y=-tghx$ in one figure, it is easily seen that p_n is an angle in the $2n$-th quadrant, and q_n an angle in the $(2n+1)$-th quadrant. For $n\to\infty$ p_n and q_n converge to the middlepoints of the intervals. From this it follows that $\cos p_n$ and $\sin p_n$ converge to $\pm\frac{1}{2}\sqrt{2}$ and it is easily seen that the absolute value of $\varphi_n(x)$ and $\psi_n(x)$ remains less than a number which is independent from x and n. Now as

$$\lim_{n\to\infty}\frac{p_n}{n}=\lim_{n\to\infty}\frac{q_n}{n}=\pi$$

the two series occurring in

$$K(x,\xi,\lambda)=\frac{\varphi_0(x)\varphi_0(\xi)+\psi_0(x)\psi_0(\xi)}{\lambda}+\sum_{n=1}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)}{\lambda+(2p_n/l)^4}+$$

$$+\sum_{n=1}^{\infty}\frac{\psi_n(x)\psi_n(\xi)}{\lambda+(q_n/l)^4}\quad\ldots\quad(3)$$

will be uniformly convergent and the right hand side therefore will be equal to $K(x,\xi,\lambda)$.

4. We now suppose y to be the required solution of (1), viz. that solution for which $y'' = y''' = 0$ in the points $x = 0$ and $x = l$ and which is continuous in $(0, l)$ as well as its first three derivates; as to y'''' it may have a saltus in a finite number of points a_i, which will be the case if $q'(x)$ has in the points a_i discontinuities for which $q'(a_i + 0)$ and $q(a_i - 0)$ exist. The points a_i and the value ξ divide the interval $(0, l)$ into a number of subintervals; in the interior of each of them we have

$$\frac{d}{dx}[y''' K(x, \xi, \lambda) - y'' K'(x, \xi, \lambda) + y' K''(x, \xi, \lambda) - y K'''(x, \xi, \lambda)] =$$
$$= y'''' K(x, \xi, \lambda) - y K''''(x, \xi, \lambda).$$

Integrating the equation over the subintervals, adding the results and regarding that $y'' = y''' = K''(x, \xi, \lambda) = K'''(x, \xi, \lambda) = 0$ for $x = 0$ and $x = l$ and that $y, y', y'', y''', K, K', K'', K'''$ are continuous everywhere except K''' in ξ, we find

$$-y(\xi) = \int_0^l \{y'''(x) K(x, \xi, \lambda) - y(x) K'''(x, \xi, \lambda)\} dx.$$

Replacing y'''' by $q' - \lambda y$ from (1) and K'''' by $-\lambda K$ from (2) we get

$$y(\xi) = \int_0^l K(x, \xi, \lambda) q'(x) dx$$

or interchanging x and ξ and observing the symmetry of $K(x, \xi, \lambda)$ with respect to x and ξ

$$y(x) = \int_0^l K(x, \xi, \lambda) q'(\xi) d\xi.$$

If the beam is not loaded by $q(x)$, but by N loads Q_i, concentrated in the points ξ_i, we have

$$y(x) = \sum_{i=1}^{N} Q'_i K(x, \xi_i, \lambda)$$

where $Q'_i = Q_i / EI$. If the beam bears both the load $q(x)$ and the loads Q_i we have

$$y(x) = \int_0^l K(x, \xi, \lambda) q'(\xi) d\xi + \sum_{i=1}^{N} Q'_i K(x, \xi_i, \lambda) \quad . \quad . \quad . \quad (4)$$

From (4) it follows that y is a meromorphic function of λ with the poles 0, $-({}^{2}p_{n/l})^4$ and $-({}^{2}q_{n/l})^4$. This is easily seen from sub-

stituting (3) in (4) and integrating term by term, which is permitted, the series (3) being uniformly convergent.

Expanding y in a series of ascending powers of λ (the first term will in general contain λ^{-1}) the expansion will generally be convergent for $|\lambda| < ({}^{2p}{}_{1/l})^4$; only if the term with the denominator $\lambda + ({}^{2p}{}_{1/l})^4$ cancels, the expansion will be valid for larger values of λ. In case the Q_i's are zero this occurs if $q(x)$ be orthogonal to $\varphi_1(x)$. We thus see that if the expansion of paper 1 be exact and if not by chance

$$\int_0^l q(x)\,\varphi_1(x)\,dx = 0$$

it converges only if

$$\frac{kl^4}{EJ} \leq (2p_1)^4 = 500{,}54665 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From (4) we deduce a formula which will be of use further on. Supposing the beam to bear only a load $p(x)$ pro unit of length and to be in equilibrium, we will have

$$\int_0^l p(x)\,dx = \int_0^l x\,p(x)\,dx = 0$$

or which is the same

$$\int_0^l p(x)\,\varphi_0(x)\,dx = \int_0^l p(x)\,\psi_0(x)\,dx = 0.$$

Now from (4), in which $q'(x)$ is to be replaced by $p(x)/EI$ and in which $Q'_i = 0$, we have

$$y(x) = \int_0^l \overline{K}(x, \xi, \lambda)\,p(\xi)\,d\xi,$$

where $\overline{K}(x, \xi, \lambda)$ arises from $K(x, \xi, \lambda)$ by omitting in (3) the term with the denominator λ. Putting $\lambda = 0$ $\overline{K}(x, \xi, \lambda)$ changes into

$$K(x, \xi) = \sum_{n=1}^{\infty} \frac{\varphi_n(x)\,\varphi_n(\xi)}{({}^{2p}{}_{n/l})^4} + \sum_{n=1}^{\infty} \frac{\psi_n(x)\,\psi_n(\xi)}{({}^{2q}{}_{n/l})^4}, \quad . \quad . \quad . \quad (6)$$

and we get

$$y(x) = \frac{1}{EI}\int_0^l K(x, \xi)\,p(\xi)\,d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This represents the deflection of the beam under the conditions that the beam be in equilibrium and that the ground be absent; it is such that

$$\int_0^l y\,(x)\,dx = \int_0^l x\,y\,(x)\,dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

since $K(x, \xi)$ is orthogonal with respect to $\varphi_0(x)$ and $\psi_0(x)$. By the conditions (8) the deflection is perfectly determined and (7) represents it.

5. We shall now prove that the series deduced from (4) agrees for $\lambda = k'$ with the series of paper I. Representing the iterations of $K(x, \xi)$ by $K_2(x, \xi)$, $K_3(x, \xi) \ldots$ we get for $|\lambda| < (2p_1/l)^4$

$$\overline{K}\,(x, \xi, \lambda) = K\,(x, \xi) - \lambda\,K_2\,(x, \xi) + \lambda^2\,K_3\,(x, \xi) \ldots,$$

$$\int_0^l \overline{K}\,(x, \xi, \lambda)\,q'\,(\xi)\,d\xi = \int_0^l K\,(x, \xi)\,q'\,(\xi)\,d\xi - \lambda \int_0^l K_2\,(x, \xi)\,q'\,(\xi)\,d\xi + \ldots,$$

as is proved in the theory of integral equations. From this it follows that (4) for $|\lambda| < (2p_1/l)^4$ takes the form

$$y\,(x) = y_0\,(x) + y_1\,(x) + y_2\,(x) + \ldots, \quad . \quad . \quad . \quad (9)$$

where

$$y_0\,(x) = \frac{1}{k}\left\{\varphi_0\,(x) \int_0^l \varphi_0\,(\xi)\,q\,(\xi)\,d\xi + \psi_0\,(x) \int_0^l \psi_0\,(\xi)\,q\,(\xi)\,d\xi + \right.$$

$$\left. + \varphi_0\,(x) \sum_{i=1}^{N} Q_i\,\varphi_0\,(\xi_i) + \psi_0\,(x) \sum_{i=1}^{N} Q_i\,\psi_0\,(\xi_i)\right\}$$

$$y_1\,(x) = \frac{1}{EI} \int_0^l K\,(x, \xi)\,q\,(\xi)\,d\xi + \frac{1}{EI} \sum_{i=1}^{N} Q_i\,K\,(x, \xi_i),$$

$$y_2\,(x) = -\,k' \left\{\frac{1}{EI} \int_0^l K_2\,(x, \xi)\,q\,(\xi)\,d\xi + \frac{1}{EI} \sum_{i=1}^{N} Q_i\,K_2\,(x, \xi)\right\} =$$

$$= -\,\frac{k}{EI} \int_0^l K\,(x, \xi)\,y_1\,(\xi)\,d\xi,$$

. . . . . .

$$y_{n+1}(x) = -\frac{k}{EI}\int_0^l K(x, \xi)\, y_n(\xi)\, d\xi,$$

. . . . . .

Each of the functions $y_n(x)$, except $y_0(x)$, satisfies (8). We shall now prove the terms $y_0, y_1, y_2, \ldots$ to be the same as the corresponding quantities of I, 5, from which it will follow that the series $y_0 + y_1 + \ldots$ agrees with the series of I, 4. Indeed in the first place $y_0(x)$ is a linear function of x; the function $k y_0(x)$ represents the linear load $\alpha x + \beta$, which is defined in I, 5 and is statically equivalent to the given load. For we have

$$k\int_0^l y_0(x)\, \varphi_0(x)\, dx = \int_0^l \varphi_0(\xi)\, q(\xi)\, d\xi + \sum_{i=1}^{N} Q_i\, \varphi_0(\xi_i),$$

$$k\int_0^l y_0(x)\, \psi_0(x)\, dx = \int_0^l \psi_0(\xi)\, q(\xi)\, d\xi + \sum_{i=1}^{N} Q_i\, \psi_0(\xi_i),$$

or substituting in it the expressions found for the functions $\varphi_0(x)$ and $\psi_0(x)$

$$\int_0^l k\, y_0(x)\, dx = \int_0^l q(\xi)\, d\xi + \sum_{i=1}^{N} Q_i$$

$$\int_0^l (x - \tfrac{1}{2})\, k y_0(x)\, dx = \int_0^l (\xi - \tfrac{1}{2})\, q(\xi)\, d\xi + \sum_{i=1}^{N} Q_i\, (\xi_i - \tfrac{1}{2}),$$

which proves the proposition.

Omitting from (9) the deflexion y_0, the remaining terms represent the remaining deflexion. This becomes y_1 for $k = 0$ and so y_1 represents the deflexion which the beam, if not supported by the ground, gets under the influence of the load that remains after subtraction of $\alpha x + \beta$ from the given load. As besides $y_1(x)$ satisfies (8), it is identical with the quantity y_1 of I, 5.

The reaction of the ground, arising from the deflexion y_1, represents a load $-ky_1$ of the beam; by this load the beam, if not supported by the ground, would get a deflexion, which we may calculate from (7) viz.

$$-\frac{k}{EI}\int_0^l K(x,\xi)\, y_1(\xi)\, d\xi.$$

This represents the deflexion $y_2(x)$; it is seen to be the same as the quantity y_2 of I, 5. In the same way we continue and so we may prove that (9) agrees term by term with the series of paper I.

7. In case the expansion do not converge, it may happen that the method of graphical integration, communicated in paper I, remains still valid (vid I, 7); this depends on the approximate proportionality of the functions $y_n(x)$ for large values of n. We shall prove this now; more exactly: we shall prove

$$\lim_{n\to\infty} \frac{y_{n+1}(x)}{y_n(x)} = -\mu$$

where μ is independent from x.

Now $K_n(x, \xi)$ is represented by the absolutely and uniformly convergent series

$$K_n(x, \xi) = \sum_{m=1}^{\infty} \frac{w_m(x)\, w_m(\xi)}{\lambda_m^n},$$

where the quantities λ_m represent the numbers $(2\rho_i/l)^4$ and $(2q_i/l)^4$ in the order of their magnitude and the functions $w_m(x)$ are the corresponding normal orthogonal functions. Putting

$$\int_0^l w_m(\xi)\, q'(\xi)\, d\xi + \sum_{m=1}^{N} Q'_i\, w_m(\xi_i) = P_m, \quad . \quad . \quad . \quad . \quad (10)$$

we get the absolutely and uniformly convergent series

$$y_n(x) = (-k')^{n-1} \sum_{m=1}^{\infty} \frac{P_m\, w_m(x)}{\lambda_m^n} \quad (n = 1, 2, \ldots)$$

Supposing h to be the smallest value of m for which $P_m \neq 0$, we can write

$$y_n(x) = \frac{(-k')^{n-1}}{\lambda_h^{n-1}} \left\{ P_h\, w_h(x) + \left(\frac{\lambda_h}{\lambda_{h+1}}\right)^n \sum_{m=1}^{\infty} \left(\frac{\lambda_{h+1}}{\lambda_{h+m}}\right)^n P_m\, w_m(x) \right\}.$$

The series in the right hand member of this equation has an absolute value which is less than the sum of the series

$$\sum_{m=1}^{\infty} \frac{\lambda_{h+1}}{\lambda_{h+m}}\, P_m\, w_m(x)\,|,$$

a quantity which is independent from n. From this and from

$$\lim_{n\to\infty} \left(\frac{\lambda_h}{\lambda_{h+1}}\right)^n = 0$$

we get

$$\lim_{n\to\infty} \frac{\lambda_h^n}{(-k')^{n-1}}\, y_n(x) = P_h\, w_h(x)$$

In this way we find

$$\lim_{n\to\infty} \frac{y_{n+1}(x)}{y_n(x)} = -\frac{k'}{\lambda_h} \lim_{n\to\infty} \frac{\dfrac{\lambda_h^{n+1}}{(-k')^n}\, y_{n+1}(x)}{\dfrac{\lambda_h^n}{(-k')^{n-1}}\, y_n(x)} = -\frac{k'}{\lambda_h},$$

which proves the proposition; we see that

$$\mu = -\frac{k'}{\lambda_h}.$$

Now, if in drawing the successive deflexions $y_0, y_1, y_2, \ldots$ it is found that $y_{n+1} : y_n$ is sufficiently independent from x, it will be permitted occasionally to consider

$$\bar{y}_n = y_0 + y_1 + \ldots + y_{n-1} + \frac{y_n}{1 + \dfrac{k'}{\lambda_h}}$$

to be the deflexion y. For we have

$$y_1 + y_2 + \ldots + y_{n-1} + \frac{y_n}{1 + \dfrac{k'}{\lambda_h}} = \sum_{\nu=1}^{n-1} (-k')^{\nu-1} \sum_{m=h}^{\infty} \frac{P_m w_m(x)}{\lambda_m^{\nu}} + \frac{y_n}{1 + \dfrac{k'}{\lambda_h}}$$

$$= \sum_{m=h}^{\infty} \frac{P_m w_m(x)}{\lambda_m + k'} \left\{ 1 - \left(-\frac{k'}{\lambda_m} \right)^{n-1} \right\} + \sum_{m=h}^{\infty} \frac{P_m w_m(x)}{\lambda_m^n} \cdot \frac{(-k')^{n-1}}{1 + \dfrac{k'}{\lambda_h}}$$

$$= \sum_{m=h}^{\infty} \frac{P_m w_m(x)}{\lambda_m + k'} - \sum_{m=h}^{\infty} P_m w_m(x) \left(\frac{-k'}{\lambda_m} \right)^{n-1} \left\{ \frac{1}{\lambda_m + k'} - \frac{\lambda_h}{\lambda_m(\lambda_h + k')} \right\}.$$

and as

$$y = y_0 + \sum_{m=1}^{\infty} \frac{P_m w_m(x)}{\lambda_m + k'}$$

we get

$$\bar{y}_n - y = -\sum_{m=h+1}^{\infty} P_m w_m(x) \left(\frac{-k'}{\lambda_m} \right)^{n-1} \left\{ \frac{1}{\lambda_m + k'} - \frac{\lambda_h}{\lambda_m(\lambda_h + k')} \right\},$$

since $m = h$ gives zero. If $k' < \lambda_{h+1}$, the series has zero as a limit for $n \to \infty$, which is easily seen by writing it in the form

$$\bar{y}_n - y = -\left(\frac{-k'}{\lambda_{h+1}} \right)^{n-1} \sum_{m=h+1}^{\infty} P_m w_m(x) \left(\frac{\lambda_{h+1}}{\lambda_m} \right)^{n-1} \left\{ \frac{1}{\lambda_m + k'} - \frac{\lambda_h}{\lambda_m(\lambda_h + k')} \right.$$

since the absolute values of the series occurring in the right hand member is less than the sum of the convergent series

$$\sum_{m=h+1}^{\infty} \frac{| P_m w_m(x) |}{\lambda_m + k'}$$

It thus appears that we may consider $\overline{y_n}$ to be the required deflexion y, supposed n be large enough and $k' < \lambda_{h+1}$. If g is, after h, the first value of m such that $P_m \neq 0$, the condition $k' < \lambda$. must been satisfied if we wish to replace y by $\overline{y_n}$ for large values of n.

Chemistry. — "*The Phenomenon of Electrical Supertension.*" III. [1])
By Prof. A. SMITS. (Communicated by Prof. P. ZEEMAN.)

(Communicated at the meeting of February 24, 1923.)

In my book "Die Theorie der Allotropie" [2]), and also in the preceding communications I have treated the electrical supertension only very briefly. Therefore I will discuss this important phenomenon somewhat more at length here.

We imagine the case that a palladium or platinum electrode is made cathode. For the explanation of the phenomenon that will now

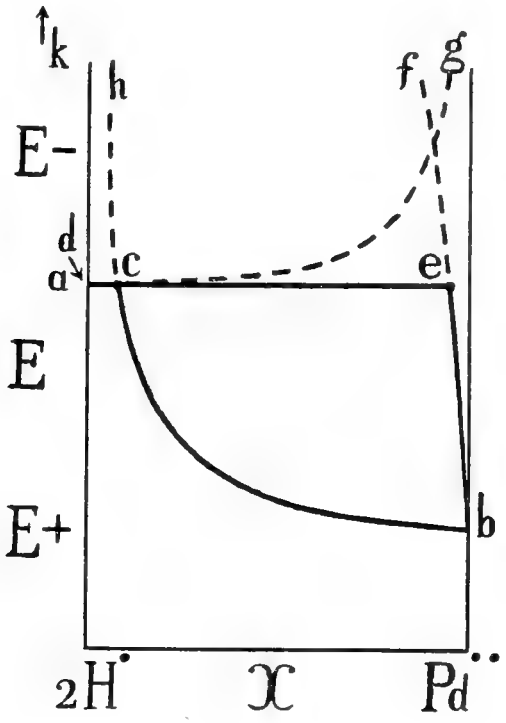


Fig. 1.

appear, we shall make use of the E,X-diagram, in which the experimental electric potential of the electrodes is plotted as function

[1]) These Proc. Vol. XXI No. 3, p. 375 (1918); Vol. XXI, No. 8, p. 1106 (1919).
[2]) JOHANN AMBROSIUS BARTH, Leipzig. 1921.
English edition LONGMANS, GREEN and Co. London 1922.
French edition GAUTHIER VILLARS. Paris. 1923.

of the concentration; on the assumption that the pressure (1 atm.), temperature, and total ion-concentration (metal ions + hydrogen ions) are constant. In the foregoing figure 1 hydrogen is taken for one electrode, and palladium for the other, but instead of the latter platinum might, of course, have been chosen just as well.

Line *bh* indicates the potentials of the series of electrolytes that can coexist with different palladium phases. These phases of the palladium are different, because palladium dissolves the hydrogen in quantities which increase with the hydrogen-ion concentration of the electrolyte.

Line *bf* indicates the potentials of the different palladium phases containing hydrogen [1]), which coexist with the different electrolytes. In our *E,X*-figure the potential of the metal-phase can be read on the *E*-axis, but it is clear that on this axis also the potential of the electrolyte can be read, when we reverse the sign.

The line *ag* represents the potentials of the different electrolytes coexisting with the gaseous hydrogen phases. These hydrogen phases consist of pure hydrogen, and lie, therefore, on the hydrogen axis. Accordingly the portion *ak* of the hydrogen axis gives the potentials of the hydrogen phases coexisting with the different electrolytes.

The point of intersection *c* of the lines *bh* and *ag* represents the electrolyte which can coexist at the same time with the palladium phase (*e*) and with the hydrogen phase (*d*), so that it also shows the potential of this three-phase equilibrium. The situation of this point of intersection follows from the solubility products of hydrogen and palladium: [2])

$$L_{H_2} = (H^{\cdot})^2 (\theta)^2 = 10^{2\times -48}$$
$$L_{Pd} = (Pd^{\cdot\cdot}) (\theta)^2 = 10^{2\times -62.2}.$$

At the three-phase equilibrium

$$(\theta)_{H_2} = (\theta)_{Pd}$$

from which follows:

$$\frac{(Pd^{\cdot\cdot})}{(H^{\cdot})^2} = \frac{L_{Pd}}{L_{H_2}} = 10^{2\times -14.2}.$$

If $(H^{\cdot})$ is put $= 1$, then $(Pd^{\cdot\cdot}) = 10^{2\times 14.2}$.

From this it is seen that the point *e* lies very much on one side, and that when a palladium electrode was immersed in a 1-N sulphuric

[1]) This line indicates the gross hydrogen concentrations, and gives, therefore, no information about the state in which the hydrogen is.

[2]) Compare with regard to the smallness of these products the remarks in "The Theory of Allotropy" in the chapter: "Small concentrations" p. 172.

acid solution, and the palladium was and remained in inner equilibrium, this metal would dissolve a little, till the palladium concentration of $10^{2 \times 14.2}$ was reached, while a corresponding inappreciable quantity of hydrogen would have been generated. In this it is assumed that both platinum and hydrogen continue to be in inner equilibrium, for the value used for L_{H_2} agrees with the value for hydrogen in inner equilibrium, and we shall for the moment assume the value used for L_{Pd} also to agree with the condition of inner equilibrium of *Pd*. *Pd* is, however, an inert metal, so that the solubility product of this metal will in reality have decreased through the slight attack, and the dissolving will have already stopped, before the palladium ion concentration $10^{2 \times -15.2}$ has been reached [1]).

For the sake of simplicity we shall, however, assume here that no disturbance of the *Pd* takes place, and that the three-phase equilibrium is established, in which the *Pd*-phase *e* coexists with the electrolyte *c* and with the hydrogen phase *d* at a pressure of one atmosphere. When now the *Pd*-electrode is made cathode, or in other words, when electrons are added to the *Pd*, hydrogen and palladium ions in the ratio of $1 : 10^{2 \times -14.2}$ or practically only hydrogen ions will be separated at this electrode. It will now depend on the velocity with which the inner equilibrium

$$2H^{\cdot}_{G} + 2\theta_{G} \rightleftarrows H_{2G}$$

sets in, if the hydrogen formed will coexist in a state of internal equilibrium or in a state of formation. In this condition the solubilityproduct of the hydrogen is greater, and the point that now denotes the coexisting hydrogen phase, will lie on a potential curve that lies at more negative values, and is represented by a'g' in fig. 2. We must, however, not forget that this line could only be realised when the state of formation of the hydrogen discussed just now could coexist unchanged in electro-motive equilibrium with a series of solutions. This is, however, not the case; only one point can be realised on this curve, and this is the point indicating the liquid layer that coexists with the hydrogen phase *d'*, which is in a state of formation, and with the palladium phase *e'*. The heterogeneous equilibrium between the metal boundary layer and the hydrogen boundary layer, just as that with the liquid boundary layer, having been immediately established, the palladium boundary layer will also contain too many hydrogen ions and electrons, which means that also the hydrogen dissolved in this metal boundary layer, will be in a state of formation.

[1]) The potential + 0.82 V., from which the solubility product $L_{Pd} = 10^{2 \times -62\,2}$ has been calculated, is most probably already a potential of a disturbed state of the metal palladium.

We may, of course, also start from the *Pd*, and say that only in the *Pd*-electrode to which electrons are added, and in which hydrogen ions dissolve, hydrogen is formed in a state of formation, and that *afterwards* gaseous hydrogen occurs in a state of formation, but this only implies a difference so far as the first moments are concerned, for when once electrolytic generation of hydrogen has set in, this

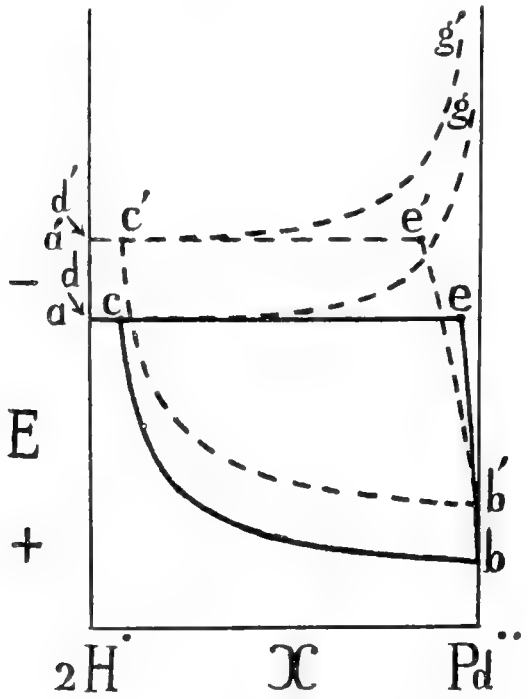


Fig. 2.

will occur in a state of formation at the same time in the gas phase and in the metal phase.

It should be pointed out here that when we have a homogeneous phase, as the solid solution of hydrogen in palladium, the electrical potential of these two components with respect to the coexisting electrolyte must be the same. This applies also to the solid solutions lying on the line *be*, but in the solid solution lying on this line there is equilibrium between hydrogen molecules, hydrogen ions, and electrons, whereas this is not the case in the *Pd*-boundary layer which coexists with hydrogen in a state of formation.

This is, therefore, the reason that the *Pd*-phase *e'* coexisting with the hydrogen phase *d'*, does not lie on the prolongation of the line *be*.

The hydrogen dissolved in the *Pd*-phase *e'* is in the state of formation, and consequently this phase is richer in hydrogen ions and electrons than when the hydrogen is in inner equilibrium. The

potential of the dissolved hydrogen in e' is more strongly negative, and the same must, therefore, hold for the *Pd*. It is now, however, the question in what way the potential of the palladium has undergone this change.

It is clear that the *Pd* must have become richer in *Pd*-ions and electrons. We have already seen that this phase has become richer in electrons through addition of hydrogen in a state of formation, so that only the question is still to be answered how the concentration of the *Pd*-ions can have been increased. This must have taken place through the reaction

$$2H_S^{\cdot} + Pd_S \rightarrow Pd_S^{\cdot\cdot} + 2H_S$$

in which, therefore, hydrogen ions have ceded their charge to *Pd*-atoms. We, therefore, come to the conclusion that the palladium boundary layer, which coexists with hydrogen in a state of formation, will possess too many hydrogen ions, palladium ions and electrons, or in other words, that it will contain both hydrogen and palladium in a state of formation.

If palladium could coexist in the same state of formation with a 1.N. solution of a palladium salt, the electric potential would, of course, possess a more strongly negative value than corresponds to point b in fig. 2. This more strongly negative potential is indicated by b'. And when, therefore, the same state of formation of *Pd* could continue to exist also in contact with the whole series of solutions, the line $b'e'$ would indicate the solid solutions which can coexist with the electrolytes lying on the line $b'c'$. The new three-phase equilibrium that is found when *Pd* is made cathode at a definite density of current, and in which hydrogen escapes in a state of formation, is denoted by the points $d'c'e'$. The line $a'c'g'$ rising very little throughout the greater part of the concentration region, it is clear that the value of the negative potential in this new three-phase equilibrium would be equally great when the point c' lay on the prolongation of the line bc, and the point e on the prolongation of the line be, but as we demonstrated above, the points c' and e' belong to other lines than those that are mentioned here. It follows from these considerations that in the case of electrolytic generation of hydrogen the state of formation of the hydrogen in the coexisting hydrogen and palladium phases are *very closely* related. This makes it clear that the cathode metal can exert influence on the degree of super-tension. The state of formation is a state of non-equilibrium, and the different cathode metals will, to a different degree, accelerate the conversion of this state of non-equilibrium in the direction of the inner

equilibrium. This is the reason why the so-called super-tension of hydrogen is different, when different metal cathodes are used.

It is self-evident that when the state of formation of the hydrogen does not vanish too quickly, the hydrogen must possess an abnormally high conductivity for electricity immediately after the escape. This phenomenon was, indeed, found long ago[1]), but it was tried to explain it in another way; it is, however, probable that this phenomenon is for the greater part to be attributed to the state of formation.

The activity of the hydrogen dissolved in the metal phase, is in perfect harmony with the considerations given here. As regards the temperory variations of the super-tension, they will have to be explained by the slow change in constitution of the coexisting phases. The heterogeneous equilibrium between the boundary layers is established with great velocity, but the composition of the phases changes slowly, and this must be the reason that the three-phase equilibrium metal-electrolyte-hydrogen changes slowly.

In conclusion I will still point out that analogous considerations, of course, apply to oxygen and other non-metals. As is discussed in "The Theory of Allotropy" p. 160 et seq, the extension of this theory to non-metals, necessitated the assumption that the atoms of all elements can split off and receive electrons.[2]) The difference between the solubilities of the positive and the negative ions in elements with pronounced metal- resp. metalloid character, is so great that for the explanation of the electro-motive behaviour as a rule only the positive or the negative ions need be taken into account. But as was also already stated the supposition mentioned must very certainly be used when the positive charges of non-metals with regard to electrolytes, and likewise the small electric conductivity of non-metals in electrically neutral condition, is to be explained. Further the said supposition is also required to make clear the formation of compounds between metals.[2])

When we now return to the non-metals and choose oxygen as example, we have to consider the two following reactions:

$$O_2 \rightleftarrows 2O^{\nu_1'} + 2\nu_1\,\theta$$

and

$$O_2 + 2\nu_2\,\theta \rightleftarrows 2O^{\nu_2''}$$

As $\nu_2 = 2$, the latter reaction may be written:

$$O_2 + 4\theta \rightleftarrows 2O''.$$

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[1]) Becker. Jahrb. der Radioaktivität. **9**, 52 (1912).

[2]) Theory of Allotropy p. 160.

The latter equation is sufficient to explain the electric super-tension of the oxygen. It was stated [1]) that in the case of anodic polarisation of an unattackable electrode or an inert metal the separated oxygen must relatively contain too few electrons and too few negative oxygen ions, so that oxygen in a state of formation or in other words oxygen in super-tension would have to possess an abnormally small electric conductivitly immediately after its formation, at least when no other phenomena neutralise this effect.

When we have an inert metal, i. e. a metal that can be easily disturbed, and we make this anode, polarisation will take place. If the disturbance of the metal goes so far that oxygen is separated, then, the metal boundary layer being poor in ions and electrons, also the coexisting oxygen phase will be abnormally poor in electrons. Besides the other substances coexisting in the liquid, the metal boundary layer will also contain oxygen dissolved, and it is evident that the state of this oxygen, dissolved in the metal, will depend on the state of the oxygen in the coexisting oxygen layer.

Laboratory for General and Inorganic Chemistry of the University.

Amsterdam, Februari 1923.

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[1]) Theory of Allotropy p. 164.

Chemistry. — "*The Influence of Intensive Drying on Internal Conversion*". I. By Prof. A. SMITS. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of March 24, 1923).

In December 1921 a communication was published in the 100th volume of the Z. f. physik. Chemie under the same title as is given above. In manuscript this communication was at first more extensive, for it also contained a possible explanation of the great influence found by BAKER of intensive drying on the chemical reactivity of gases, and besides a discussion of the sa-ammoniac problem [1]). The reason why for the present I withheld this part was as follows.

I was at the time still in doubt whether in intensive drying it should be assumed that a fixation or a shifting of the inner equilibrium takes place. The results of BAKER's researches [2]) published then spoke greatly in favour of a shifting, but at first this assumption seemed open to objections, because it is then necessary to assume that the slightest trace of moisture can give rise to a great displacement of the inner equilibrium.

Afterwards, when BAKER had published [3]) a new series of experiments, it seemed nevertheless the most probable conclusion that here a shifting of the inner equilibrium takes place, which from a thermodynamic standpoint means that very much work is required to withdraw the last traces of water from a system.

Accordingly I showed in the English and in the French edition of the Theory of Allotropy, in which I devoted a chapter to BAKER's experiments, that in my opinion intensive drying gives rise to a displacement of the internal equilibrium. Since then my own investigation, which I carried out with some of my pupils, has confirmed this supposition.

The explanation of the influence of intensive drying on reactivity, which I left unpublished so far, is exceedingly simple, for we

[1]) Also the influence of intensive drying on the properties of Sal ammoniac, becomes explicable, when this substance is assumed to contain two kinds of molecules, one of which is dissociable, and the other is not.

[2]) Trans. Chem. Soc. **51**, 2339 (1903).

[3]) Trans. Chem. Soc. **121**, 568 (1922).

have only to apply the theory of allotropy, i.e. we have to assume that every phase of these substances contains at least two different kinds of molecules, which are of course in inner equilibrium in the case of unary behaviour, to which we add the supposition that at least one of these kinds of molecules is chemically inactive. This is very well possible, since the mechanism of the transformation into another type of molecule will be an intirely different one from that of chemical action with other substances. To represent the case as simply as possible we can then assume that there are only two different kinds of molecules, one of which is active, the other inactive. When for ammonia we denote them by $NH_3\alpha$ and $NH_3\beta$, we have in each phase in the case of unary behaviour, the following inner equilibrium:

$$NH_{3\alpha} \rightleftarrows NH_{3\beta}$$

My supposition was this that on intensive drying this inner equilibrium is shifted towards the inactive side, and in this case, *completely*, so that in the ammonia remains that only contains the inactive kind of molecules.

I will just mention here that I emphatically pointed out before that the expression "different kinds of molecules" should be taken in its widest sense. It should comprise not only the isomer and polymer molecules, but also the electrically charged dissociation products, ions + electrons, and it stands to reason that in many cases the difference between the different kinds of molecules lies in a difference in the atomic structure.

It is particularly the more recent views of atomic structure that have brought to light that between the different atoms very subtle differences are possible, which are e.g. in connection with a change of the quanta values of the valency-electron-paths, and this leads to kinds of molecules with more subtle differences than those which are assumed to exist between the ordinary isomers. The fact, however, remains that also these different kinds of molecules may be ranged under this category when the sense in which the idea "isomery" is taken, is very wide.

During my investigation there appeared a publication by Baly and Duncan [1]), in which they communicate among other things that the rapidity at which gaseous ammonia, withdrawn from an iron cylindre with liquid ammonia, is decomposed by a platinum spiral heated at a definite temperature, is dependent on the velocity of evaporation of the liquid ammonia. On rapid evaporation ammonia gas

[1]) J. Chem. Soc. **121** en **122**, 1008 (1922).

was obtained of much smaller velocity of decomposition than on slow evaporation. BALY and DUNCAN expressed the opinion that this difference is probably caused by this, that on rapid evaporation there is formed a gas phase rich in the kind of molecules that preponderate in the liquid phase, whereas on slow evaporation there has been a possibility for the conversion of this kind of molecules into another, of which the gas phase chiefly consists in ordinary circumstances.

One kind of molecules, which chiefly occurs in liquid ammonia, would then be the inactive kind, and the other kind of molecules, of which the ordinary ammonia gas chiefly consists, the active one. They further pointed out that the existence of inactive and active kinds of molecules probably accounts for the chemical inactivity of the gas dried by BAKER.

So we see that in this paper BALY and DUNCAN already express the supposition at which I had also arrived, though I did not publish it because my investigation was not yet sufficiently advanced. BALY-DUNCAN's results, however, are not very convincing, as BRISCOE [1]) observed, because they can also be explained in another way. He says: "It is known, that ordinary commercial ammonia, dried over lime, contains about 1 per cent of water [2]), and that rapid, irreversible destillation, such as may occur by free discharge of gas from a cylinder of liquid, is a very effective means of separating the constituents even of a constant boiling mixture [3]), so that the gas thus obtained may well be considerably drier than that in real equilibrium with the cylinder liquid. BALY has found that the addition of water vapour to ordinary ammonia increases its reactivity, drying certainly decreases its reactivity, and so the greater dryness of the "inactive" form would appear to be capable of explaining the whole of the observations, including the "recovery" of the gas in cylinders on standing (by acquisition of the equillibrium content of water vapour) identity of slowly released cylinder gas with laboratory preparations dried by lime, recovery of inactive gas in the experimental tube, when the wire is heated at 200° (release of absorbed water from the wire or walls) and the increase in reactivity of "inactive" ammonia with increase of temperature of the wire".

These remarks of BRISCOE's, which are very true in my opinion, deprive BALY's published experiments for the present of all their

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[1]) Annual Reports of the Progress of Chemistry vol. **19** 1922, p. 37.

[2]) Briscoe refers here to WHITE T. **121**, 1688 (1922), but this must be a mistake for WHITE has not found this.

[3]) MULLIKEN J. Amer. Chem. Soc. **44**, 2389 (1922).

cogency as a proof of the existence of an active and an inactive kind of molecules in ammonia.

I wanted to test my supposition in another way and took, accordingly, an entirely different course.

After having convinced myself that the pure P_2O_5 which I prepared by BAKER's method, had really the same properties as that of BAKER[1]), I began with some of my pupils an investigation of the influence of intensive drying on the point of transition, the melting-point, the vapour tension of the solid and liquid state, and the electrical resistance of the liquid phase of a great number of substances, and among them those substances, of which BAKER found that the chemical activity disappeared by intensive drying, occupy a very particular place on account of the great importance of this phenomenon. Of this latter group first of all NH_3, HCl, CO, and O_2 were taken in hand.

In a following communication our results and the particulars of the experiments will be discussed.

Laboratory of General and Inorg. Chemistry of the University.

Amsterdam, March 20th 1923.

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[1]) I became acquainted with this method through a private communication by Prof BAKER before it was published, which saved me a great deal of trouble and time. I will avail myself of this apportunity to express my cordial thanks to Prof. BAKER for his kindness.

Chemistry. — "*The System Sulphur Trioxide*" I. By Prof. A. SMITS. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of March 24, 1923).

For some years the examination of sulphur trioxide has been on my programme, because I surmised that this substance would yield suitable material to test the theory of allotropy. As, however, other investigations had to go first, this examination could not be taken in hand until a short time ago.

In the meantime BERTHOUD [1]) LE BLANC with RÜTLE [2]) published each a treatise on vapour tensions and melting-points of this substance. Though these two papers will be discussed more at length later on, I will make here already a few remarks, and more particularly in connection with the latter publication.

The results published there prove with the greatest clearness that SO_3 is really a substance which not only can be used as a test of the above-mentioned theory, but which is so eminently fit for it that in this respect it is unequalled by any other. For the results obtained show that both the liquid and the solid phases of the SO_3 can behave as phases of more than one component, which without any doubt must be attributed to the complexity of this phase.

This complexity is owing to the occurrence of different kinds of molecules in the same phase, which molecular-species are in internal equilibrium with each other in the case of unary behaviour. I emphatically pointed out on an earlier occasion that the term "different kinds of molecules" should be taken in as *wide a sense* as possible [3]). By them we should understand not only the isomer and the polymer molecules, but also the electrically charged dissociation products, ions + electrons, and it is self-evident, that in many cases the difference between molecular-species mentioned here lies in a difference between the atoms. It is in particular the more recent views on the atomic structure, that bring to light, that there are very subtle differences possible between the different atoms, which e.g. are in connection with a change of the quanta-values of the valency-electron-paths, and this leads to kinds of molecules with

[1]) Helvetica Chem. Acta **5**, 513 (1922).

[2]) Ber. d. Sächs. Akad. v. Wiss. Leipzig **74**, 106 (1922).

[3]) The theory of Allotropy p. 2.

more *subtle* differences than those, which are assumed between the ordinary isomers. Nevertheless when the idea of "isomery" is taken in a wider sense, also these different kinds of molecules may be classed under this category.

We cannot say as yet what kinds of molecules occur in the different phases of the pure SO_3. The molecular size in the vapour phase agrees about with SO_3, but it is very well possible that there occur isomer molecules of SO_3 at the same time, and it is also possible that there is also a polymer kind of molecules present in small concentration. The kinds of molecules that occur in the gas phase, will also be present in the liquid phase, hence according to the theory of allotropy also in the solid phase, though in a different proportion, when the idea molecular of conception is taken in a wide sense [1]). Up to now we have been completely in the dark as far as the internal state of solid SO_3 is concerned. The measurements of the surface tension can, indeed, extend our knowledge concerning the complexity of the liquid phase somewhat, but we still lack means to decide whether a unary solid phase is a mixed crystal in internal equilibrium or not.

Contrary to LE BLANC's opinion it is not possible to conclude to the molecular size of a substance in the solid state in a solvent from the found mol. weight of this substance. [2])

With a view to supplementing our methods of research with those that make use of RÖNTGEN rays in the hope of learning something more in the end about the more delicate inner state of equilibrium in the solid phase, I instituted a department for the RÖNTGEN investigation of the solid substance in my laboratory some years ago. Though the way which I had decided to follow, leads to the typical allotropic substances, it seemed desirable first to examine some simple, but nevertheless very interesting, substances, in which results were to be expected which might be of great importance for getting a clearer insight into the nature of the chemical bond. Accordingly Messrs J. M. BIJVOET and A. KARSSEN have studied Li, LiH, $NaClO_3$, $NaBrO_3$, in which it was possible to determine the structure and the binding of the particles on definite suppositions. [3]) Now the investigation of HgI_2 has been taken in hand, though we know that by means of this investigation we shall not be able to decide whether the solid phase in a mixed crystal.

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[1]) Cf. "The Theory of Allotropy" p. 220.

[2]) Loc. cit.

[3]) Partly published in These Proc. **23**, 644, 1365 (1921); **25**, 27 (1922); Zeitschr. f. Physik. **14**, 291 (1923).

The investigation by means of RÖNTGEN rays is by no means so powerful as it is often supposed to be. Thanks to the researches of BAKHUIS ROOZEBOOM and his pupils we have got to understand the behaviour of the mixed crystal phases in binary systems to a great extent, but what does the RÖNTGEN investigation teach us about these mixed crystals?

Let us e.g. take the simple system KCl,KBr, a system of which we know that the solid components are homogeneously mixable in all proportions, and let us now suppose an arbitrary mixed crystal from this continuous series to be given to a RÖNTGEN analyst. If this investigator is under the impression that he has to do with a solid phase of a simple substance, he will interpret the intensities found in the usual way, and will find them in very good agreement with the image of the system that was supposed by him to be *monocomponential*. For the intensities can only serve as a test of an already assumed model, and as there are still so many factors that are not sufficiently accurately known in the interpretation of these intensities, and because besides there are nearly always some parameters that have to be chosen so as to suit, a good agreement can be found, even when the supposition is erroneous.

Partly in consequence of these circumstances, partly in consequence of the impossibility to give already now a sharp image of the complexity, as this has also been assumed by me for the solid phase, the RÖNTGEN investigation, in its present stage of development, cannot serve as yet for a further elaboration of the theory of allotropy, and it will, no doubt, be still some years before the RÖNTGEN research will be able to throw new light on the inner equilibria, which have already been found in the solid state.

All the same we have started the RÖNTGEN study of the interesting HgI_2, because we wished in any case to ascertain if any changes occur in the RÖNTGEN spectrum of these compounds in the temperature interval of 130—255°, and, if so, what changes, hoping that some conclusions may be drawn from this with some probability.

I have thought it necessary to publish the above discussion, because a great many mistaken ideas still prevail in this region.

When we now return to LE BLANC's investigation, I will remark that he found, among other things, that on cooling of the supercooled liquid below 13.9° solidification suddenly sets in, on which the vapour tension appeared to have risen, also after the temporary rise of temperature had disappeared. Hence at the same temperature the solid phase formed presented a higher pressure than the supercooled liquid, and LE BLANC thought this phenomenon comparable

with the action of oxygen on phosphorus, in which ozone and a phosphoro-oxygen compound was formed.

This, statement shows very clearly the insuperable difficulty with which one is confronted, when with phenomena which so clearly point to the complex character of the phases, one yet continues to occupy the old standpoint.

I will not treat the phenomena found in the examination of SO_3 more at length here, but leave the discussion of them to the following communication.

Amsterdam, March 1923.

Laboratory of General and Inorganic Chemistry of the University.

Geology. — *"Geological data derived from the region of the "Bird's head" of New-Guinea"*. By Prof. L. RUTTEN.

(Communicated at the meeting of March 24, 1923).

The great northwestern Peninsula of New-Guinea is one of the least known parts of the Indian Archipelago. In recent times some data concerning it have been published by R. D. M. VERBEEK in his "Molukken Verslag" [1]), and C. E. A. WICHMANN, when journeying from the east coast to Horna, discovered a folded coal-bearing formation [2]) which proved to be of tertiary age [3]).

In the last few years (between 1917 and 1921), however, explorations were made on a large scale in Northern New-Guinea and also in the "Birds head" for oil and coal, by the officers of the Mining Department in the Dutch East Indies. The results of these explorations have not been published as yet [4]), but some years ago I received from the Director of the Mining Department in the Dutch East-Indies a rather large collection of limestones and marls for examination. The study of this collection has been finished, but there would be little sense in expatiating on it here, a fortiori as a description will probably be published elsewhere. It may be of interest though, to summarize the obtained results.

Although we are not quite sure that all the rocks we examined, are of tertiary age, this may yet be assumed for the great majority. Now, when observing on the subjoined sketch-map the localities of "Bird's head" from which the examined rocks are derived, we realize at once that *tertiary deposits have a wide distrbution in the north-west part of New-Guinea*. However, *eocene rocks seem to be scarce* among the tertiary deposits, which is quite in keeping with what we know about the other parts of New-Guinea. They were found only in two regions: in the first place between the island of Rumberpon and Horna, where, in two localities, Nummulites-Alveolina limestone and Alveolina-Lacazina limestone have been

[1]) Jaarboek Mijnwezen Ned. Indië 1908. Wetensch. Gedeelte.

[2]) Nova Guinea. IV. 1917.

[3]) Nova Guinea. VI. 2. 1914.

[4]) I. C. O.-Commissie, The history and present state of scientific research in the Dutch East Indies. Geology. p. 28. 1923.

collected, as well as oligomiocene limestones; while Lacazina-limestones have been found near the Campong Horna; in the second

Eoceen	= Eocene.
Oud Neogeen	= Older Neogene.
Jong Neogeen	= Younger Neogene.
Indifferente Gesteenten, meest tertiair	= Indifferent Rocks, mostly tertiary.
Marine Arkose	= Marine Arcose.
Marmer	= Marble.
Oude Schist-materiaal in Tertiair	= Old Schist-material in Tertiary.

place in the northwestern part of the "Bird's head", where Lacazina-limestones have been collected, at one locality. From this it is evident that eocene is only sparingly distributed; moreover it should be observed that the rocks of the two localities, where Lacazina alone is found, cannot on that account be referred to the eocene with absolute certainty, however probable this may be. From the region between Rumberpon (Amberpon) and Horna rocks have been described by me formerly that pointed to the boundary strata between eogene and neogene [1]).

On the contrary *limestones of littoral facies from the older neogene* have been found in *a large* number of localities, characterized by the occurrence of Lepidocyclina, Miogypsina and Cycloclypeus. Similar limestones from the region between Rumberpon and Horna and from the Andai-river near Menokwari, have been previously described. They now appear to occur to the west of Rumberpon in a broad zone, running north-south, and to extend farther south

[1]) Nova Guinea. VI. 2. 1914.

than Andai, while they can be recognized in a zone running all along the north coast of "the Bird's head" as far as the island of Batanta. It will be seen at a glance that we have to do here with a comparativily narrow zone of older-neogene, which follows the east coast and the north coast of the "Bird's head". It may be that older-neogene still occurs also in the more western and southern region of "Bird's head", but it is remarkable that among the numerous rocks from those regions that were examined by me, there was not a single one that could positively be referred to the older neogene. We shall see lower down that this is partly due to the facies of the discovered rocks being indifferent, to our having to do either with non-fossiliferous rocks or with rocks that have been deposited in a deeper sea, in which the fossils, so characteristic of the littoral older neogene, cannot be expected to occur. But beyond these also rocks occur repeatedly in the southern part of the "Bird's head", that are of littoral facies, in which e. g. Lithothamnium, Operculina and Amphistegina, the companions of Lepidocyclina in the older neogene etc., occur, but in which the Foraminifera, which are characteristic of the older neogene, are lacking. In such cases we no doubt have to do with *younger neogene which indeed is often borne out by the habitus of the rocks.* As an instance we point to the basin of the Aer Beraur and of the Aer Klasaman, in which a series of rocks occur that are referable to the younger neogene. Another region of probably young-neogene rocks, partly with true littoral habitus, is situated North of lake Amaru. Between lake Amaru and the Aer Beraur a number of rocks have been found: globigerina marls, fine grained lime sandstones and the like, which are completely indifferent, so that nothing can be said about their age. The same applies to some rocks from the region south of lake Amaru. A long list of rock samples, collected in a west-east zone far north of lake Amaru, are undoubtedly referable to the neogene, but their fossils and their facies are not typical enough to say whether they belong to the older or to the younger neogene. In some rocks, however, doubtful Lepidocyclina were recognized; the others have been classed under the "indifferent rocks". Lastly among the rocks from the basin of the Aer Sebjar there are some littoral limestones, in which no "older" forms are to be found, so that here also we have probably to do with younger neogene. On the other hand, a number of very fine grained lime sandstones and globigerina limes, collected east of Muturi-river have to be classed under the "indifferent rocks". They may be of older-neogene age, because in the adjacent region towards the east (west of Rumberpon) a few transition rocks were

found among true littoral Lepidocyclina-limes and Globigerina-limes.

Lastly presumably young-neogene rocks are to be found to the North and West of Menokwari. Here Globigerina marls and loose limesands, occur, which indeed do not include typical fossils, but which on account of their quite young habitus are most likely to be reckoned to the younger neogene. This in fact agrees with the circumstance that some limestones in this region are of littoral facies but do not contain Lepidocyclina, Cyclocypeus or Miogypsina. Before this a description was published of limestones from the island of Manaswari, near Menakwari, that were considered to be younger-neogene [1]).

Between the localities of old-neogene limes south of Menokwari and those west of Rumberpon are situated the high Arfak Mountains, which according to VERBEEK [3]) and WICHMANN [2]) are composed of granular eruptive rocks, schists and slates. From the region of the Arfak mountains I received three rocks most likely tertiary and built up of *detritus from the Arfak Mountains*. They are coarse-grained arcoses of marine origin, which together with Corals also contain a very few Globigerina. The minerals represented here are much quartz, orthoclase, perthite and less plagioclase and biotite: apparently we have to do here with the detritus of acid granites.

Coarse-grained *detritus of old rocks* occurs also frequently in the *northern part of "the Bird's head"* in the rocks of tertiary age — notably in the old-neogene rocks. This goes to show that below, and perhaps also at the surface, *there must exist a mountain range of older rocks*. The localities marked on the map by an σ are those where in the limestones transported fragments of quartzite and phyllite occur. A rock from the basin of the Aer Sebjar contained grains of perthite and orthoclase, which remind us of the detritus rocks of the Arfak mountains.

The future reports of the Mining Department will undoubtedly contain interesting information on these "older rocks" in the" Bird's head".

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[1]) Nova Guinea. VI. 2. p. 29. 42.

[2]) Nova Guinea IV. p. 97.

[3]) Tijdschr. Kon. Ned. Aardr. Gen. (2). 21. 1904.

Mathematics. — "*A theorem concerning power-series in an infinite number of variables, with an application to* DIRICHLET's [1]) *series.*" By H. D. KLOOSTERMAN. (Communicated by Prof. J. C. KLUYVER.)

(Communicated at the meeting of March **24**, 1923).

§ 1. An important relation between the theory of DIRICHLET's series and the theory of power-series in an infinite number of variables (for abbreviation we shall write: power-series in an i. n. of v.) has been discovered by H. BOHR [2]). Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad , \quad s = \sigma + it \; . \; . \; . \; . \; . \; . \; . \; . \; (1)$$

be an ordinary DIRICHLET's series. Put $x_1 = \frac{1}{2^s}$, $x_2 = \frac{1}{3^s}$, ... $x_m = \frac{1}{p_m^s}$, . . . (where p_m is the m-th prime-number, and let $n = p_{n_1}^{\nu_1} p_{n_2}^{\nu_2} \ldots p_{n_r}^{\nu_r}$, where p_{n_1}, p_{n_2}, . . . p_{n_r} are the different primes which divide n. Then the series (1) can *formally* be written as a power-series in an i. n. of v., thus:

$$P(x_1, x_2, \ldots x_m, \ldots) = \sum_{n=1}^{\infty} a_n x_{n_1}^{\nu_1} x_{n_2}^{\nu_2} \ldots x_{n_r}^{\nu_r} =$$

$$c + \sum_{\alpha=1,2,\ldots} c_\alpha x_\alpha + \sum_{\substack{\alpha,\beta=1,2,\ldots \\ \alpha \leq \beta}} c_{\alpha,\beta} x_\alpha x_\beta + \sum_{\substack{\alpha,\beta,\gamma=1,2,\ldots \\ \alpha \leq \beta \leq \gamma}} c_{\alpha,\beta,\gamma} x_\alpha x_\beta x_\gamma + \ldots$$

This relation has been applied by BOHR to the so-called absolute-convergence-problem for DIRICHLET's series, that is to say the determination of the abscissa of absolute convergence of (1) (the lower bound of all numbers β, such that the series (1) converges for $\sigma \geqslant \beta$, in terms of (preferably as simple as possible) analytic properties of the function represented by (1). Let B be the abscissa of absolute convergence of (1), and D the lower limit of all numbers α, such that $f(s)$ is *regular* and *bounded* for $\sigma \geqslant \alpha$. The absolute-convergence-problem will be solved, if the difference $B - D$ is known. BOHR proves that $B = D$ for any DIRICHLET's series that can be *formally* represented in one of the following forms:

[1]) A more detailed proof of the theorem will be published elsewhere.

[2]) Göttinger Nachrichten, 1913.

$$f(s) = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{a_{p_m}^{l}}{(p_m^l)^s}$$

or

$$f(s) = \prod_{m=1}^{\infty} \left(1 + \sum_{l=1}^{\infty} \frac{a_{p_m}^{l}}{(p_m^l)^s} \right),$$

or, what comes to the same thing, for any DIRICHLET's series for which the connected power-series in an i. n. of v. has one of the forms

$$P(x_1, x_2, \ldots x_m, \ldots) = \sum_{n=1}^{\infty} Q_n(x_n) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or

$$P(x_1, x_2, \ldots x_m, \ldots) = \prod_{n=1}^{\infty} (1 + Q_n(x_n)) \quad . \quad . \quad . \quad (3)$$

where $Q_n(x_n)$ $(n = 1, 2, \ldots.)$ is a power-series in x_n without a constant term. The equality $B = D$ is a consequence of the theorem:

If: *a*. The series is *bounded* [1]) for $|x_n| \leqslant G_n$ $(n = 1, 2, \ldots.)$, then

b. it is absolutely convergent for $|x_n| \leqslant \theta G_n$, where θ is an arbitrary positive number in the interval $0 < \theta < 1$ [2]).

Now, if we consider the power-series (2) and (3), we see that the variables x_n occur to some extent separated from one another. This led BOHR to the conjecture, that the equality $B = D$ would hold for any DIRICHLET's series, for which the variables in the connected power-series in an i. n. of v. do not occur too much mixed up. Confirmation of this conjecture is the purpose of the present com-

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[1]) According to HILBERT (Wesen und Ziele einer Analysis der unendlich vielen unabhängigen Variabeln, Palermo Rendiconti, vol. 27, p. 67) a power-series in an i. n. of v. is defined to be *bounded* if:

1°. The power-series $P_m(x_1, x_2, \ldots x_m)$ (*Abschnitte*), that may be obtained from the power-series in an i. n. of v. by putting $x_{m+1} = x_{m+2} = \ldots = 0$, are, for all values of m, absolutely convergent in the region $|x_1| \leqslant G_1$, $|x_2| \leqslant G_2, \ldots. |x_m| \leqslant G_m$.

2°. There exists a number K, independent of m, such that, for every m, the inequality

$$|P_m(x_1, x_2, \ldots x_m)| < K$$

holds in the region $|x_1| \leqslant G_1$, $|x_2| \leqslant G_2, \ldots. |x_m| \leqslant G_m$.

[2]) It is well known, that *b* follows from *a* for *any* power-series in a *finite* number of variables. Originally HILBERT had assumed this also, as being self-evident, for an i. n. of v. But BOHR showed that this could not be true by constructing an example to the contrary.

munication. In fact it can be proved that $B = D$ holds for any DIRICHLET's series that can *formally* be written in the form

$$f(s) = \varphi\left(\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{a_{p_m^l}}{(p_m^l)^s}\right),$$

where φ is an arbitrary (non-constant) [1]) integral function. As a consequence of the relation, already mentioned above several times, the following theorem concerning power-series in an i. n. of v. is equivalent to this statement.

Theorem. If φ is an integral function and $Q_n(x_n)\,(n = 1, 2, \ldots)$ a *formal* [2]) power-series in x_n, without a constant term, and if the power-series in an i. n. of v. $P(x_1, x_2, \ldots x_m, \ldots) = \varphi(Q_1(x_1) + Q_2(x_2) + \ldots + Q_m(x_m) + \ldots)$ is bounded for $|x_n| \leqq G_n\,(n = 1, 2, \ldots)$, then it is absolutely convergent for $|x_n| \leqq \theta\, G_n$, if $0 < \theta < 1$.

In the following pages an outline of the proof of this theorem will be given.

§ 2. For the sake of simplicity we take $G_1 = G_2 = \ldots = G_n = \ldots = G > 1$, but $\theta\, G < 1$.

Because the given power-series in an i. n. of v. is bounded, there exists a number K, not depending on m, such that

$$|\varphi(Q_1(x_1) + Q_2(x_2) + \ldots + Q_m(x_m))| < K. \quad . \quad . \quad . \quad (4)$$

The first part of the proof of the theorem of § 1 discusses the power-series $Q_n(x_n)\,(n = 1, 2, \ldots)$. It is proved that it follows from (4) that all these power-series possess a certain region of convergence. Further research shows that two cases may occur:

1°. The functions $Q_n(x_n)$ are all regular for $|x_n| < G$. This is the general case.

2°. If the integral function $\varphi(y)$ has the form $V\left(\frac{y}{e^M}\right)$ (where V is again an integral function), then it is only possible to conclude that the functions $Q_n(x_n)$ are logarithms of functions regular for $|x_n| < G$, namely that they have the form $Q_n(x_n) = log\,(1 + R_n(x_n))$, where $R_n(x_n)$ is regular for $|x_n| < G$, and $R_n(0) = 0$ [3]).

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[1]) If φ is a constant, the theorem is trivial.

[2]) That is to say, the existence of a region of convergence is not assumed, but will appear to be a consequence of the other assumptions.

[3]) It is interesting to observe, that obviously the series (2), with $\varphi(y) = y$, falls under the first case, and the series (3), with $\varphi(y) = e^y$, $V(z) = z$, under the second case.

For shortness' sake we confine ourselves to the first case. (The proof in the second case is not essentially different, though in details more intricate). Then the functions $Q_n(x_n)$ are, because $G > 1$, all regular in their resp. circles $|x_n| \leqslant 1$.

For any function $f(z)$, regular for $|z| \leqslant 1$, and for which $f(0) = 0$, we now define a number r as follows: r is the radius of the largest circle, of which all points represent numbers assumed by $f(z)$ in the circle $|z| \leqslant 1$. Let r_n $(n = 1, 2, \ldots)$ be the corresponding quantity for $Q_n(x_n)$. Then we first prove, that the series $\sum_{n=1}^{\infty} r_n$ converges.

For this purpose we consider (4), valid for all sets of values of $x_1, x_2, \ldots . x_m$, satisfying $|x_n| \leqslant G$ $(n = 1, 2, \ldots . m)$, and, *a fortiori*, for all satisfying $|x_n| \leqslant 1$. Because $\varphi(y)$ is an integral function, it is possible to choose a number L so large, that the maximum value of $|\varphi(y)|$, on the circle $|y| = L$, is $> K$. Now suppose that, for some value of m, $r_1 + r_2 + \ldots . + r_m > L$. Then the maximum value of $\varphi(y)|$ on the circle $|y| = r_1 + r_2 + \ldots . + r_m$ would be $> K$. Now if we let the variables x_n $(n = 1, 2, \ldots . m)$ describe their resp. circles $|x_n| \leqslant 1$, then $Q_n(x_n)$ assumes all values satisfying $|Q_n(x_n)| = r_n$. Hence $y = Q_1(x) + Q_2(x_2) + \ldots . + Q_m(x_m)$ assumes all values satisfying $|y| = r_1 + r_2 + \ldots . + r_m$. Therefore it would be possible to find a set of values $x'_1, x'_2, \ldots . x'_m$ such that

$$y = Q_1(x'_1) + Q_2(x'_2) + \ldots + Q_m(x'_m) = (r_1 + r_2 + \ldots + r_m)e^{i\psi},$$

where $(r_1 + r_2 + \ldots . + r_m)\, e^{i\psi}$ represents that point of the circle $|y| = r_1 + r_2 + \ldots + r_m$ where $|\varphi(y)|$ assumes its maximum value. Therefore we should have

$$|\varphi(Q_1(x'_1) + Q_2(x'_2) + \ldots + Q_m(x'_m))| > K,$$

contradictory to (4). Therefore the supposition $r_1 + r_2 + \ldots + r_m > L$ can not be true. Since L is independent of m, this proves the convergence of $\sum_{n=1}^{\infty} r_n$.

We now apply the following theorem of BOHR [1]):

Let the function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ $(f(0) = 0)$ be regular for $|z| \leqslant 1$.

Let $M(\varrho)$ be the maximum value of $|f(z)|$ on the circle $|z| = \varrho$ $(0 < \varrho < 1)$. Then, if r is the quantity defined above, we have $r \geqslant k\, M(\varrho)$, where k is a number which depends on ϱ only (k is

[1]) Not yet published.

therefore the same for all functions satisfying the assumptions of the theorem).

Hence, if $M_n(\varrho)$ is the maximum value of $|Q_n(x_n)|$ on the circle $|x_n| = \varrho$ $(n = 1, 2, \ldots)$, we have $r_n \geqslant k\, M_n(\varrho)$. Since we have proved that $\sum_{n=1}^{\infty} r_n$ is convergent, it now follows that the series $\sum_{n=1}^{\infty} M_n(\varrho)$ converges also (for $\varrho < 1$). From this fact the theorem of § 1 can be easily deduced.

For let $Q_n(x_n) = \sum_{p=1}^{\infty} a_p^{(n)} x_n^p$ $(n = 1, 2, \ldots)$. Then

$$|a_p^{(n)}| \leq \frac{M_n(\varrho)}{\varrho^p} \begin{pmatrix} n = 1, 2, \ldots \\ p = 1, 2, \ldots \end{pmatrix} (\varrho < 1).$$

If $\Theta = \theta\, G$ (where θ is the constant of § 1), then it follows that, if $\Theta < \varrho < 1$, $\left(\text{we take for example } \varrho = \frac{1+\Theta}{2}\right)$,

$$\sum_{p=1}^{\infty} |a_p^{(n)}|\, \Theta^p \leq \frac{2\,\Theta\, M_n(\varrho)}{1-\Theta}.$$

Hence the series

$$\sum_{n=1}^{\infty} \sum_{p=1}^{\infty} |a_p^{(n)}|\, \Theta^p,$$

is also convergent. This proves a *fortiori* the convergence of the given power-series in an i. n. of v. for $|x_n| \leqslant \Theta = \theta\, G$ $(n = 1, 2 \ldots)$.

It cannot be denied that the assumption, that φ is an integral function, is somewhat unaesthetic. However, the author has not succeeded in dealing with the more general problem, where φ is an arbitrary (purely formal) power-series. In any case the method described does not give the required result in the more general case.

Copenhagen, November 1922.

Chemistry. — "*In-, mono- and divariant equilibria.*" XXIII. By Prof. F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of March 24, 1923).

Equilibria of n components in n + 1 phases, when the quantity of one of the components approaches to zero. The influence of a new substance on an invariant equilibrium. (Continuation).

We write the isovolumetrical reaction of an equilibrium $E(x=0)$:

$$\lambda_1 F_1 + \lambda_2 F_2 + \ldots = 0 \quad \Sigma(\lambda H)_V > 0 \quad \Sigma(\lambda V) = 0. \quad . \quad (1)$$

and the isentropical reaction:

$$\mu_1 F_1 + \mu_2 F_2 + \ldots = 0 \quad \Sigma(\mu H) = 0 \quad \Sigma(\mu V)_H > 0. \quad . \quad (2)$$

Consequently in reaction (1) are formed on addition of heat and in reaction (2) on increase of volume those phases, which have a negative reaction-coefficient. We have, therefore:

$$\Sigma(\lambda x)_V = -\lambda_1 x_1 - \lambda_2 x_2 - \text{ en } \Sigma(\mu x)_H = -\mu_1 x_1 - \mu_2 x_2 - \ldots$$

When we subtract both reaction-equations (1) and (2) from one another, after having multiplied the first one with μ_1 and the second one with λ_1, then we find the reaction:

$$(\mu_1 \lambda_2 - \lambda_1 \mu_2) F_2 + (\mu_1 \lambda_3 - \lambda_1 \mu_3) F_3 + \ldots = 0 \quad . \quad . \quad (3)$$

wherein the change of entropy is $\mu_1 \Sigma(\lambda H)_V$

and the change of volume is $-\lambda_1 \Sigma(\mu V_H)$.

As (3) represents the reaction, which may occur in the equilibrium $(F_1) = F_2 + F_3 + \ldots$, we have

$$\left(\frac{dP}{dT}\right)_1 = -\frac{\mu_1}{\lambda_1} \cdot \frac{\Sigma(\lambda H)_V}{\Sigma(\mu V)_H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Herein $\left(\frac{dP}{dT}\right)_1$ indicates the direction of curve (F_1) in the invariant point. In the same way we find:

$$\left(\frac{dP}{dT}\right)_2 = -\frac{\mu_2}{\lambda_2} \cdot \frac{\Sigma(\lambda H)_V}{\Sigma(\mu V)_H}; \left(\frac{dP}{dT}\right)_3 = -\frac{\mu_3}{\lambda_3} \cdot \frac{\Sigma(\lambda H)_V}{\Sigma(\mu V)_H} \text{ etc.} \quad (5, 6)$$

As we are able to deduce from (1) and (2) also the direction of temperature and pressure of the different monovariant curves, the P,T-diagram is, therefore, quantitatively defined.

Now we add to the equilibrium a new substance X, which occurs

in the phases $F_1\ F_2 \ldots$ with the concentrations $x_1\ x_2 \ldots$ In accordance with (13) and (15) (XXII) we now have:

$$\frac{\Sigma\,(\lambda H)_V}{RT}.\,(dT)_x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots = -\,\Sigma\,(\lambda x)_V. \quad . \quad (7)$$

$$\frac{\Sigma\,(\mu V)_H}{RT}.\,(dP)_x = -\,\mu_1 x_1 - \mu_2 x_2 \ldots = \Sigma\,(\mu x)_H \quad . \quad (8)$$

With the aid of (4) etc. we may also write for this:

$$\frac{\Sigma\,(\mu V)_H}{RT}.\,(dT)_x = -\,x_1\mu_1\left(\frac{dT}{dP}\right)_1 - x_2\,\mu_2\left(\frac{dT}{dP}\right)_2 - \quad . \quad . \quad (9)$$

$$\frac{\Sigma\,(\lambda H)_V}{RT}.\,(dP)_x = x_1\,\lambda_1\left(\frac{dP}{dT}\right)_1 + x_2\,\lambda_2\left(\frac{dT}{dP}\right)_2 + \quad . \quad . \quad (10)$$

It follows from (8) and (9):

$$\left(\frac{dT}{dP}\right)_x = -\,\frac{x_1\,\mu_1}{\Sigma\,(\mu x)_H}\cdot\left(\frac{dT}{dP}\right)_1 - \frac{x_2\mu_2}{\Sigma\,(\mu x)_H}\cdot\left(\frac{dT}{dP}\right)_2 - \quad . \quad . \quad (11)$$

from (7) and (10) it follows:

$$\left(\frac{dP}{dT}\right)_x = -\,\frac{x_1\,\lambda_1}{\Sigma\,(\lambda x)_V}\cdot\left(\frac{dP}{dT}\right)_1 - \frac{x_2\,\lambda_2}{\Sigma\,(\lambda x)_V}\cdot\left(\frac{dP}{dT}\right)_2 - \quad . \quad . \quad (12)$$

and from (7) and (8):

$$\frac{\Sigma\,(\mu V)_H}{\Sigma\,(\lambda H)_V}\cdot\left(\frac{dP}{dT}\right)_x = -\,\frac{\mu_1 x_1 + \mu_2 x_2 + \ldots}{\lambda_1 x_1 + \lambda_2 x_2 + \ldots} \quad . \quad . \quad . \quad (13)$$

From (7) we see that we are able to express $(dT)_x$ with the aid of the isovolumetrical reaction (1); it is apparent from (9) that, however, we cannot express $(dT)_x$ with the aid of the isentropical reaction (2) only, but that we must know also the directions of the monovariant curves $(F_1)\ (F_2) \ldots$ of the equilibrium $E\,(x = 0)$.

It appears from (8) that we are able to express $(dP)_x$ with the aid of the isentropical reaction (2); we see, however, from (10) that we cannot define $(dP)_x$ with the aid of the isovolumetrical reaction only but that we must know for this also again the directions of the curves $(F_1)\,(F_2) \ldots$

The direction of the monovariant curve E can be defined, as is apparent from (13), with the aid of the isovolumetrical and isentropical reaction; it follows from (11) and (12) that it can also be defined with the aid of the directions of the curves $(F_1)\ (F_2) \ldots$. and one of both reactions.

When we add a new substance X which occurs in one of the phases only, f.i. in F_1 than we must put in (7)—(13) $x_2 = 0\ x_3 = 0 \ldots$ As now $\Sigma\,(\lambda x)_V = -\,\lambda_1 x_1$, it follows from (12):

$$\left(\frac{dP}{dT}\right)_x = \left(\frac{dP}{dT}\right)_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which follows of course immediately from (11). Consequently curve E and (F_1) have the same tangent in the invariant point. It follows from (7) and (8) that they go also in the same direction of temperature and pressure, starting from this point. When viz. λ_1 is positive, then it follows from reaction (1) that curve (F_1) goes towards higher temperatures, starting from the invariant point. As it follows, however, from (7) that $(dT)_x$ is then positive also, consequently curve E goes also towards higher T. When λ_1 is negative, then the curves (F_1) and E go both towards lower T. It follows from (2) and (8) that both curves have also the same direction of pressure.

In accordance with previous papers (Communication XXII) we, therefore, find: when the new substance occurs in the phase F_1 only, then curve E coincides with curve (F_1).

When the new substance occurs in the phases F_1 and F_2 only, then (12) passes into:

$$\left(\frac{dP}{dT}\right)_x = \frac{\lambda_1}{\lambda_1 + K\lambda_2}\left(\frac{dP}{dT}\right)_1 + \frac{K\lambda_2}{\lambda_1 + K\lambda_2}\left(\frac{dP}{dT}\right)_2 \quad . \quad . \quad . \quad (15)$$

wherein $K = \frac{x_2}{x_1}$. Hence it follows:

$$d\left(\frac{dP}{dT}\right)_x = \frac{\lambda_1\lambda_2}{(\lambda_1 + K\lambda_2)^2}\left[\left(\frac{dP}{dT}\right)_2 - \left(\frac{dP}{dT}\right)_1\right]dK. \quad . \quad . \quad (16)$$

For fixing the ideas we assume that $\left(\frac{dP}{dT}\right)_2$ is greater than $\left(\frac{dP}{dT}\right)_1$. Now we distinguish two cases.

1. λ_1 and λ_2 have the same sign. The following is apparent from (15) and (16). When K changes from 0 tot ∞ then $\left(\frac{dP}{dT}\right)_x$ increases from $\left(\frac{dP}{dT}\right)_1$ to $\left(\frac{dP}{dT}\right)_2$ without becoming maximum, minimum or discontinuous.

2. λ_1 and λ_2 have opposite sign. When K changes from 0 to ∞, then $\left(\frac{dP}{dT}\right)_x$ decreases without becoming maximum or minimum from $\left(\frac{dP}{dT}\right)_1$ till $-\infty$, then it proceeds discontinuously towards $+\infty$ and afterwards it decreases to $\left(\frac{dP}{dT}\right)_2$.

When λ_1 and λ_2 are both positive, then, in accordance with reaction (1) both curves (F_1) and (F_2) go towards higher tempera-

tures starting from the invariant point; when λ_1 and λ_2 are both negative, then both curves go towards lower T; when λ_1 and λ_2 have opposite sign, then both curves go, starting from the invariant point in opposite direction of temperature.

It follows from all this that the tangent to curve E is situated within the angle, which is formed by the curves (F_1) and (F_2). [Of course we mean that angle wich is smaller than 180°]. As in the case of $K = 0$ (consequently $x_2 = 0$) curve E coincides with (F_1) and in the case of $K = \infty$ (consequently $x_1 = 0$) curve E coincides with (F_2) consequently the property follows, which we have deduced already in the previous communication also, viz:

Curve E is situated between the curves (F_1) and (F_2) or in other words: in the region $(F_1\ F_2)$.

Yet also we find, however:

Curve E is situated nearer curve (F_1) in proportion as the concentration of the new substance in the phase F_1 is larger with respect to that in F_2; curve E is situated nearer to curve (F_2) in proportion as the concentration of the new substance in the phase F_2 is greater with respect to that in F_1.

When the new substance occurs only in the phases $F_1\ F_2$ and F_3, then we find, in accordance with previous papers that curve E is situated in the region $(F_1\ F_2\ F_3)$.

When one of the curves, f.i. (F_3) is between the other two (F_1) and (F_2) then curve E is situated also between (F_1) and (F_2). When, however, none of the three curves is situated between the other two, then curve E may go, starting from the invariant point in every arbitrary direction.

Now we consider the binary equilibrium

$$E\,(x = 0) = F + L_1 + L_2 + G$$

we represent the composition, the entropy and the volume of

$$\begin{array}{llllll} F & \text{by} & y & 1-y & H & \text{and } V \\ L_1 & ,, & y_1 & 1-y_1 & H_1 & \text{and } V_1 \\ L_2 & ,, & y_2 & 1-y_2 & H_2 & \text{and } V_2 \\ G & ,, & y_3 & 1-y_3 & H_3 & \text{and } V_3 \end{array}$$

When we add a new substance X, then we call its concentration in those phases $x\ x_1\ x_2$ and x_3.

In order to deduce the isovolumetrical and isentropical reaction we take two arbitrary reactions; for this we choose:

$$F + a\,L_1 \rightleftarrows (1+a)\,L_1 \qquad \Delta H\ \Delta V \quad . \quad . \quad . \quad . \quad (17)$$
$$(1+b)\,L_2 \rightleftarrows F + b\,G \qquad \Delta H'\ \Delta V' \quad . \quad . \quad . \quad (18)$$

Herein is:

$$\Delta H = (1+a)\,H_1 - H - aH_2 \qquad \Delta H' = H + bH_2 - (1+b)\,H_2$$
$$\Delta V = (1+a)\,V_1 - V - aV_2 \qquad \Delta V' = V + bV_2 - (1+b)\,V_2$$

In (17) and (18) a and b may be as well positive as negative. It follows from (17) and (18) for the isovolumetrical reaction:

$$(\Delta V + \Delta V')\,F - (1+a)\,\Delta V'\,L_1 + [a\Delta V' - (1+b)\,\Delta V]\,L_2 + b\Delta V.\,G = 0$$
$$\Delta H\,\Delta V' - \Delta H'.\,\Delta V \quad 0 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and for the isentropical reaction:

$$-(\Delta H + \Delta H')\,F + (1+a)\Delta H'.\,L_1 - [a\Delta H' - (1+b)\Delta H]\,L_2 - b\Delta H.\,G = 0$$
$$0 \qquad \Delta H\,\Delta V' - \Delta H'\,\Delta V \quad . \quad . \quad . \quad . \quad . \quad (20)$$

We now add to this equilibrium $E(x=0)$ a new substance X_2 which occurs in the two liquids L_1 and L_2 only. With the aid of (19) and (20) it then follows from (7) and (8):

$$M.\,(dT)_x = -(1+a)\,\Delta V'.\,x_1 + [a\Delta V' - (1+b)\,\Delta V]\,x_2 \quad . \quad (21)$$
$$M.\,(dP)_x = -(1+a)\,\Delta H'.\,x_1 + [a\Delta H' - (1+b)\,\Delta H]\,x_2 \quad . \quad (22)$$

wherein:

$$M = (\Delta H.\,\Delta V' - \Delta H'.\,\Delta V) : R\,T$$

It follows from (21) and (22): when we add to the equilibrium $E(x=0)$ a new substance which occurs only in the two liquids, then the temperature as well as the pressure may be increased or decreased.

We now shall assume that the four phases are situated with respect to one another, as on the line YZ in fig. 1. Then we have:

$$y > y_1 > y_2 > y_3.$$

It follows from (17) and (18) for the determination of a and b:

$$y + a\,y_2 = (1+a)\,y_1 \qquad (1+b)\,y_2 = y + by_2$$
$$a = \frac{y - y_1}{y_1 - y_2} \qquad b = \frac{y - y_2}{y_2 - y_3} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

so that a and b are positive. Further we assume that F and L_1 and also that L_1 and L_2 are not situated very close to one another, so that a is neither very small nor very large. When F and L_2 and also L_2 and G are not situated very close to one another, then also b is not very small and not very large.

As now $\Delta V'$ is positive and very large with respect to ΔV, M is positive.

Further we may distinguish the following cases.

$$\left.\begin{array}{llll} a) & \Delta H > 0 & \Delta V \gtrless 0 & \Delta H' > 0 \qquad \Delta V' > 0 \\ & a\,\Delta H' - (1+b)\,\Delta H > 0 & & a\,\Delta V' - (1+b)\,\Delta V > 0 \end{array}\right\} \quad (24)$$

$$\left.\begin{array}{llll} b) & \Delta H > 0 & \Delta V \gtrless 0 & \Delta H' > 0 \qquad \Delta V' > 0 \\ & a\,\Delta H' - (1+b)\,\Delta H < 0 & & a\,\Delta V' - (1+b)\,\Delta V > 0 \end{array}\right\} \quad (25)$$

$$\left.\begin{array}{llll} c) & \Delta H > 0 & \Delta V \gtrless 0 & \Delta H' < 0 \qquad \Delta V' > 0 \\ & a\,\Delta H' - (1+b)\,\Delta H < 0 & & a\,\Delta V' - (1+b)\,\Delta V > 0 \end{array}\right\} \quad (26)$$

In each of the three cases, mentioned above, is in (21) the coefficient of x_1 negative and of x_2 positive; consequently we have:

$$(dT)_x \gtrless 0 \text{ when } \frac{x_2}{x_1} \gtrless \frac{(1+a)\,\Delta V'}{a\,\Delta V' - (1+b)\,\Delta V} \quad . \quad . \quad . \quad (27)$$

As $\Delta V'$ is very large with respect to ΔV it follows from this approximately with the aid of (23):

$$(dT)_x \gtrless 0 \text{ when } \frac{x_2}{x_1} \gtrless \frac{y-y_2}{y-y_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

In the case, mentioned sub b in (22) the coefficients of x_1 and x_2 are negative, so that $(dP)_x$ is also negative; consequently the pressure is lowered.

In order to examine more in detail the sign of $(dP)_x$ we write for (22)

$$M\,(dP)_x = \left[x_2 - - \frac{\Delta H'}{\Delta H' - \frac{1+b}{a}\Delta H} \cdot \frac{1+a}{a} x_1 \right] N \quad . \quad . \quad (29)$$

wherein:

$$N = a\,\Delta H' - (1+b)\,\Delta H$$

When we put herein the value of a from (23) then we may write for (29):

$$M\,.\,(dP)_x = \left[\frac{x_2}{x_1} - \frac{\Delta H'}{\Delta H' - \frac{1+b}{a}\Delta H} \cdot \frac{y-y_2}{y-y_1} \right] N x_1 \quad . \quad . \quad (30)$$

When we consider the three cases a, b and c mentioned above, then we may write for (30):

$$a) \qquad (dP)_x = \left[\frac{x_2}{x_1} - (1+K)\frac{y-y_2}{y-y_1} \right] L \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$b) \qquad (dP)_x = -\left[\frac{x_2}{x_1} + K\frac{y-y_2}{y-y_1} \right] L \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$c) \qquad (dP)_x = \left[-\frac{x_2}{x_1} + (1-K)\frac{y-y_2}{y-y_1} \right] L \quad . \quad . \quad . \quad . \quad (33)$$

wherein L, K, $1+K$ and $1-K$ are positive. In each of the three formula's L and K have different values.

In order to apply the above we take the figs. 1 and 2, wherein XY is a side of the components-triangle XYZ. The points F, L_1, L_2 and G represent the four phases of the invariant binary equilibrium $E(x=0)=F+L_1+L_2+G$. When we add a new substance X then the ternary equilibrium $E=F+L_1+L_2+G$ arises. The liquids L_1 and L_2 then proceed along the curves $L_1 q_1 r_1$ and $L_2 q_2 r_2$; as the new substance is not volatile, G follows a part of the line XZ. When we add only a little of the new substance, then the liquids are represented by the points q_1 and q_2 in the immediate

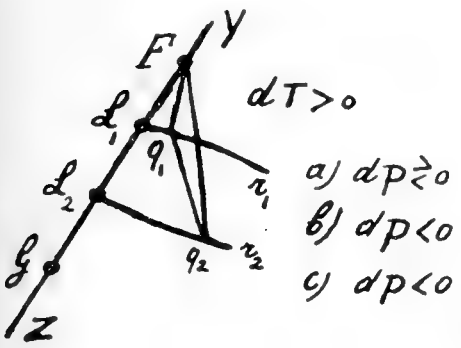

Fig. 1.

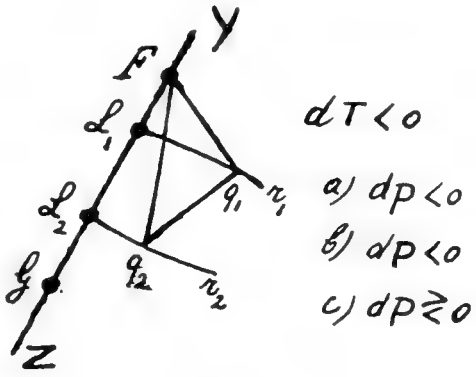

Fig. 2.

vicinity of L_1 and L_2; for the sake of clearness they have been drawn in the figures on greater distance.

In fig. 1 is:

$$\frac{x_2}{y-y_2} > \frac{x_1}{y-y_1} \quad \text{or} \quad \frac{x_2}{x_1} > \frac{y-y_2}{y-y_1} \quad . \quad . \quad . \quad . \quad (34)$$

consequently in accordance with (28): $(dT)_x > 0$ as is also indicated in the figure. It follows from (31)—(33):

$$\begin{aligned} &\text{in case } a \text{ is } (dP)_x \gtrless 0 \\ &\text{,, ,, } b \text{ ,, } (dP)_x < 0 \\ &\text{,, ,, } c \text{ ,, } (dP)_x < 0 \end{aligned}$$

as is also indicated in fig. 1.

In fig. 2 is:

$$\frac{x_2}{y-y_2} < \frac{x_1}{y-y_1} \quad \text{or} \quad \frac{x_2}{x_1} < \frac{y-y_2}{y-y_1} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

It follows from (28): $(dT)_x < 0$. From (31)—(33) it follows:

$$\text{in case } a \text{ is } (dP)_x < 0$$
$$\text{,, ,, } b \text{ ,, } (dP)_x < 0$$
$$\text{,, ,, } c \text{ ,, } (dP)_x \gtrless 0$$

as is indicated also in fig. 2.

In fig. 1 the pressure may as well increase as decrease in the case a; it is apparent from (31) that $(dP)_x$ shall be positieve for large values of $x_2 : x_1$. As L_1 (and consequently also q_1) is the liquid which contains the most of the solid substance F we shall call L_1 (and consequently also q_1) the concentrated and L_2 the diluted solution.

We, therefore, find the following:

when the threephases-triangle solid-liquid-liquid turns its concentrated solution towards the side of the components-triangle (fig. 1) then the temperature increases and the pressure generally decreases; only when the concentration of the new substance in the diluted liquid (consequently x_2) is much larger than in the concentrated liquid consequently x_1), then in the case a the pressure may increase also.

In fig. 2 in the case c the pressure may as well increase as decrease; it appears from (33) that $(dP)_x$ shall be positive for small values of $x_2 : x_1$.

Consequently we find the following:

when the threephases-triangle solid-liquid-liquid turns its concentrated solution away from the side of the components-triangle (fig. 2) then the temperature decreases and generally the pressure also.

Only when the concentration of the new substance is much larger in the concentrated solution (x_1) than in the diluted solution (x_2), then in the case c the pressure may also increase.

We may obtain the previous results also by using the P,T-diagram of the equilibrium $E(x = 0)$. We may deduce this in the following way.

The direction of temperature of the equilibrium $(G) = F + L_1 + L_2$ is defined by the sign of the coefficient of the phase G in the isovolumetrical reaction (19). As $b\,\Delta\,V$ may be as well positive as negative, curve (G) may go, starting from the invariant point i, as well towards higher as towards lower temperatures.

The direction of pressure of the equilibrium (G) is defined by the sign of the coefficient of G in the isentropical reaction (20). As $-b\,\Delta H$ is negative in each of the cases a, b and c, curve (G) proceeds, starting from the invariant point i, towards higher pressures.

As further, in accordance with (17):

$$\left(\frac{dP}{dT}\right)_G = \frac{\Delta H}{\Delta V}$$

and ΔV is very small, curve (G) is ascending, starting from point i fast vertically. In figs 3 and 4 this curve is drawn vertically upwards; the double arrow indicates that starting from i, it may run either towards the right or to the left.

As the coefficient $-(1 + a)\Delta V'$ of the phase L_1 is negative in each of the cases a, b and c, in accordance with (19) curve $(L_1) = F + L_2 + G$ is going starting from point i towards lower pressures (figs 3 and 4).

In the cases a and b the coefficient $(1 + a)\Delta H'$ of phase L_1 is positive in equation (20) so that curve (L_1) is going, starting from i, towards lower pressures (fig. 3). In the case c is $(1 + a)\Delta H'$ negative and curve (L_1) is going, therefore, starting from i, towards higher pressures (fig. 4). This is in accordance also with that which follows from (18) viz.

$$\left(\frac{dP}{dT}\right)_{L_1} = \frac{\Delta H'}{\Delta V'}.$$

Consequently we have defined the direction of the curves (G) and (L_1); fig. 3 is true for the cases a and b, fig. 4 for the case c.

With the aid of (19) and (20) we should be able to determine also the position of the curves (F) and (L_2) and then we could prove that the four curves are situated with respect to one another as in figs 3 and 4. [Compare f. i. Communication XIII]. As we know, however, the situation of the curves (G) and (L_1) we can find the position of curves (F) and (L_2) much more easily by using the rule for the position of the four monovariant curves of a binary equilibrium [Compare Communication I fig. 2].

In accordance with this rule we must meet, when we go, starting

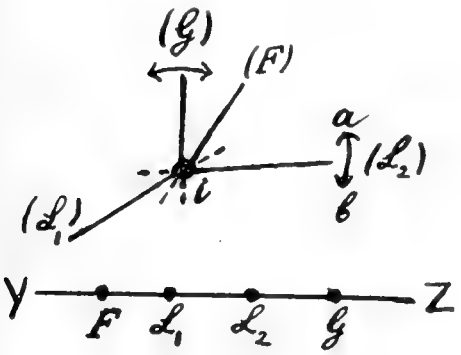

Fig. 3.

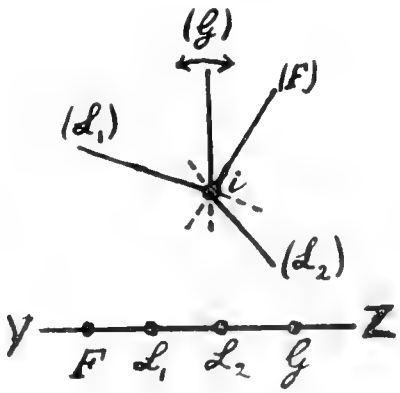

Fig. 4.

from curve (G) in the direction of the hands of a clock towards curve (L_1) firstly curve (F) and afterwards curve (L_2). As further (G) and (F) must form a bundle and their prolongations must be situated between (L_1) and (L_2) and as the angle between two succeeding curves, must be always smaller than 180°, hence follows for the curves (F) and (L_2) a situation as in the figures 3 and 4.

In fig. 3 curve (L_2) is drawn horizontally; starting from i it may run either upwards or downwards; this has been indicated by the double little arrow. When it goes upwards, starting from i, then its prolongation must yet always be situated above curve (L_1). It appears from the coefficient of the phase L_2 in reaction (20) that curve (L_2) must go in case a starting from i upwards and in case b, starting from i downwards. This has also been indicated in fig. 3.

As we know the P, T-diagram of the equilibrium $E(x = 0)$ we can easily determine the situation of curve E. It follows viz. from our general considerations in the beginning of this communication, that curve E must be situated between the curves (L_1) and (L_2). For $x_2 : x_1 = \infty$ curve E coincides with (L_2) for $x_2 : x_1 = 0$ with curve (L_1). When $x_2 : x_1$ changes from ∞ towards 0 than curve E moves in the direction of the hands of a clock from (L_2) towards (L_1).

Firstly we now take the case a, so that we must imagine in fig. 3 curve (L_2) to be drawn upwards starting from i. When we do change now $x_2 : x_1$ from ∞ to 0, then it follows from the different positions which curve E may obtain, that the following cases may occur:

$$(dT)_x > 0 \quad \text{and} \quad (dP)_x > 0$$
$$(dT)_x > 0 \quad \text{and} \quad (dP)_x < 0$$
$$(dT)_x < 0 \quad \text{and} \quad (dP)_x < 0$$

In case b we must image in fig. 3 curve (L_2) to be drawn downwards starting from i. When we do change $x_2 : x_1$ from ∞ to 0, then it follows from the situation of curve E:

$$(dT)_x > 0 \quad \text{and} \quad (dP)_x < 0$$
$$(dT)_x < 0 \quad \text{and} \quad (dP)_x < 0$$

In case c fig. 4 is true. When $x_2 : x_1$ changes again from ∞ to 0, then it follows from the position of curve E:

$$(dT)_x > 0 \quad \text{and} \quad (dP)_x < 0$$
$$(dT)_x < 0 \quad \text{and} \quad (dP)_x < 0$$
$$(dT)_x < 0 \quad \text{and} \quad (dP)_x > 0$$

We see that those deductions are in accordance with the previous ones and with the figs 1 and 2.

Our previous considerations are all valid in the supposition that the four phases $F\,L_1\,L_2$ and G are situated with respect to one another as is indicated in the figs 1—4. When the four phases are situated otherwise with respect to one another, the reader my deduce all in similar way.

We now shall assume that the new substance is volatile, so that it occurs in the phases $L_1 L_2$ and G with the concentrations $x_1 x_2$ and x_2.

We find with the aid of (7) and (19):

$$M\,(dT)_x = -(1+a)\,\Delta V' x_1 + [a\Delta V_1 - (1+b)\,\Delta V]\,x_2 + b\,\Delta V\,.\,x_2 \qquad (36)$$

and with the aid of (8) and (20):

$$M\,.\,(dP)_x = -(1+a)\,\Delta H' x_1 + [a\Delta H' - (1+b)\Delta H]x_2 + b\,\Delta H\cdot x_2 \qquad (37)$$

wherein

$$M = (\Delta H\,.\,\Delta V' - \Delta H'\,.\,\Delta V) : RT$$

so that the direction of temperature and pressure of curve E are defined by (36) and (37).

As ΔV is very small in comparison with $\Delta V'$ we may neglect in (36) the terms with ΔV as long as x_2 is not very large, then it follows with approximation:

$$(dT)_x \gtrless 0 \quad \text{voor} \quad \frac{x_2}{x_1} \gtrless \frac{y-y_2}{y-y_1} \;.\;.\;.\;.\;.\;. \qquad (38)$$

Only for very great values of x_2 in comparison with x_1 and x_2 the term $b\,\Delta V\,.\,x_2$ in (36) will be of great importance and will be approximately

$$(dT)_x = \frac{RT\,\Delta V}{\Delta H\,.\,\Delta V'}\,b x_2 = \frac{RT}{\Delta V'}\cdot\left(\frac{dT}{dP}\right)_G b x_2 \;.\;.\;. \qquad (39)$$

In (37) ΔH is not small in comparison with $\Delta H'$ and the term $b\,\Delta H\,.\,x_2$ will assert its influence already with values of x_2 which are not too small.

Consequently, in general the influence of the new substance on $(dT)_x$ and $(dP)_x$ will be larger in proportion as the new substance is more volatile and it will assert its influence sooner on $(dP)_x$ than on $(dT)_x$.

We may also deduce anything about the position of curve E with the aid of the general considerations at the beginning of this communication. Hence it follows viz that curve E must be situated either between the curves (L_1) and (L_2) or between (L_1) and (G) or between (L_2) and (G). As in the figs 3 and 4 the prolongation of each of those curves is situated between both the other curves, curve E may go, therefore, starting from point i in every direction.

Consequently the temperature may as well increase as decrease, and the pressure may increase or decrease as well at rising as at falling temperature, dependent on the position of curve E.

It follows from (12):

when x_1 is extremely small with respect to x_2 and x_3 then curve E is situated between (G) and (L_2);

when x_2 is extremely small with respect to x_1 and x_3 then curve E is situated between (G) and (L_1);

when x_3 is extremely small with respect to x_1 and x_2 then curve E is situated between (L_1) and (L_2);

when x_1 is extremely large with respect to x_2 and x_3 then curve E is situated in the vicinity of L_1);

when x_2 is extremely large with respect to x_1 and x_3 then curve E is situated in the vicinity of (L_2);

when x_3 is extremely large with respect to x_1 and x_2 then curve E is situated in the vicinity of (G).

In each of those cases we can see at once from the figs 3 and 4 which signs $(dT)_x$ and $(dP)_x$ may have.

When f.i. x_2 is very small with respect to x_1 and x_3 then curve E is situated between (L_1) and (G); when now fig. 4 is valid then the pressure shall, therefore, always increase and the temperature shall decrease. In the special case only, when x_3 is still also extremely large with respect to x_1 and when at the same time $\Delta V > 0$ [then curve (G) proceeds, starting from i, a little to the left] then the temperature may fall a little.

When we add a new substance which is not volatile, but which forms mixed crystals with the solid substance F, then we have in figs. 3 and 4 the curves (F) (L_1) and (L_2). It appears from the position of those curves with respect to one another that the previous considerations are also valid in this case.

When we wish to calculate $(dT)_x$ then, as is apparent from (19) we have to substitute in (36) $b \Delta V x_3$ by $(\Delta V + \Delta V')x$. When we neglect again the terms with ΔV then we find:

$$M(dT)_x = [x - (1+a)x_1 + ax_2] \Delta V'$$

or:

$$(dT)_x = \frac{RT}{\Delta H} \cdot \frac{x(y_1 - y_2) - (y - y_2)x_1 + (y - y_1)x_2}{y_1 - y_2} \quad . \quad . \quad (40)$$

In the figs 5 and 6 YZ represents a side of the components-triangle, $F L_1 L_2$ and G the four phases of the invariant binary equilibrium $E(x = 0)$. When we add a new substance then the ternary equilibrium $E = F + L_1 + L_2 + G$ arises. The solid sub-

stance F and the liquids L_1 and L_2 then proceed along the curves $F\,q\,r$, $L_1\,q_1\,r_1$ and $L_2\,q_2\,r_2$. When we add only little of the new substance, then the 3 phases are represented by the points $q\,q_1$ and q_2 which we must imagine in the immediate vicinity of the side YZ.

When we put $t = x\,(y_1 - y_2) - (y - y_2)x_1 + (y - y_1)x_2$ and when we consider x and y as running coordinates, then $t = 0$ represents the equation of the straight line which goes in fig. 5 and 6 through q_1 and q_2.

When the point q is situated on the line $q_1 q_2$ then $t = 0$; the sign of $(dT')_x$ is then determined by the terms which have been neglected in (40).

When q is situated at the right side of the line $q_2 q_1$ (viz. when we go from q_2 towards q_1) as in fig. 5, then $t > 0$; when q is situated at the left side of the line $q_2\,q_1$, as in fig. 6, then $t < 0$. Hence it follows, therefore, that in fig. 5 the temperature increases and in fig. 6 the temperature decreases, as is also indicated in both figures.

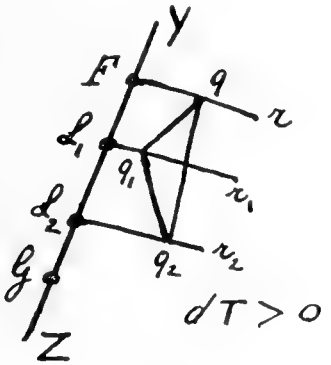

Fig. 5.

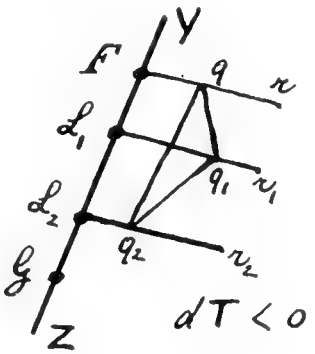

Fig. 6.

Consequently we find the following:

when we add to the invariant binary equilibrium $E\,(x = 0) = F + L_1 + L_2 + G$ a substance which is not volatile and which forms mixed crystals with the solid substance F, then

the temperature rises, when the threephases-triangle solid-liquid liquid turns its concentrated liquid towards the side of the components-triangle (fig. 5)

the temperature falls when the threephases-triangle turns its concentrated solution away from this side (fig. 6).

Comparing fig. 1 with fig. 5 and fig. 2 with fig. 6, the reader will see that for the change of temperature the same rules are true, independent of the fact whether the new substance forms mixed crystals with F or not.

Finally we could still treat the general case that the new substance forms not only mixed crystals with F but that it is volatile also.

It follows from figs. 3 and 4, in connection with the theories discussed in the beginning of this communication that curve E can go in all directions, starting from point i.

In order to define $(dT')_x$ we must still include in (36) the term $(\Delta V + \Delta V')_x$; then we get again (40) approximately unless x_2 is extremely large.

Consequently in this case also the figs. 5 and 6 remain valid, unless the threephases-triangle $q\,q_1\,q_2$ becomes very narrow and the concentration of the new substance in the vapour is extremely large.

(*To be continued*).

Leiden, *Inorg. Chem. Lab.*

Anatomy. — "*The Development of the Shoulder-blade in Man*". By O. H. DIJKSTRA. (Communicated by Prof. L. BOLK).

(Communicated at the meeting of March 24, 1923).

Unlike the development of the clavicula that of the scapula has received comparatively little attention. The textbooks of anatomy (CUNNINGHAM, GEGENBAUER, RAUBER—KOPSCH, MERKEL, POIRIER—CHARPY, TESTUT) contain only general notions such as the information that the ossification of the shoulder-blade begins in the vicinity of the collum scapulae at the end of the second or in the beginning of the third month. POIRIER and CHARPY speak of an incipient ossification between the 40th and 50th day. BARDELEBEN reports a periostal ossification (such as occurs with the bones of the cranial vault) beside and under the spina scapulae at the end of the 10th week.

BRYCE alone enters into more details in QUAINS's Elements of Anatomy. According to his description the rudiment of the shoulder-blade is in the 6th week entirely cartilaginous, proc. acromialis and proc. coracoïdeus are present, but the spina scapulae is wanting. (Nevertheless BRYCE reproduces the diagram of LEWIS [1]), in which a spina is really indicated). In the 8th week ossification begins with a centre near the collum scapulae, developing into a triangular plate, at whose upper margin the spina appears in the 3rd month as a low ridge. At birth coracoid and acromion, margo vertebralis and the margin of the spina are still made up of cartilage. This description by BRYCE agrees fairly well with the one we find in BROMAN's textbook of Embryology and in that of KEIBEL and MALL, in which BARDEEN deals with this subject. BROMAN, like BRYCE, states that no spina is to be found at the cartilaginous scapula. Nonetheless he reproduces the figure of LEWIS, in which there is indeed a spina. KOLLMANN, SCHENCK, MINOT, PARKER do not speak of the first development of the shoulder-blade and only dwell on stadia of advanced ossification. In HERTWIG's Entwickelungsgeschichte BRAUS and also HERTWIG himself report a separate centre of ossification in the spina scapulae; according to the latter the spina in the neonatus still consists of cartilage sometimes; according to KÖLLIKER (quoted by BADE, Arch. f. mikr. Anat. LV) this is even always the case.

[1]) Am. Journ. Anat. Vol. I. 1901—'02.

The most detailed report concerning the development of the shoulder-blade is that by BRYCE and BROMAN. From their figures it is evident that they derive their data from LEWIS, who published in the American Journal of Anatomy (Vol. I 1901—'02) a minute description of the development of the arm in man. Broadly stated his data agree with those of BRYCE, mentioned above. They differ, however, as to the spina scapulae. According to LEWIS the spina probably takes origin in the upper margin of the scapula. This margo superior thickens and then splits into a medial and a lateral lip. The medial lip is the future margo superior, the lateral one is the first beginning of the spina scapulae.

HAGEN [1]) describes a shoulder-blade of an embryo 17 mm. in length. The spina scapulae is absent, the proc. coracoïdeus is large, the proc. acromialis small. The latter statement cannot be reconciled with LEWIS's communication, which, on the contrary, speaks of a relatively large proc. acromialis.

This review of the literature would not be complete without mentioning the interesting study by RUTHERFORD [2]) who entered into many details of the development of the shoulder-blade. Like LEWIS he constructed wax models of the skeleton of the shoulder-girdle, and i. a. found that the spina scapulae originates in very early ossification of derivates of cartilage cells, situated between M. supra- and infraspinatus.

From this review it is clear that our knowledge of the modus of development of the shoulder-blade in man is still limited. The shape in the initial stages of development is described differently. Conflicting views are held as to the genesis of the spina and from the contents of this paper it will be seen that these are not the only points of controversy.

With a view to trace the development of the shoulder-blade in man, I constructed wax models of various stages of development. Fig. 1 represents the wax model of the shoulder-blade of the youngest embryo, 16 mm. in length. The scapula is drawn from the lateral side and from above.

The reconstruction shows:

1°. that the shoulder-blade lies in a sagittal plane, so that the lower half is in contact with the three upper ribs. Processus acromialis and clavicula are not in contact as yet.

2°. that the processus coracoïdeus is large; the processus acromialis is relatively small. The joint-cavity rests chiefly on the processus coracoïdeus.

[1]) Arch. f. Anat. u. Entwickel. Gesch. 1900.

[2]) Journal of Anatomy and Physiology 1914.

3°. There is no indication of a spina scapulae. The margo superior is neither thickened nor split into two labia.

Fig. 1.

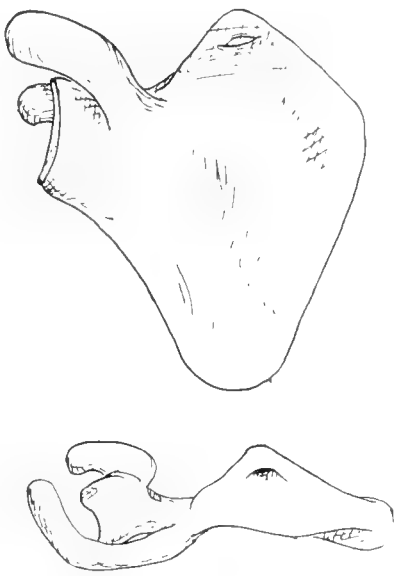

Fig. 2.

4°. The margo superior is straight, so there is no incisura scapulae.

5°. For the rest the shape of the scapula fairly well agrees with that of an adult shoulder-blade. In reconstructing the scapulae of two monkey embryos (viz. Macacus cynomolgus 17 mm. in length, and Semnopithecus maurus) it became evident that, also in these primates, the embryonic shoulder-blade already in its first beginning resembles that of the adult. Here also a spina was absent.

6°. Close beneath the angulus superior we observe a well-defined fovea where a foramen is found in older stages of development. To this we shall revert when discussing the following stage.

This stage is illustrated in fig. 2. It concerns the shoulder-blade of an embryo, 25 mm. in length. Also in this stage any indication of a spina scapulae or of a thickening of the margo superior is lacking. Nevertheless when compared with the first stage some modifications can be recognized.

1°. The shoulder-blade does not lie any more in a sagittal plane, but makes an angle with it, as is also the case with the adult. The joint-cavity lies at the level of the first rib. Acromion and clavicula have joined.

2°. The processus coracoïdeus has comparatively decreased, the processus acromialis, on the other hand, has increased. It appears, then, that the processus coracoïdeus, which is phylogenetically the oldest part, is most strongly developed in the youngest stage, whereas the processus acromialis, which is phylogenetically younger, comes more to the fore in the older stages. The joint-cavity now lies for the greater part on the planum scapulae.

3°. The margo vertebralis consists of a shorter upper portion and a longer lower portion. They are at an obtuse angle to each other.

4°. The portion of the scapula from which afterwards the fossa supraspinata develops, makes an angle with the future subspinal portion. This deviation of the upper part, which also occurs in the adult shoulder-blade (since fossa supra- and infraspinata do not lie in one and the same plane), had not yet taken place in the 16 mm. embryo.

5°. In the cranial part of the shoulder blade a foramen occurs under the angulus superior, which extends at the costal plane of the scapula as a groove along the margo superior in the direction of the joint-cavity. In fig. 3 we give a cross-section of this foramen, which is filled with connective tissue.

The existence of this foramen is no doubt surprising; yet it was not entirely unknown, as already RUTHERFORD has described it (l. c.). However, according to this author it proceeds in a groove, which reaches as far as the margo vertebralis. Now, in all the serial sections in which I also met with a groove as well as with the foramen, it proceeded along the margo superior in the direction of the joint-cavity.

RUTHERFORD explains this foramen as follows. He considers the part of the scapula, cranial to the foramen (resp. groove), as a separate piece of cartilage, which he terms *praescapula*, and which, according to his account, is connected by a strand of mesenchyma tissue with the sternal half of the clavicula. In this way he believes an inner shoulder-girdle to have developed, while he supposes the acromion-clavicula to build up the outer girdle. He adduces various arguments to prove this; however, they are weak. In my judgment the hypothesis is of no value, because a connection of the so-called praescapula with the sternal half of the clavicula does not occur. At all events in my preparations I never found a cell-strand like the one described by RUTHERFORD.

This foramen is not present in all cases. Its development also differs with various individuals, as shown by the following data. I could establish its presence either as a true foramen, or as a deep groove in human embryos of the length of 16, 17.5, 18, 19.6, 21, 22, 25 (see fig. 3), 26, 27, 56, and 90 mm. On the other hand I did not

recognize it in embryos of 12, 18, 18, 24, 26, 40, 120 mm. From this it follows that it is not infrequently absent. In some embryos the portion of the planum scapulae cranial to the foramen, i.e. RUTHERFORD's praescapula, made an angle with the rest of the planum, a fact that lends support to RUTHERFORD's view, viz that it is really a separate piece

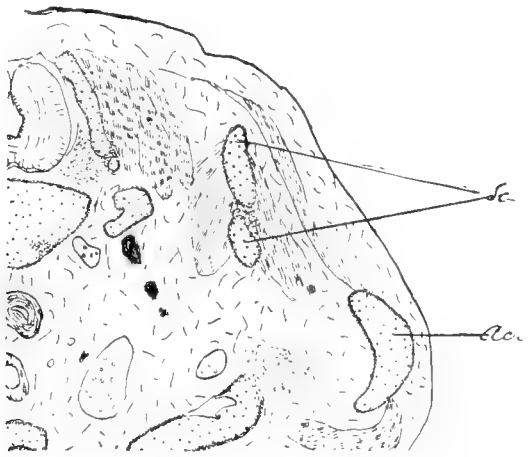

Fig. 3. Homo 25 mm. transverse. *Sc* = Scapula; *Acr* = Processus acromialis.

of cartilage. The foramen which, in young embryos, is situated rather closely to the margo superior, as observable in fig. 3, migrates in older embryos towards the margo vertebralis. Consequently RUTHERFORD's praescapula is relatively enlarged.

Now it is an interesting fact that this foramen does not occur in any other mammal, neither in reptiles, nor in amphibians. At least I never detected any. The following embryos I have examined for the occurrence of this foramen.

Semnopithecus maurus 20 mm. (C. R.)

Macacus cynomolgus 17 mm. (C. R.)

Cercopithecus 2 stages.

Sus scrofa N. T. (Keibel) 83—85, N. T. 88, N. T. 88, N. T. 91, 24 mm. (C. R.) 26 mm. (C. R.) In the last two embryos two foramina were recognized in the fossa infraspinata. It is not quite impossible that these foramina are analoga of the foramen in the human shoulder-blade.

Bos taurus 21 mm. (C. R.)

Ovis aries 19.5, 20.5, 21.5, 22.5, 23, 23.5, 26, 27, 29, 35, 45 mm. (C. R.)

Canis familiaris 12, 12, 22, 23.5 mm. (C. R.)
Sciurus vulgaris 12, 30 mm. (C. R.)
Mus decumanus 11.5, 12, 13, 13, 13.2, 14.5, 16, 18, 20, 22 mm. (C. R.)
Lepus cuniculus 17, 20 mm. (C. R.)
Spermophillus citillus 15 mm. (C. R.)
Rousettus amplexicaudatus 7.5, 10.5, 11, 11, 11.5, 12, 12, 14.5, 15,5, 16, 18 mm. (C. R.)
Talpa europea 8.5, 9, 9, 10, 12, 13, 16.5, 20 mm. (C. R).
Perameles obesula 50 mm. (C. R.)
Perameles spec. 38 mm. (C. R.)
Dasyurus viverrinus 19.6, 33, 36, 40, 53, 63 mm. (C. R.)
Sminthopsis crassicaudatus 13, 25 mm. (C. R.)
Phascalogale pennicillata 37 mm. (C. R.)
Trichosurus vulpecula 32 mm. (C. R.)
Didelphys cancrivora, 4 embryos of 25 mm. length.
Lacerta agilis N. T. (Keibel) 117, 118, 120, 123, 123, 124, 125, 126.
Calotes iubatus, length of the head $5^1/_2$ mm.; 7 mm.
Lagysoma 27.5 mm.
Hemidactylus fren. length of the head 4.5 mm.
Salamandra mac. 11, 13, 15, 16, 16, 24 mm.
Pipa Americana, 12 mm.
Rana . 2 embryos.

So far as I am able to judge foramina in adult shoulder-blades occur only with Homo and with various Edentata, in which they are always formed by bridging of the Incisura scapulae, and with Delphinus delphis. In the latter the character of the foramen is not known. Rutherford (l. c.) has described it.

A conceivable connection, that might exist between the praescapula of Rutherford and the attachment of the clavicula (not only the sternal half of the clavicula, as Rutherford supposed) to the margo superior scapulae, as it occurs in reptiles, echidna and ornithorynchus, could not be ascertained, since a connection of the praescapula of Rutherford to the acromial part of the clavicula could not be detected either.

It appears, then, that the foramen, present in the majority of human embryos in the cranial part of the shoulder-blade, does not occur in other vertebrates, (except in Delphinus delphis, which, however, is of such a pronounced specificity that this foramen cannot be looked upon as a homologue of that of man). Neither did I find any attachment of the praescapula of Rutherford to any other

skeletal bone. The significance of this foramen is unknown as yet.

As to the ossification of the scapula my experience proved it not to be so simple as is represented in the literature.

The earliest ossification I observed in an embryo of 40 mm. I constructed a wax model (fig. 4) of the scapula of this embryo.

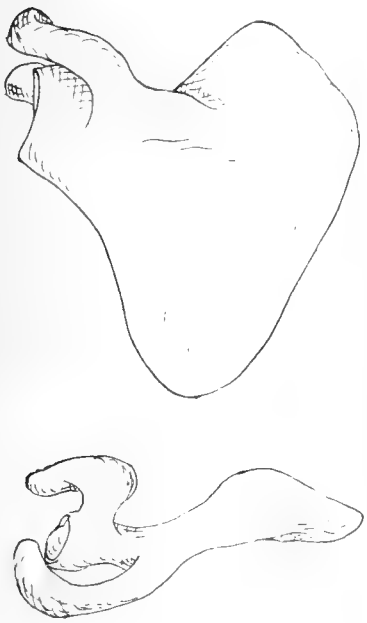

Fig. 4.

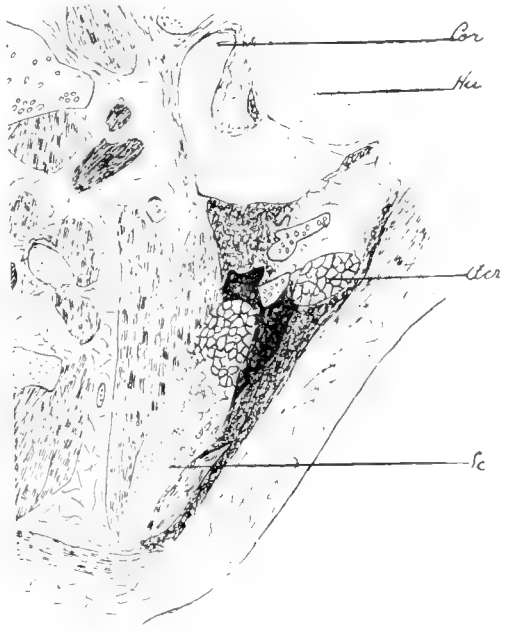

Fig. 5. Homo 40 mm. Transversal.
Cor = Processus coracoideus; *Hu* = Humerus;
Acr = Processus acromialis; *Sc* = Scapula.

Like the preceding model this also is viewed from above and from the dorso-lateral side. What this reconstructed model shows us may follow here:

The joint-cavity, lying at the level of the first rib, is now located almost entirely on the planum scapulae (as with the adult scapula). Of the spina not a trace is visible as yet, the margo superior is not thickened. To the basis of the processus acromialis an area of closely packed mesenchyma is attached, which extends between the muscular tissue and separates the rudiment of Musc. supra-, and infraspinatus.

This area of mesenchyma is cut in a cross section as represented in fig. 5. Behind the root of the processus acromialis begins a perichondrial ossification, which continues into this condensed mesenchyma. This ossification is the first formation of the spina. We see,

therefore, that it is formed by a perichondrial ossification, for although no ossifying perichondrium is visible here, the fact that the bone is formed from the surrounding mesenchyma co-ossifying with cartilage, established the character of the ossification. In fig. 5 we give a cross section of this first stage of the spina.

I have not been able to recognize two centres of ossification in the cartilaginous scapula, described by RAMBAUD and RENALT (quoted by POIRIER [1]), which, according to these authors, arise between the 40th and 50th day and fuse in the third month.

In the scapula of an older embryo (56 mm. in length) this perichondrial ossification appears to be largely extended. The margo anterior scapulae is almost reached. The cartilage of the planum

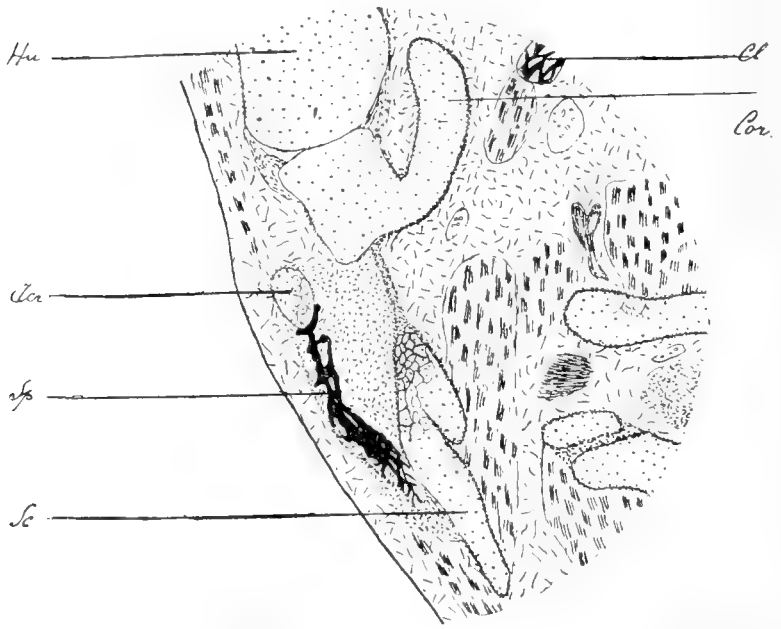

Fig. 6. Homo 56 mM. Transversal. *Hu* = Humerus; *Cl* = Clavicula; *Cor* = Processus coracoideus; *Acr* = Processus acromialis; *Sp* = Spina scapulae; *Sc* = Scapula.

scapulae, however, has been distinctly calcified over a considerable area already. The marked enlargement of the spina scapulae is shown in fig. 6. Besides the spina this figure also shows part of the foramen described above. The spina is formed by a growth of bone between

[1]) POIRIER et CHARPY, Traité d'Anatomie humaine.

M. supra- and infraspinatus, between acromion and planum scapulae. It cannot be denied, however. that in the mesenchyma, in which this bone develops, very young cartilage-cells are noticeable here and there. These cells, however, have no intermediate matter as yet; they are little differentiated and it is difficult to distinguish them from the mesenchyma-cells. So it is evident that besides bone-cells also cartilage-cells develop in the mesenchyma.

In an embryo of 90 mm. enchondrial as well as perichondrial ossification takes place, the boundary between the two being no

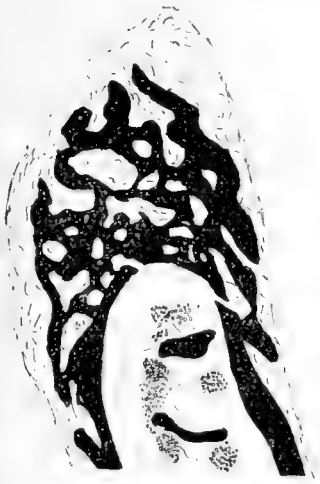

Fig. 7. Homo 90 mm. Margo anterior scapulae transversal.

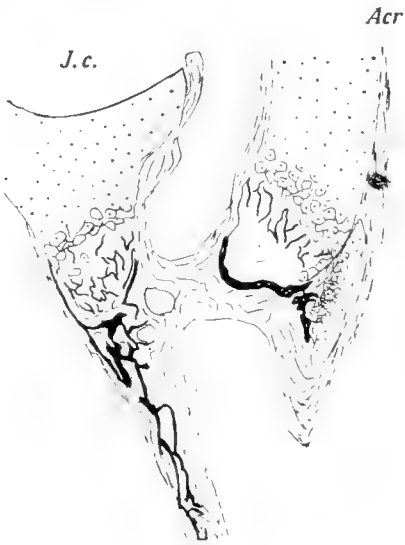

Fig. 8. Homo 90 mm. Scapula transversal *Acr.* = Processus acromialis *J.c.* = Joint-cavity. *Pl. Sc.* = Planum scapulae.

longer perceivable. The peculiar character of the perichondrial ossification along the margo anterior is remarkable. In the place of the formation of compact bone, which in other cases occurs with perichondrial ossification e.g. that of the long bones, we see here a bony framework encircled by mesenchyma. Fig. 7 shows a cross section through the margo anterior.

The study of this object (embryo of 90 mm.) shows remarkable pecularities of the growth of the spina scapulae. In the mesenchyma between M. supra- and infraspinatus a distinct cartilage is now recognizable. It is quite independent of the other mass of cartilage

of the scapula. It is younger than the remaining part of the shoulder-blade; nevertheless it has already calcified to some degree and forms bone of the spina.

The cartilage has been cut in three different cross sections, as represented in the figures 8, 9 and 10. Fig. 8 illustrates a section through the scapula above the place of attachment of the processus acromialis. In the mesenchyma, which extends from the processus acromialis towards the margo vertebralis, lies the cartilage which is already partly calcified. In fig. 9 we give a section at a lower level.

The processus acromialis attaches itself at this level to the planum scapulae. Here also we observe the cartilage of the spine, independent of the remaining cartilage of the shoulder-blade. Fig. 10 shows a section through the scapula at the level of the lowest place of

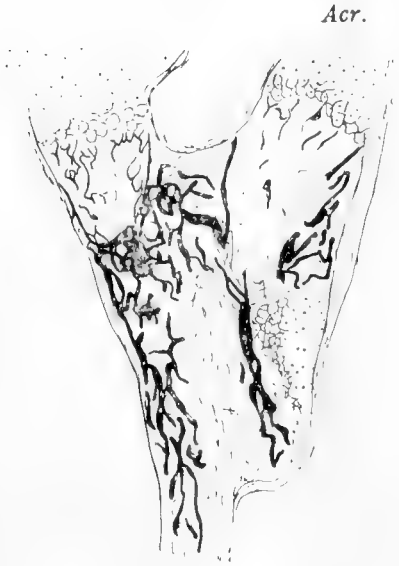

Fig. 9. Homo 90 m.m. Scapula transversal. *Acr.* = Processus acromialis. *Pl. Sc.* = Planum scapulae.

Fig. 10. 90 m.m. Scapula transversal. *C.* = cartilage of the spine. *Pl. Sc.* = Planum scapulae.

attachment of the spina. The young cartilage, which forms the spina, has here been cut over a large area. The cartilage will be seen to

be partly calcified, while bone has been formed, uniting with this calcified area.

So while the first beginning of the spina is formed by perichondrial bone in the mesenchyma between M. supra-, and infraspinatus, its further development is effected by chondrial bone, which originates in the younger cartilage. This cartilage has been generated between the afore-said muscles by the same mesenchyma.

A peculiar feature is still to be observed at the shoulder-blade of the embryo of 90 mm. Bone is developed at the margo superior as well enchondrially as perichondrially. In the mesenchyma that forms the perichondrial bone, and into which this bone extends over some distance, there are two cartilaginous nuclei, made up of the same young tissue from which the cartilage of the spina has been built up. Fig. 11 shows in cross section these nuclei, which are not in contact with the remaining cartilage of the shoulder-blade. These cartilage-islets appear to be already calcified and ossified here and there. It is impossible to draw a boundary-line between the bone formed in this process and the perichondrial bone of the scapula. This ossifying process, in which (besides the enchondrial ossification of the scapula) both perichondrial and chondrial ossification of a cartilage nucleus, situated outside the perichondrial bone, are present, agrees completely with the formation of the spina scapulae. This is striking, since the spina scapulae and the definitive margo superior are the two parts of the shoulder-blade, which are missing in the first rudiment of the cartilaginous scapula. This deficiency vertebral of the place destined for the future incisure, is indeed accounted for by the fact that the margo superior in young embryos is still straight and displays no incisure. The missing parts are apparently supplied by the perichondrial bone that reaches far into the mesenchyma, together with the bone formed by the afore-said cartilage-nuclei. At the shoulder-blade of an embryo of 120 mm in length, in which the ossification had considerably advanced, the incisure was indeed present.

Of course, the question arises, how the cartilage of the spina as well as the cartilage nuclei are further developing. In both places the cartilage is soon transformed completely into bone. In an embryo of 120 mm. only a very few remnants of the cartilage of the spina were still left. The rest had been ossified.

After this the development of the shoulder-blade proceeds in the way described in the text-books of embryology.

Now let us review once more the current opinions of the development of the spina scapulae. It will be seen, then, that however

divergent they may be, most of them cannot be deemed incorrect, when we bear in mind that they concern different stages.

Fig. 11.
Homo 90 m.m.
Margo superior scapulae transversal.

Rutherford's view of the very early ossification of cartilaginous cells is no doubt correct, but holds good only for young stadia. Neither is the conception of Hertwig and Braus about a separate centre of ossification quite incorrect, since there is a stage in which an independent cartilage is forming bone. Bardeleben's record about an ossification under and beside the spina cannot altogether be disqualified either, but it only applies to a brief stage of development. However, ossification like that of the bones of the cranial vault does not occur in the development of the shoulder-blade. In the neonatus a few cartilage may possibly sometimes be found at the spina (Bryce), but it is certain that the spina scapulae in the new-born child does not consist of cartilage. (Kölliker and Hertwig advocate the opposite view). Lewis's conception, however, (doubling of the margo superior) is altogether wrong. The diagram borrowed from Lewis by Broman, Bryce and Bardeen represents a faulty reconstruction of the shoulder-blade.

Zoology. — *"Secondary sex-characters and testis of the ten-spined Stickleback (Gasterosteus pungitius L.)."* By Dr. G. J. VAN OORDT. (Communicated by Prof. J. BOEKE).

(Communicated at the meeting of March 24, 1923)

It is generally known that the sex-glands strongly influence the so-called secondary sex-characters. This is apparent from the marked somatic and psychic differences, which e. g. Mammals or Birds, castrated at an early age, show, when compared with normal animals.

At present it is generally accepted that in Vertebrates this effect, resulting from the gonads, takes place by internal secretion, that is by the influence of certain substances, which pass into the blood ("hormones"). As the correlation between the secondary sex-characters and the gonads generally is most distinct in male Vertebrates, I will speak only of the formation of these hormones in the testis for convenience' sake.

Recently it has been especially attempted to ascertain, by which part of the male gonad these hormones are formed. The numerous investigators, treating this subject, chiefly hold the two following, contradictory opinions.

According to STIEVE (1922) and others these hormones are exclusively formed by the sexual cells, whereas BOUIN and ANCEL (1903), STEINACH (1920), LIPSCHÜTZ (1919), BASCOM (1923), their collaborators and others are of opinion that these hormones originate in the interstitial cells (LEYDIG's cells), situated in the interstitium of the male gonad. According to STIEVE these cells are only thropic elements for the sperm cells. Consequently no value must be attached to the name „Puberty Gland", which name was given to the collective LEYDIG's cells by STEINACH and LIPSCHÜTZ.

Up till now the investigators, when treating the subject above mentioned, have chiefly examined Mammals, Birds and Amphibia. For that reason I resolved to trace the changes in the testis at the appearance of the secondary sex-characters in a Fish, and so I chose the ten-spined Stickleback (*Gasterosteus pungitius* L.), which was easy to obtain.

During breeding time, in spring, the males of this species possess

a number of secondary sex-characters (cf. TITSCHACK 1922), of which the following are distinctly perceptible.

In spring a very distinct black pigmentation (red in the three-spined species) can be observed at the throat and at the abdomen, which soon spreads over the rest of the body, so that the animals become dark-black, except for their pectoral spines. Outside breeding-time it is difficult to distinguish the males from the females: then both show dark spots on a pale green ground. Individual colour-differences occur.

Every male makes a nest, in which the eggs are deposited. The material of which the nest consists (parts of waterplants etc.) is collected by the male and fastened by means of a secretion, formed by the kidney-tubules and Wolffian Ducts (TITSCHACK 1922, COURRIER 1922*b*, both in *Gasterosteus aculeatus* L.). This peculiar secretion occurs exclusively in the male during breeding time; for that reason in spring the kidney strongly increases in size, the kidney-tubules and the Wolffian Ducts get a larger diameter and exercise a different function.

The male guards his nest and drives off all intruders fiercely. When the eggs have been deposited in the nest, they are at once fertilized. During the development of the eggs, the male takes care that they are constantly provided with oxygen by conducting fresh water to the nest with his pectoral fins. Sometimes, when eggs drop out of the nest, they are again collected by the male and taken back to the nest in his mouth. Whether the young are guarded by the male, after they have left the nest, in nature, is not known to me: care must be taken to separate the young, living in prison, from their father and the other inhabitants of the aquarium, as the young will otherwise be eaten.

The aim of my investigation, begun in September 1922, was to trace the changes, occurring in the testes of the Stickleback at the appearance of the secondary sex-characters. So it was my intention to catch a number of Sticklebacks at fixed times during the winter and the succeeding spring and to examine their sex-glands. At that time I thought that nothing was known as yet about the relation between the secondary sex-characters and the testis of the Stickleback, but it soon appeared to me that COURRIER had already investigated the three-spined Stickleback (*Gasterosteus aculeatus* L.) and had published some papers, regarding this point (1922*a*, 1922*b*).

I therefore changed my original plan and resolved to trace what influence a rather high temperature, about the temperature of

ditchwater in spring (12°—20° C.), would have on the appearance of the secondary sex-characters and what changes would take place in the testes of these animals simultaneously. The sex-glands of control-animals, caught in nature, could serve at the same time to verify the results of COURRIER. In this paper I will only communicate the results, obtained in animals, kept in a temperature of 12°—20° C. during last winter.

In September and October 1922 I caught a large number of specimens of *Gasterosteus pungitius* L. at Rotterdam. They were kept in an aquarium of which the water was often renewed, and they were copiously fed with *Chironomus*-larvae.

All the testes of the Sticklebacks, killed in autumn, contained a more or less large number of spermatozoa. The number of spermatogonia is always small, the number of spermatocytes and spermatids varies in the different specimens. In all cases, examined by me, small groups of interstitial cells (LEYDIG's cells) were present, close to the hilus or there where three or more tubules come together. In a few testes, in which the interstitium is somewhat wider, these cells are also situated between the seminiferous tubules. They were absent in none of the cases examined.

In one specimen (n°. 6), a rather dark-coloured male, not showing the black pigmentation of males during breeding time, however, the interstitium is much wider than in the other males, caught at the same time. The number of interstitial cells is also larger in this specimen, while in the seminiferous tubules spermatozoa are almost exclusively found.

Oblong connective tissue-nuclei are observed everywhere in the interstitium of the testes of animals, caught in autumn; blood-vessels are present, but they are not numerous; they are narrow and contain few blood-cells.

This testis-structure is shown by animals, caught in September and the beginning of October, and which were kept in an aquarium of which the water then agreed in temperature with ditchwater.

The testes of Sticklebacks, kept for two, three and even four months, i. e. till the end of January 1923, in a temperature of 12°—20° C., all increase in size and show the following structure. The spermatogenesis is very intensive. In all testes this process takes place from the exterior to the interior, i. e. the spermatozoa are situated as a rule more in the centre, the spermatogonia and spermatocytes more at the periphery of the gonad. The interstitium of such animals does not change; it remains narrow, the number

of LEYDIG's cells is generally small and they are especially present near the hilus and there where three or more seminiferous tubules come together.

Till the end of January it was difficult to distinguish the males from the females. In the last days of January, however, one of my specimens showed at throat and abdomen a faint black pigmentation, which soon increased strongly. Besides, this animal became very agressive and in the beginning of February he began to collect material for the nest. On the 14th or 15th of February the eggs were laid in the completed nest; (I cannot give the exact date, as the female was not seen in this nest). On the 16th of February this male was killed.

The nuptial colours successively developed in the other males, which soon began to prepare their nests. After the eggs had been deposited in them, they were carefully guarded by the males, which constantly conducted fresh water to the nests.

On comparing the testes of animals killed in the end of December or in January with the testes of these males, we see that the latter have greatly changed.

The spermatogenesis has totally come to an end. The seminiferous tubules are entirely filled with a large number of spermatozoa. Moreover, at the periphery of the tubules small groups of spermatozoa are to be seen, the heads of which are directed to the wall and the tails to the centre of the tubules. The number of spermatogonia and spermatocytes has strongly decreased.

The interstitium is no longer narrow but is enlarged; the number of LEYDIG's cells has strongly increased; the blood-vessels have become more numerous and larger.

So we see that the high temperature of the water in winter favours the spermatogenesis and that consequently after four months a testis originates of which the seminiferous tubules practically contain spermatozoa exclusively. Then the secondary sex-characters distinctly develop, the interstitium is enlarged and the cells of LEYDIG and the blood-vessels increase in number.

So I have observed a coincidence of the occurrence of the secondary sex-characters and the termination of the spermatogenesis, while simultaneously an enlargement of the interstitium with increase in number of the LEYDIG's cells and of the bloodvessels takes place. This does *not* prove, however, that a correlation exists between these phenomena.

According to COURRIER (1922*a*, 1922*b*) it does. This investigator observed in the three-spined Stickleback that after the spermato-

genesis the interstitium increases considerably in size. In it a strong augmentation of the number of LEYDIG's cells and of the bloodvessels has taken place. According to COURRIER the testes of Sticklebacks, caught in winter, only contain a few interstitial cells here and there. The spermatogenesis, which is very intensive in spring till the end of March, has no influence on the development of the secondary sex-characters. The latter occur not earlier than at the end of April, simultaneously with the strong development of the interstitial cells. As he, moreover, observes the same granules in the cells of LEYDIG and in the bloodvessels, situated close to them, he assumes that the hormones which influence the development of the sex-characters are formed in the interstitial cells and pass from the latter into the blood. In my opinion it might be that the granules, observed by COURRIER, are transmitted by the blood to the interstitial cells.

COURRIER has also kept his fishes in water of 17° C. (1922*a* and 1922*b*, p. 137) during a part of the winter. After two months and a half the structure of the seminiferous tubules of these animals resembles that of animals during breeding time i. e. they are entirely filled with spermatozoa and contain only a few spermatogonia, spermatocytes and cells of SERTOLI. Changes in the interstitium have not occurred. Consequently, the secondary sex-characters have not developed in these animals. COURRIER thinks, however (1922*a*, in a note), on the ground of experiments, which were in progress at that time, that the interstitium would increase in size, when exposed longer to a high temperature and that consequently the sex-characters would also develop in these animals.

I think I am justified to conclude from my investigations, described above, that the correlation of interstitial cells and secondary sex-characters is not so easy to establish.

In the first place *all* testes of *Gasterosteus* possess a more or less large number of interstitial cells. These evidently do not cause the development of the secondary sex-characters. Here I must especially point to the male above described (N°. 6) of which the testes contain a wide interstitium with many LEYDIG's cells and of which the seminiferous tubules are entirely filled with spermatozoa. The secondary sex-characters had not developed in this animal, however. Among the testes of control-animals, caught in nature in winter, I also found some of which the tubules almost exclusively contained spermatozoa and of which the interstitium with numerous interstitial cells is rather strongly developed. These animals, however, did not show sex-characters either.

In a very recent paper CHAMPY (C. R. Soc. de Biologie, Séance du 17 Février 1923) communicates that he has obtained Sticklebacks (*aculeatus*) with nuptial colours last winter and that in the testes of these animals he had not observed a well-developed interstitial tissue. As he has not found any interstitial cells in the testes of various species of fishes with distinct secondary sex-characters, CHAMPY is of opinion that these cells have no influence on the development of those sex-characters and that the formation of the hormones responsible for the development of these characters would take place by means of the sexual cells.

Finally, I will once more call attention to the fact that the testes, examined by me, in which the spermatogenesis has almost come to an end, possess a more strongly developed interstitium than testes, in which the spermatogenesis is still in full swing. Possibly this fact points to a correlation between spermatogenesis and interstitial cells. Whether the sex-hormones are formed in the seminiferous tubules as well, I cannot decide at this moment. Later on, when I have more material at my disposal, I hope to recur to this subject in a more detailed paper.

Zoölogical Laboratory of the Veterinary College. Utrecht, March 1923.

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KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXVI

Nos. 5 and 6.

President: Prof. F. A. F. C. WENT.
Secretary: Prof. L. BOLK.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vol. XXXII).

CONTENTS.

Mathematics. — "*On* EULER's *Constant*". By Prof. J. C. KLUYVER.

(Communicated at the meeting of May 26, 1923).

In calculating the value of EULER's constant C the summation formula or any other asymptotic series is used, and one term at least in the expansion is always a transcendental quantity. It would be preferable to represent C as a convergent expression containing rational terms only, because such a representation of the number C perhaps eventually will furnish the means to establish its irrationality. As yet VACCA's series [1])

$$C = 1\left(\frac{1}{2}-\frac{1}{3}\right)+2\left(\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}\right)+$$
$$+3\left(\frac{1}{8}-\frac{1}{9}+\ldots-\frac{1}{15}\right)+4\left(\frac{1}{16}-\frac{1}{17}+\ldots-\frac{1}{31}\right)+\ldots$$

is the only result in the desired direction, and as a second I will add the proof that $C-\frac{1}{2}$ can be expanded in a convergent continued fraction

$$\frac{1}{|a_1|}+\frac{1}{|a_2|}+\frac{1}{|a_3|}+\frac{1}{|a_4|}+\ldots,$$

the quantities a_k being throughout positive and rational.

Following STIELTJES' method [2]) for converting an integral into a continued fraction, I consider the integral

$$J(z)=\int_0^\infty \frac{du}{u+z}\quad f(u)=\int_0^\infty \frac{du}{u+z}\cdot\frac{1}{e^{2\pi\sqrt{u}}-1},$$

supposing $z>0$. Expanding the integrand in powers of $\frac{1}{z}$, term-by-term integration gives the divergent series

$$\frac{c_0}{z}-\frac{c_1}{z^2}+\frac{c_2}{z^3}\ldots\ldots+(-1)^h\frac{c_h}{z^{h+1}}+\ldots\ldots,$$

the coefficients of which are determined by the equation

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[1]) Q. J. Math., London, vol. XLI, p. 363.

[2]) Recherches sur les fractions continues. Oeuvres complètes, II, p. 402.

$$c_h = \int_0^\infty u^h f(u)\, du = \frac{2}{(2\pi)^{h+1}} \int_0^\infty \frac{v^{2h+1}}{e^v - 1}\, dv = \frac{B_{h+1}}{2h+2}.$$

Hence c_h, directly deduced from the Bernoullian number B_{h+1}, is a positive and rational quantity.

In order to evaluate the integral $J(z)$, we write

$$J(z) = \int_0^\infty \frac{du}{u+z} \left\{ e^{-2\pi\sqrt{u}} + e^{-4\pi\sqrt{u}} + e^{-6\pi\sqrt{u}} + \ldots + e^{-2m\pi\sqrt{u}} \right\} +$$

$$+ \int_0^\infty \frac{du}{u+z} \cdot \frac{e^{-2m\pi\sqrt{u}}}{e^{2\pi\sqrt{u}} - 1},$$

and substituting in the remainder $u = v^2$, we find

$$\int_0^\infty \frac{du}{u+z} \cdot \frac{e^{-2m\pi\sqrt{u}}}{e^{2\pi\sqrt{u}} - 1} = 2 \int_0^\infty \frac{dv}{v^2+z} \cdot \frac{v e^{-2\pi v}}{e^{2\pi v} - 1} < \frac{1}{2\pi^2 m z}.$$

Hence we have

$$J(z) = \int_0^\infty \frac{du}{u+z} \cdot \sum_{k=1}^{k=\infty} e^{-2\pi k\sqrt{u}} = \int_0^\infty du\, e^{-2\pi\sqrt{u}} \sum_{k=1}^{k=\infty} \frac{1}{u + k^2 z},$$

and, putting $u = \frac{z v^2}{4\pi^2}$, we get

$$J(z) = \int_0^\infty e^{-v\sqrt{z}}\, dv \sum_{k=1}^{k=\infty} \frac{2v}{v^2 + 4k^2\pi^2} = \int_0^\infty e^{-v\sqrt{z}}\, dv \left\{ \frac{e^v}{e^v - 1} - \frac{1}{v} - \frac{1}{2} \right\} =$$

$$= -\int_0^\infty dv \left\{ \frac{e^{-v}}{v} - \frac{e^{v(1-\sqrt{z})}}{e^v - 1} \right\} + \int_0^\infty \frac{e^{-v} - e^{-v\sqrt{z}}}{v}\, dv - \frac{1}{2\sqrt{z}} =$$

$$= -\frac{\Gamma'}{\Gamma}(\sqrt{z}) + \log(\sqrt{z}) - \frac{1}{2\sqrt{z}},$$

a result from which we deduce at once $J(1) = C - \frac{1}{2}$.

Now according to STIELTJES' theory the integral $J(z)$ can be converted formally in a continued fraction

$$\cfrac{1}{|a_1 z|} + \cfrac{1}{|a_2|} + \cfrac{1}{|a_3 z|} + \cfrac{1}{|a_4|} + \cfrac{1}{|a_5 z|} + \ldots,$$

the quantities a_k depending on the coëfficients $c_0, c_1, c_2, \ldots$ of the

divergent series. Following the general method we consider the determinants

$$\lambda_{2n} = \begin{vmatrix} c_1 & c_2 & c_3 \dots\dots c_n \\ c_2 & c_3 & c_4 \dots\dots c_{n+1} \\ c_3 & c_4 & c_5 \dots\dots c_{n+2} \\ \dots & \dots & \dots \\ c_n & c_{n+1} & c_{n+2} \dots\dots c_{2n-1} \end{vmatrix}, \quad \lambda_{2n-1} = \begin{vmatrix} c_0 & c_1 & c_2 \dots\dots c_{n-1} \\ c_1 & c_2 & c_3 \dots\dots c_n \\ c_2 & c_3 & c_4 \dots\dots c_{n+1} \\ \dots & \dots & \dots \\ c_{n-1} & c_n & c_{n+1} \dots\dots c_{2n-2} \end{vmatrix},$$

then we shall have

$$a_1 = \frac{1}{c_0}, \quad a_2 = \frac{c_0^2}{c_1}, \dots, a_k = \frac{\lambda_{k-1}^2}{\lambda_k \lambda_{k-2}}.$$

These general formulae give no insight in the numerical values of the quantities a_k, remembering however that $c_h = \frac{B_{h+1}}{2_{h+2}}$, it is obvious that they are rational and depending on the Bernoullian numbers only. Moreover they are positive, for considering the determinant

$$D = \begin{vmatrix} c_p & c_{p+1} & c_{p+2} & \dots\dots c_{p+m} \\ c_{p+1} & c_{p+2} & c_{p+3} & \dots\dots c_{p+m+1} \\ c_{p+2} & c_{p+3} & c_{p+4} & \dots\dots c_{p+m+2} \\ \dots & \dots & \dots & \dots \\ c_{p+m} & c_{p+m+1} & c_{p+m+2} & \dots\dots c_{p+2m} \end{vmatrix},$$

with arbitrary indices p and m, we get

$$D = \frac{1}{(m+1)!} \int_0^\infty \int_0^\infty \dots \int_0^\infty f(u_1) f(u_2) \dots f(u_{m+1}) du_1 du_2 \dots du_{m+1} u_1^p u_2^p \dots u_{m+1}^p \begin{vmatrix} 1 & u_1 & u_1^2 \dots u_1^m \\ 1 & u_2 & u_2^2 \dots u_2^m \\ 1 & u_3 & u_3^2 \dots u_3^m \\ \dots & \dots & \dots \\ 1 & u_{m+1} & u_{m+1}^2 \dots u_{m+1}^m \end{vmatrix}^2$$

Hence D and in particular every determinant λ_k is positive, therefore the same conclusion holds for a_k. By direct calculation we get for the very first quantities a_k rather irregular numerical values. We shall find

$$a_1 = 12, a_2 = \frac{5}{6}, a_3 = \frac{252}{79}, a_4 = \frac{79^2}{60 \cdot 241}, a_5 = \frac{241^2 \cdot 11 \cdot 12}{79 \cdot 52489},$$

but these results give no indication about the possible convergence of

the continued fraction. In order to prove this convergence for $z > 0$, we change $f(u)$ into

$$g(u) = \frac{1}{e^{2\pi\sqrt{u}} - e^{-2\pi\sqrt{u}}}$$

and applying STIELTJES' method to the new integral

$$J_1(z) = \int_0^\infty \frac{du}{u+z}\, g(u) = -\frac{\Gamma'}{\Gamma}(\sqrt{z}) + \frac{\Gamma'}{\Gamma}(2\sqrt{z}) - \frac{1}{4\sqrt{z}} - \log 2,$$

we obtain the continued fraction

$$\frac{1}{|a'_1 z} + \frac{1}{|a'_2} + \frac{1}{|a'_3 z} + \frac{1}{|a'_4} + \frac{1}{|a'_5 z} + \cdots$$

with $a'_{2n} = \frac{1}{2n}$ and $a'_{2n+1} = \frac{16}{2n+1}$. Now both the series $\sum_1^\infty a'_{2k}$ and $\sum_0^\infty a'_{2k+1}$ evidently, diverge, hence we infer that for $z > 0$ the new continued fraction necessarily converges, and by the way we may note for $z = \frac{1}{16}$ the rather remarkable result

$$\frac{\pi}{2} - 1 = \frac{1}{|\frac{1}{1}} + \frac{1}{|\frac{1}{2}} + \frac{1}{|\frac{1}{3}} + \frac{1}{|\frac{1}{4}} + \cdots$$

Comparing the functions $f(u)$ and $g(u)$ we have

$$\frac{f(u)}{g(u)} = 1 + e^{-2\pi\sqrt{u}}$$

and accordingly everywhere in the range of integration

$$1 \leqq \frac{f(u)}{g(u)} \leqq 2,$$

therefore, again using STIELTJES' argument, we conclude to the inequalities

$$\tfrac{1}{2}(a'_1 + a'_3 + a'_5 + \ldots + a'_{2n+1}) < (a_1 + a_3 + a_5 + \ldots + a_{2n+1}) < \\ < (a'_1 + a'_3 + a'_5 + \ldots + a'_{2n+1}),$$

otherwise written

$$8\left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n+1}\right) < (a_1 + a_3 + a_5 + \ldots + a_{2n+1}) < \\ < 16\left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n+1}\right).$$

Consequently the lower limit of a_{2k+1} must be zero, and that agrees with the fact that $\frac{c_{n+1}}{c_n}$ tends to infinity, for STIELTJES shewed that in that case no upper limit can be assigned to $\frac{1}{a_n a_{n+1}}$.

The principal conclusion, however, is that the series $\sum_0^\infty a_{2k+1}$ diverges, that therefore the continued fraction

$$\frac{1}{|a_1 z} + \frac{1}{|a_2} + \frac{1}{|a_3 z} + \frac{1}{|a_4} + \frac{1}{|a_5 z} + \dots .$$

converges except when z is real and negative, and that it is equal to the integral $J(z)$. Thus then, putting $z = 1$, we have proved that $C - \frac{1}{2}$ can be expanded in the continued fraction

$$\frac{1}{|a_1} + \frac{1}{|a_2} + \frac{1}{|a_3} + \frac{1}{|a_4} + \frac{1}{|a_5} + \dots ,$$

the quantities a_k being rational and positive, whilst those of odd index have the lower limit zero. More or less we are inclined to believe that a fraction satisfying these conditions cannot represent a rational number, and so the expansion of $C - \frac{1}{2}$ again suggests the conjecture that C must be irrational.

The result obtained is of no practical value; that after some reductions we have

$$C - \tfrac{1}{2} = \frac{1}{|12} + \frac{6}{|5} + \frac{79}{|42} + \frac{2410}{|79} + \frac{262445}{|2651} + \dots ,$$

is of small service in the evaluation of the constant, and though numerator and denominator of any convergent can be expressed in the Bernoullian numbers, in approximating the constant C other methods are to be preferred.

Chemistry. — "*Researches on the Addition of Water to Ethylene and Propylene*". (Preliminary Communication). By Dr. J. P. WIBAUT and J. J. DIEKMANN. (Communicated by Prof. A. F. HOLLEMAN).

(Communicated at the meeting of March 24, 1923).

About two years ago experiments were carried out by one of us purposing to study the possibility of a direct addition of water to ethylene and propylene. The continuation of this investigation has been rendered possible by a liberal support granted me from the HOOGEWERFF-fund. I gladly avail myself of this opportunity to express my great indebtedness to the Board of Management of the HOOGEWERFF-fund for this help.

Though these investigations have not yet been completed, it seems desirable to me in connection with a short notice in the „Chemiker Zeitung" of Jan. 2nd 1923 (N°. 47, p. 7), in which H. W. KLEVER describes similar researches, to publish a preliminary communication on the results obtained by us.

J. P. WIBAUT.

§ 1. *The Action of Water-vapour on Ethylene and Propylene in the Presence of Catalysts.*

Since the investigations by IPATIEW, SENDERENS and SABATIER it has been known that at high temperature and in the gaseous condition ethyl-alcohol and some of its homologues can be decomposed in two ways:

$$C_2H_5OH \rightarrow C_2H_4 + H_2O \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I).$$

$$C_2H_5OH \rightarrow CH_3\overset{H}{C}{=}O + H_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (II).$$

Both reactions are typical catalytic reactions, which only proceed readily in the presence of certain contact-substances. Anhydrous aluminiumsulphate and aluminiumoxide are typical catalysts that split off water (reaction I). Metals like copper and iron, especially in finely divided condition, are typical catalysts for the splitting off of hydrogen (reaction II).

The range of temperature, in which particularly the first reaction

takes place, lies between 300—400°, dependent on the nature of the catalysing substance; when the temperature is raised to about 400° and higher, the formation of aldehyde becomes prominent even in the presence of substances like aluminium oxide and other catalysts that split off water.

It is well known that reaction (II) is reversible — aldehydes can be smoothly reduced with molecular hydrogen over nickel — but nothing is known about the reversibility of reaction (I).

In the extensive literature on the splitting up of alcohols into olefine and water, the question whether direct addition of water to the double bond in ethylene and propylene is actually possible, has never been examined. We have carried out a number of experiments to answer this question. A mixture of ethylene and water-vapour was led over different contact-substances at a temperature between 300° and 400° C. On use of aluminiumhydroxyde or of aluminium sulphate as catalysts, the reaction product contained acetaldehyde. We have proved the presence of acetaldehyde by the usual reactions (reduction of an ammoniacal solution of silver hydroxide); SCHIFF's reaction; reaction with nitro-prussidsodium and piperidine according to LEWIN) and also isolated as p-nitrophenylhydrazone. The quantities of acetaldehyde are very small; by far the greater part of the ethylene remains unchanged during the experiment. The quantity of acetaldehyde amounted to from 0,2 to 0,4 % at 350°—360°, calculated to the quantity of ethylene.

The presence of alcohol could not be verified [1]).

In our opinion the formation of acetaldehyde must be explained in this way that primarily ethylalcohol is formed through addition of water to ethylene, and then acetaldehyde through splitting up of hydrogen. If this second reaction proceeds much more rapidly than the addition of water to the double bond, no alcohol will be found in the reaction product. As at 350°—360° ethyl-alcohol is almost quantitatively decomposed into ethylene and water (at this temperature, however, a little hydrogen is also formed) it is clear that only at a lower temperature the inverse reaction can take place in a considerable degree. We have, however, not succeeded in finding a catalyst that causes the addition of water to ethylene below 300°.

We have proved by means of a separate experiment that no acetaldehyde is formed from mixtures of dry ethylene with about 10% of air at 360° over aluminiumoxide. It, therefore, appears from

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[1]) The analytical particulars will be given later, as also the full description of the arrangement of the experiments.

this that the formation of acetaldehyde is not the consequence of an oxidation of ethylene, e. g. according to the scheme

$$C_2H_4 + O \rightarrow \underset{\diagdown O \diagup}{CH_2—CH_2} \rightarrow CH_3\overset{H}{C}=O$$

Hence the formation of acetaldehyde cannot have been caused by the possible presence of small quantities of air in the ethylene used.

We are, therefore, of opinion that we are justified in concluding that a primary addition of water to the double bond has taken place, and that the reaction:

$$C_2H_5OH \rightleftarrows C_2H_4 + H_2O$$

may accordingly be considered as a reversible reaction.

We have obtained perfectly analogous results with mixtures of propylene and water-vapour. At 350° and in the presence of aluminiumhydroxide *acetone* was then formed in a quantity of from 0,2 to 0,3 % of the propylene. In our opinion the primary formation of isopropylalcohol by addition of water to propylene, must be assumed in this case. Afterwards the isopropylalcohol is transformed to acetone through the splitting off of hydrogen. Hence the direct addition of water proceeds analogously to the addition of hydriodic acid, in which likewise the isopropyl compound appears. Accordingly the rule of MARKONIKOW remains valid also in this case.

On the ground of these results it is probable that the addition of water to propylene and ethylene can take place under high pressure at temperatures far below 300°. We have, however, made no experiments in this direction.

§ 2. *The Hydration of Ethylene and Propylene by Means of Acids.*

The syntheses of ethyl- and isopropylalcohol from ethylene and propylene by the formation of alkyl-sulphuric acid, and subsequent hydrolysis, by M. BERTHELOT [1]) are among the classic syntheses of organic chemistry. BERTHELOT investigated the absorption of these olefines by pure sulphuric acid of 98—99 % H_2SO_4 at ordinary temperature. Afterwards the absorption of ethylene by sulphuric acid has been repeatedly studied. Particularly in the last few years several technical chemists have made experiments to absorb the ethylene from

[1]) BERTHELOT: Chimie organique fondée sur la synthèse, p. 115. c. f. Ann. de Chimie et de Physique. (7), **4**, 101 (1895). Bull. Soc. Chim. XI, 13. (1869).

coal-distillation gases by means of hot strong sulphuric acid (of 96 %), and to obtain ethylalcohol after dilution and distillation of the sulphuric acid [1]).

With regard to the action of sulphuric acid on propylene, a process of CARLETON-ELLIS [2]) has become known. In this process the waste gases formed in the preparation of light hydrocarbons from heavy petroleum-distillates (cracking-process of BURTON) are passed through sulphuric acid of 87 %; the propylene present in these is said to be transformed into isopropylsulphuric acid. After dilution and distillation of the sulphuric acid isopropylalcohol is obtained.

Systematic researches on the behaviour of ethylene and propylene towards acids of different concentrations have not been published.

On the other hand there are many instances known, in which the addition of water to a double bond takes place under the influence of diluted acids. Geraniol absorbs two molecules of water when treated with 5 % sulphuric acid. BUTLEROW [3]) found that isobutylene and heptylene were very slowly hydrated to the corresponding alcohols by means of diluted sulphuric acid and nitric acid at the ordinary temperature.

It seemed interesting to us to examine how ethylene and propylene would behave towards acids of different concentrations. If ethylsulphuric acid can be obtained through the action of ethylene on diluted sulphuric acid at high temperature, there would be a possibility that afterwards the ethylsulphuric acid should be hydrolized:

(1) $$C_2H_4 + H_2SO_4 \rightarrow C_2H_5HSO_4$$

(2) $$C_2H_5HSO_4 + H_2O \rightarrow C_2H_5OH + H_2O$$

If the two reactions proceeded rapidly enough, the experiment might be arranged so that the alcohol formed is immediately distilled off from the reaction liquid.

Such a course of the reaction would then be practically an addition of water to ethylene, in which the question whether we have to do here with a direct addition or which an intermediary

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[1]) FRITZSCHE. Chemische Industrie **20**, 266 (1897) and **21**, 27 (1898); TAU and BERTELSMANN, Glück Auf **57**, 189 (1921); BURY en OLLANDER: „Byproduct development in the Iron and Steel Industry"; Paper read before the Cleveland Institution of Engineers, 15 December 1919; cf. TIDMAN, Journ. Soc. Chem Ind. **40**, 86 T (1921); DE LOISY. Compt. Rend. Ac. d. Sc. Paris **170**, 50 (1920); DAMIENS, DE LOISY en PIETTE, Eng. Pat. 180988 (1922).

[2]) Cf. Chemical and Metallurgical Engineering. Vol. **23**, 1230 (1920).

[3]) Lieb. Ann. **180**, 245 (1876).

formation of ethylsulphuric acid, can be left undecided for the present.

We have devised an apparatus, in which an ascending stream of gas came into intimate contact with the descending acid. This washing apparatus, which is placed vertically was electrically heated by means of a coil of nichrome-wire so as to make it possible to keep the reaction temperature constant within narrow limits. The ethylene, which is led through the heated, diluted sulphuric acid will withdraw water-vapour from the liquid, for so far as it is not absorbed, which would cause the acid to become more concentrated in the course of the experiment. To prevent this we have added water-vapour to it at the same time with the ethylene; the partial tension of the water vapour in the introduced gas-mixture was about the same as the water-vapour tension of the used sulphuric acid at the temperature of the experiment. In this way the concentration of the sulphuric acid was kept about constant during the experiment.

At the top of the apparatus there escaped, therefore, water-vapour, not absorbed ethylene, and alcohol vapour, if any was formed.

It actually appeared possible to obtain alcohol from ethylene in this way. A mixture of ethylene and steam was washed with sulphuric acid of 65 % H_2SO_4 at a temperature of 150°—160°. After 5 litres of ethylene had been passed through in 5 hours' time, the distillate contained 0,21 gr. of alcohol [1]), i. e. a conversion of about 2 %.

Then the sulphuric acid used was strongly diluted and distilled out, and in this way 0,08 gr. of alcohol more was obtained. Hence a little ethylsulphuric acid was still present in the sulphuric acid after the experiment. This renders it probable that the ethylsulphuric acid is formed as an intermediate product, and that accordingly the formation of alcohol is the result of two successive reactions, as given above.

In a second similar experiment 4 % of the ethylene that was passed through, was converted into ethylalcohol.

With a mixture of sulphuric acid and water containing 55 % H_2SO_4 only 0.01 gramme of ethylalcohol was found in the distillate, when 5 litres of ethylene mixed with steam had been passed through at 140°.

With sulphuric acid of 70 % no alcohol was found in the distillate, when three litres of ethylene had been passed through. After

[1]) The analysis took place by oxidizing the reaction liquid with chromic acid, in consequence of which the alcohol present was oxidized to acetaldehyde This latter was determined colorimetrically.

dilution and distillation the sulphuric acid yielded, however, 0.32 gr. of alcohol, which was, therefore, present as ethylsulphuric acid. This corresponds with a conversion of 5 %.

In these experiments most of the ethylene passed unchanged through the sulphuric acid; only a slight carbonisation took place. Though in principle it, therefore, appears possible to convert ethylene in this way into ethylalcohol, the yield was so small that no practical significance can be assigned to these experiments.

These researches are being continued with other acids and with salts, as aluminiumsulphate and others.

§ 3. *Propylene and Sulphuric Acid.*

It is well known from BERTHELOT's investigations that propylene is very rapidly absorbed at the ordinary temperature by sulphuric acid of 98—99 %. We have first of all made some preliminary experiments on the action of sulphuric acid of different concentrations on propylene.

In a HEMPEL's gas-pipette 100 cc propylene was placed together with the sulphuric acid to be examined.

Sulphuric acid of 96 % at once absorbs the propylene, also sulphuric acid of 90 % acts very rapidly on it; with acid of 85 % the propylene is absorbed after 20 minutes' shaking, about an hour being required for this with acid of 80 %. Also sulphuric acid of 75 % still absorbs propylene, but very slowly.

We have further investigated the action of propylene on sulphuric acid of 96 % at 0°, in which we carefully guarded against rise of temperature both during the absorption of the gas, and during the pouring out of the reaction product on ice. We have only succeeded in obtaining a small quantity of isopropylalcohol from the reaction product.

Through the action of the sulphuric acid the bulk of the propylene was changed into an oily liquid, which was unsaturated, and boiled within wide limits. It is, therefore, probable that higher unsaturated hydro-carbons are formed by the condensing action of the sulphuric acid. BERTHELOT too states that such condensation products are formed, when rise of temperature takes place during the experiment. In our experiments with sulphuric acid of 96 % at 0° the bulk of the propylene was always transformed into condensed and resinous products in spite of all our precautions. With sulphuric acid of 85 % the absorption of propylene takes place very slowly at 10°. On further treatment

of the reaction product, chiefly condensation products were again obtained.

We then examined the absorption of propylene by more diluted sulphuric acid at higher temperature. The experiments were arranged in the same way as was already described for ethylene. The mixture of propylene and steam was brought in contact in counter-current with sulphuric acid of definite concentration and definite temperature in the vertical washing-apparatus; 7.5 litres of propylene mixed with steam were passed in 4 hours through sulphuric acid of 55 % H_2SO_4 at 140°. The distillate contained 0.25 gr. isopropyl alcohol. After dilution with water a distillate was obtained from the acid in which 0.27 gr. of isopropylalcohol [1]) was present. There was, therefore, evidently still isopropylsulphuric acid present in the acid. In all 2.6 % of the total quantity of propylene was, accordingly, obtained as isopropyl alcohol.

A much greater part of the propylene was, however, decomposed. Separation of carbon took place and formation of sulphur-dioxide. After the experiment 5,3 litres of the 7,5 litres of propylene was found back. Hence 9 % of the consumed quantity of propylene was changed into isopropyl alcohol.

An experiment with sulphuric acid of 45 % H_2SO_4 and at 125—130° proceeded in the same way; 6 litres of propylene were passed through, 5 litres of them were obtained after the experiment. The yield of isopropyl alcohol amounted to 0,2 gramme in the distillate and 0.1 gramme in the acid liquid, together 0,30. gr. i.e. 10 % of the consumed propylene. Here too a large part of the consumed propylene was carbonised.

It therefore, appears from these experiments that the hydration of propylene by hot diluted sulphuric acid is possible. The reaction velocity, however, is small, which renders the yield small. Besides the sulphuric acid has a decomposing action on the propylene. If on the other hand the experiment is made with concentrated sulphuric acid at low temperature, the propylene is quickly attacked, but chiefly transformed into condensation products.

We have tried therefore the action of other acids. We first investigated the action of benzene sulphonic acid. 6 litres of propylene with steam were passed through a concentrated solution of benzene sulphonic acid; in the aqueous distillate of this experiment we found 0,25 gr. isopropyl alcohol or about 1½ % of the propylene. Hence in this case too the reaction proceeds slowly.

[1]) The analysis took place by oxidation to acetone, and colorimetric determination of this substance.

The result of the experiments on the action of acids on ethylene and propylene can, therefore, be summarized as follows: It is possible to obtain ethyl alcohol, resp. isopropyl alcohol by one operation from ethylene and propylene by means of mixtures of sulphuric acid and water at 130—150°. In this reactions the alkylsulphuric acids are probably formed as intermediate products.

The yield of alcohols is, however, very small, and particularly with propylene, the hydro-carbon is decomposed in another way during the experiment. These investigations are being continued.

Physics. — *"The relation between the widening and the mutual influence of dispersion lines in the spectrum of the sun's limb."* By Prof. W. H. JULIUS and Dr. M. MINNAERT.

(Communicated at the meeting of April 28, 1923).

Introduction.

The hypothesis that the darkness of Fraunhofer lines is mainly an effect of anomalous dispersion enables one to explain, at any rate qualitatively, a great many characteristics of the solar spectrum. It thus appears possible to formulate a theoretical connection — which has then of course to be verified quantitatively — between numerous phenomena that are less easily seen as inter-dependent if we consider them from the point of view of the unmodified classical absorption theory introduced by KIRCHHOFF. Such phenomena are e.g.: the general displacement of the solar lines towards the red, differing greatly in amount from line to line; the limb-centre displacements and their dependence on intensity and wave-lenght; the widening and the change of intensity of the lines as the limb is approached; the apparent mutual repulsion of neighbouring Fraunhofer lines, generally greater at the limb than in the centre of the disk; the systematically curved shape of the lines of the spot-spectrum if the slit cuts the spot in a direction passing through the centre of the disk; the gradual increase of the distance between the components of the bright calcium lines H_2 and K_2 as the limb is approached; and various particulars of a more local character.

We shall endeavour to express mathematically the connection which, according to the dispersion theory, should exist between a few of the above-mentioned phenomena, and then to investigate how far these quantitative relations agree with the results of measurements made on solar lines.

It is evident that the *absolute magnitude* of the influence exercised by anomalous dispersion in the solar gases on the aspect of Fraunhofer lines cannot be calculated *directly* so long as the refracting and scattering power of the sun is not otherwise known. Neither can this power be safely computed starting from line displacements only. It must be remembered, however, that a similar uncertainty prevails regarding the values given for temperatures, pressures, radial velocities, intensities of magnetic or electric fields, or grades of dissociation in the sun in so far as such values are derived from

spectral phenomena; in fact, such statements are always based on the doubtful assumption that the observed spectral phenomena are entirely due to the causes mentioned. There is, of course, no objection to introducing this assumption, — provided its hypothetical character be always kept in mind.

With equal justification we may assume that Fraunhofer lines are mainly "dispersion lines"; the essential point will then be to examine whether the deductions from this hypothesis result in an adequate theory, covering a substantial proportion of observational data. In this paper we confine ourselves to showing that the dispersion theory of the solar spectrum connects quantitatively two at first sight independent groups of observed phenomena, namely the well-established general *widening* of the Fraunhofer lines at the limb, and the increase, also at the limb, of the *mutual influence* of neighbouring lines. This relation proves to be independent of the unknown laws that govern the weakening of any given kind of light on its way through the solar gases; it enables us to indicate an *upper limit* of the mutual influence that may be expected, thus lending fresh support to our fundamental hypothesis. It will be shown, indeed, that the average value of the mutual influence as deduced from the dispersion theory is perfectly consistent with the actual observations.

The dispersion lines which, according to our hypothesis, envelop the exceedingly narrow [1]) true absorption lines of the solar spectrum arise from two dimming processes, viz.: irregular refraction and molecular scattering. For although light of *any* wave-length is subject to refraction and scattering on its long way through selectively absorbing gases, it is well known that these causes of darkening specially affect waves in the immediate vicinity of absorption lines. As the two processes weaken the transmitted light according to different laws, we shall treat them separately.

I. On the weakening of light in passing through extensive masses of gas.

§ 1. *Spreading of light by irregular ray-curving in a mixture of gases.*

Suppose we have in a given space a mixture of gases which, if they were each of them alone to fill the space, would show the absolute refractive indices n_1, n_2 .... n_i ....; then, according to ex-

[1]) Our assumption that real absorption is restricted to *very* small ranges of wave-lengths is in harmony with views recently derived from the quantum theory by N. Bohr (Zeitschr. f. Physik **13**, 162, 1923).

periments of BIOT and ARAGO (confirmed by modern observations), the refracting power of the mixture equals the sum of the refracting powers of the constituents:

$$n - 1 = \Sigma (n_i - 1).$$

The condition is implied that the gases do not act on each other. We shall assume this law to be valid also in those spectral regions where one of the constituents causes anomalous dispersion, although no very accurate direct measurements concerning such cases are as yet available. (The exceedingly narrow regions of true absorption are not considered here).

If the gaseous mass is very extensive and of unequal optical density, with irregular gradients in all directions, it will make every beam of light spread out like a bunch of feathers. According to ORNSTEIN and ZERNICKE [1]) the rate of this kind of scattering is determined by "the average square of the spreading per unit of length" $\frac{\alpha^2}{l}$; to any short path l corresponds an angle α depending on the average value of the irregular density gradients, and proportional to $n-1$ of the mixture. The weakening of the transmitted light will therefore be a function of

$$(n-1)^2 = [\Sigma (n_i - 1)]^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

that has the property of increasing and decreasing with this quantity. A characteristic difference between scattering by irregular refraction, and molecular scattering, is, that in the latter process a considerable part of each beam passes straight, and a small part of it disperses in all directions, whereas in refractional scattering every beam itself widens like a plume.

§ 2. *Scattering of light by the molecules of a gaseous mixture.*

If a beam of light of intensity I_0 has travelled a distance z through a medium containing N scattering particles per cube cm., its intensity has diminished to $I = I_0 e^{-hz}$, where, according to RAYLEIGH, h has the value

$$h = \frac{32 \pi^3 (\nu - 1)^2}{3 N \lambda^4}.$$

In this expression ν is explicitly stated to represent the refractive index of the medium as modified by the scattering particles against the unmodified medium [2]). Denoting the absolute index of the latter

[1]) ORNSTEIN and ZERNICKE, These Proceedings, Vol. 21, p. 115 (1917).

[2]) RAYLEIGH, Phil. Mag. **47**, 375, 1899. — Scientific Papers IV, 400.

by n, the absolute index of the modified medium by n', we have

$$h = \frac{32\,\pi^3 \left(\frac{n'}{n} - 1\right)^2}{3\,N\,\lambda^4} = \frac{32\,\pi^3\,(n'-n)^2}{3N\,\lambda^4\,n^2} = \frac{32\,\pi^3\,(n'-n)^2}{3N\,\lambda^4}$$

because for thin gases we may put $n^2 = 1$.

We shall take for granted that this expression for h remains valid in those regions of the spectrum where the scattering particles produce anomalous dispersion. It is precisely in those regions that h will assume considerable values.

Now suppose there be a mixture consisting of $N_1, N_2, \ldots N_i, \ldots$ scattering particles of the kinds $1, 2, \ldots i, \ldots$ For each kind the mixture of the *remaining* kinds forms the "unmodified medium", whilst the "modified medium" is the same in all cases, viz. the complete mixture. We are concerned, therefore, with a single quantity n' and several values $n_{(1)}, n_{(2)}, \ldots n_{(i)}, \ldots$ of n, if $n_{(i)}$ denotes the absolute refractive index of the mixture without the constituent i.

The scattering-coefficient h of the complete mixture will be the sum of the scattering-coefficients peculiar to the separate constituents, each in its proper medium:

$$h = h_1 + h_2 + \ldots h_i + \ldots = \frac{32\,\pi^3}{3\,\lambda^4} \Sigma \frac{(n' - n_{(i)})^2}{N_i}$$

This expression may be simplified because the above-mentioned law of Biot and Arago requires, that

$$n' - 1 = (n_{(i)} - 1) + (n_i - 1)$$

n_i representing the absolute refractive index which the gas i would show if it were alone in the given space. We, therefore, have $n' - n_{(i)} = n_i - 1$, and may write:

$$h = \Sigma\, h_i = \frac{32\,\pi^3}{3\,\lambda^4} \Sigma \frac{(n_i - 1)^2}{N_i}.$$

A beam of light, having travelled a long way through such a mixture of gases, will emerge with a loss of intensity expressible as a certain function of h which has the property of increasing and decreasing with h. In regions of the spectrum sufficiently small to permit of neglecting the change of λ^4 in them, we see that now

$$\Sigma \frac{(n_i - 1)^2}{N_i} \quad \ldots\ldots\ldots \quad (2)$$

is the variable quantity determining the loss of light. (Compare this expression (2) with the corresponding one (1) which applies to refractional scattering).

§ 3. *How anomalous refraction and anomalous scattering act in producing dispersion lines.*

It appears from the above remarks that the distribution of the intensity in a dispersion line is determined by two darkening laws which, it is true, depend on local circumstances (dimensions and shape of the source of light, condition of the medium, etc.), and to that extent are unknown, but which we do know will change with *wave-length* in accordance with the functions (1) and (2). We shall first deal with the share which irregular refraction, and thereafter with the share which molecular scattering has in the formation of dispersion lines.

A. Imaginary pure refraction lines.

Imagine a selectively absorbing gaseous mixture, lacking the faculty of molecular scattering, but with many irregular gradients of density; let a beam of white light travel through that medium, and attention be confined to a small part of the spectrum where only one characteristic frequency, i.e. one ideally sharp absorption line, is in evidence.

If the said line were absent, the mixture would, in this narrow range of wave-lenghts, show a refracting power n_0-1 varying only very slowly with λ, but to this will now be added the anomalous refracting power n_1-1 of the constituent producing the absorption line, thus determining the resultant refracting power:

$$n-1=(n_0-1)+(n_1-1).$$

The term (n_0-1) will, as a rule, preserve the same (generally positive) sign throughout the region considered, whereas (n_1-1) is negative on the violet side of the line, positive on the red side. Light on the violet side of a line will be called V-light, on the red side R-light. All effects of refraction in a gaseous mixture are, therefore, on an average greater for R-light than for V-light, because they depend on $(n-1)^2$ or on the *absolute* value of $n-1$, i.e. on $|(n_0-1)+(n_1-1)|$.

Fig. 1*a* shows the course of n_0-1 and n_1-1 each separately; Fig. 2*a* gives $n-1=(n_0-1)+(n_1-1)$; in Fig. 3*a* is represented the course of $|n-1|=|(n_0-1)+(n_1-1)|$ which determines the distribution of the light in our "refraction line".

The sharp absorption line will thus be enveloped in an *asymmetric* refraction line, whose "centre of gravity" is displaced towards the red if n_0-1 has the positive sign.

(The general displacement of the Fraunhofer lines towards the

red, which increases on approaching the sun's limb, may be considered in connection with these inferences).

Let us now imagine our small spectral region to contain *two* neighbouring sharp absorption lines, then we have:

$$n-1 = (n_0-1) + (n_1-1) + (n_2-1),$$

where n_0-1 is again assumed to be nearly constant, and the other two terms are strongly variable with λ. In the region between the lines, (n_1-1) and (n_2-1) have opposite signs (cf. Fig. 1,*b*, where the three terms are represented separately). The resultant $n-1 = f(\lambda)$

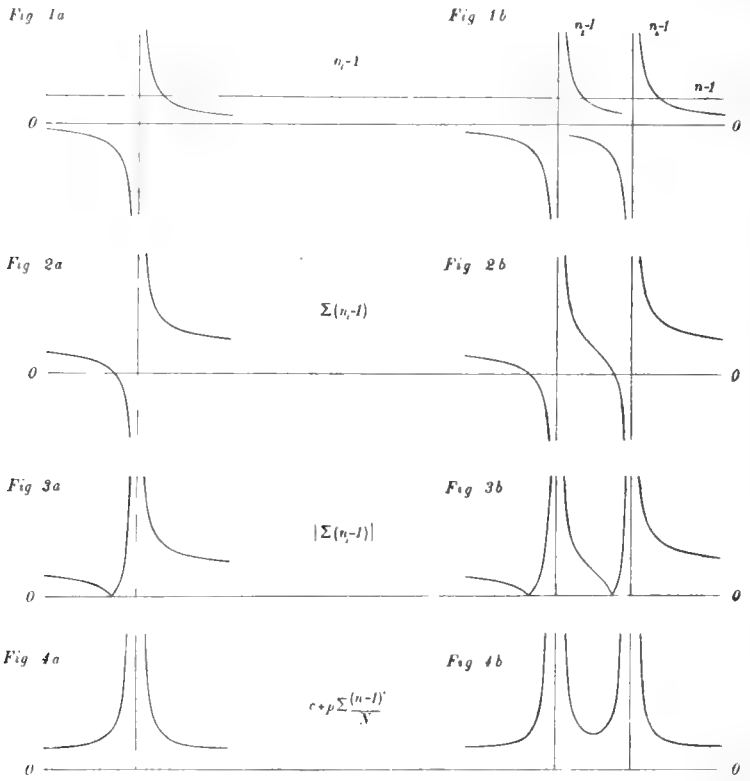

Fig. 1*a*—4*b*.

shows a point of inflexion there (Fig. 2,*b*); and, owing to the opposite signs of (n_1-1) and (n_2-1), the modulus $|(n_0-1)+(n_1-1)+(n_2-1)|$ is smaller than $|(n_0-1)+(n_1-1)|$ in the left section, and smaller than $|(n_0-1)+(n_2-1)|$ in the right section of the interval (cf. Fig. 3,*b*), so that on the two sides of the refraction lines that face each other the weakening of the light is less than it would be if the lines stood wide apart. The "centres of gravity" of two neighbouring refraction lines are, therefore, a little more

distant from each other than their cores, i.e. the true absorption lines: we observe an apparent repulsion.

In Fig. 3 we see, moreover, that on the violet side of each line there appears a point where $n-1=0$. (If n_0-1 were negative, such a point would be found on the red side of the line). Light of the corresponding wave-length would not be weakened by irregular refraction and should, therefore, show an intensity in the spectrum, surpassing the average intensity of regions clear from lines. JEWELL[1]) seems indeed to have observed casually such phenomena in the solar spectrum. It is not surprising, however, that similar places of greater brillancy are not very conspicuous there; for we can scarcely doubt that in the sun the proportion of the components of the mixture varies with depth, so that the values of λ for which $n-1=0$ will not be the same on the entire paths of the beams. Moreover, the Fraunhofer lines are partly due to molecular scattering, and it will presently be shown that this process does not involve the appearance of such narrow regions of greater brillancy in the spectrum (at least not in the central parts of the solar disk). Both circumstances tend to obliterate the brighter places near refraction lines.

B. Imaginary pure scattering-lines.

Now suppose the density of a gaseous mixture to be so uniform, that rays of light pass through it in straight lines, then the true absorption lines will *yet* be enveloped into dispersion lines, because for kinds of light belonging to the nearest environment of the distinctive frequenties the coefficient of molecular scattering has greater values. Let us analyse, indeed, how

$$h=\Sigma h_i=\frac{32\,\pi^3}{3\,\lambda^4}\,\Sigma\,\frac{(n_i-1)^2}{N_i}$$

varies with λ in a narrow spectral region containing a single absorption line of the constituent j. All terms of the sum but one may there be treated as constants, so that

$$h=C+\frac{32\,\pi^3}{3\lambda^4}\cdot\frac{(n_j-1)^2}{N_j}$$

This quantity varies with λ in the manner represented in Fig. 4,*a*; the curve is symmetrical with respect to the absorption line, provided that the dispersion curve associated with the line has the regular

[1]) JEWELL, Astroph. Journ. III, 99, 1896. Cf. also: ABBOT, The Sun, p. 115, where analogous observations of EVERSHED are mentioned in addition.

shape, and that the change of λ^4 in the small region may be neglected. The distribution of the light in the scattering line will then also be symmetrical; this we may infer without knowing the exact form of the law of darkening [1]).

In contrast to what characterizes pure refraction lines, the symmetry of pure scattering lines is *not* disturbed by the addition of a similar cause of weakening that is constant in the region considered. (The above expressions (1) and (2) explain this difference).

Anomalous molecular scattering, or diffusion of light, cannot therefore have any share in the production of the general displacements of the solar lines towards the red [2]).

Let us now consider the case that our small spectral region contains *two* absorption lines. The scattering coefficient will then take the form

$$h = C + p\left[\frac{(n_j-1)^2}{N_j} + \frac{(n_k-1)^2}{N_k}\right],$$

for we may replace the factor $\frac{32\pi^3}{3\lambda^4}$ by the constant quantity p.

Fig. 4*b* represents h as a function of λ. To this will correspond a darkening curve whose ordinates grow and decline with h. We see that in the interval between the absorption lines the superposition of their individual scattering effects must produce a greater increase of the darkening than outside the pair; so the centres of gravity of the two diffusion-lines will be a little less distant from each other than the absorption lines proper (apparent attraction).

Summarizing the above qualitative results with a view to their application in the spectroscopy of celestial bodies, we may state:

1. The general but very unequal displacements of the Fraunhofer-lines towards the red can be explained by the properties of refraction lines, but not by those of diffusion lines. This also applies to the limb-centre displacements.

2. The mutual influence of neighbouring Fraunhofer lines, which increases, as a rule, from the centre towards the limb of the solar disk, may be the result of either scattering process; but irregular refraction causes apparent repulsion, molecular diffusion of light gives apparent attraction.

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[1]) The law of darkening through molecular scattering in the sun has been amply studied by J. SPIJKERBOER in a dissertation, published in Utrecht, 1917; cf also Arch. néerl. III A, **5**, p. 1—115, 1918.

[2]) To this point our attention has first been drawn in a conversation with EINSTEIN.

II. The relation between width and mutual influence in dispersion lines and in Fraunhofer lines.

In this chapter formulae will be deduced expressing the connection between mutual influence and width of dispersion lines. If Fraunhofer lines are in the main dispersion lines, it will thus be possible, starting from data concerning the widening of the lines in the spectrum of the sun's limb, to derive values for the probable increase of the mutual influence in passing from the centre to the limb. We may then compare these theoretical results with the data obtained from observations regarding limb-centre displacements of Fraunhofer lines.

The respective shares which irregular refraction and molecular scattering may have in the production of the lines will again be treated separately.

§ 1. *Refraction lines in the spectrum of the centre of the solar disk.*

The distribution of the luminosity in a refraction line depends on the values of $|n-1| = f(\lambda)$. We owe to Roschdestwensky [1]) the most accurate measurements concerning the form of this function. He found that in region of the two yellow sodium lines Sellmeier's formula:

$$n-1 = \frac{a_1\lambda^2}{\lambda^2-\lambda_1^2} + \frac{a_2\lambda^2}{\lambda^2-\lambda_2^2} + c \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

represents the observations almost exactly. We shall suppose this formula to be applicable to the cases we are considering.

If the difference between λ_1 and λ_2 is rather considerable and if we only pay attention to the surroundings of one of the lines, we may unite the latter two terms of (1) into a single, nearly constant refracting power (n_0-1), and moreover put $\lambda + \lambda_1 = 2\lambda$. The expression thus simplifies itself, if we write k for $\frac{a_1}{2}$, to:

$$n \quad 1 = \frac{k}{\lambda-\lambda_1} + (n_0-1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The intensity at any place in the spectrum depends on the *absolute* value $|n-1|$. On either side of the absorption line we mark the values of λ where $n-1 = \pm H$ (cf. Fig. 5), H being provisionally an arbitrary constant. These places in the spectrum will be called the "H-boundaries" of the dispersion line; their distance (to be

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[1]) Roschdestwensky, Anomale Dispersion im Natriumdampf. Ann. d. Phys. **39**, 307, 1912.

indicated by B) is the "H-width" of the line. So B signifies the width an observer would assign to the line if he estimated its

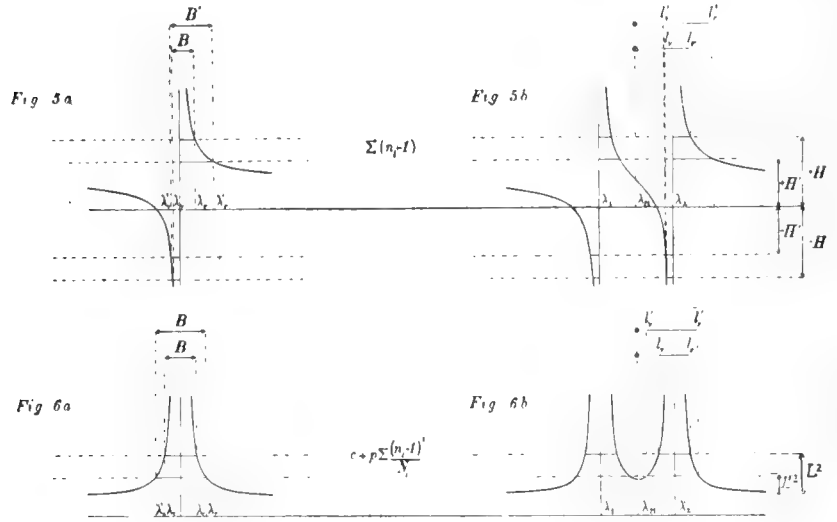

Fig. 5a—6b.

boundaries to be situated at the wave-lengths where the relative intensity has the value corresponding to H. (By "relative intensity" we understand the proportion between the intensity at the selected point of the dispersion line and the intensity in the surrounding continuous spectrum).

Suppose the "H-boundaries" of the line to be situated at λ_R and λ_V, then we obtain from equation (2);

$$\lambda_R-\lambda_1=\frac{k}{H-(n_0-1)} \text{ and } \lambda_V-\lambda_1=-\frac{k}{H+(n_0-1)}, \quad . \quad . \quad (3)$$

$$B=\lambda_R-\lambda_V=\frac{2Hk}{H^2-(n_0-1)^2}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or, inversely, expressing H in B,

$$H=\frac{k}{B}+\sqrt{\frac{k^2}{B^2}+(n_0-1)^2} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We may leave the negative value of the radical out of account.

Now proceeding to the case of two neighbouring equally strong lines, we prefer to indicate the places in the spectrum by the quantity

$$l=\lambda-\lambda_M$$

(cf. Fig. 5,b) in which λ_M represents the wave-length corresponding to the point halfway between the two absorption lines, so that this

middle-point becomes the zero in our scale of l-values. If the distance between the lines is $2A$, we have in the new notation: $l_2 = A$ for the line on the red side, $l_1 = -A$ for the line on the violet side, and we get, in analogy with (2), the relation

$$n-1 = \frac{k}{l-A} + \frac{k}{l+A} + (n_0-1) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For each of the lines we may again define two "H-boundaries", to be found by taking (6) equal to $\pm H$, wherein H has the value fixed by the relation (5).

We consider the red-facing line of the pair. Its H-boundaries l_R and l_V are found by substituting in (6) H for $n-1$, l_R or l_V for l. We thus obtain, according as the $+$ sign or the $-$ sign is chosen:

$$l_R \text{ or } l_V = \frac{k}{\pm H-(n_0-1)} + \sqrt{\frac{k^2}{[\pm H-(n_0-1)]^2} + A^2} = S_R + T_R \text{ or } S_V + T_V. \quad (7)$$

Similarly it follows, that the violet-facing component of our pair has for its H-boundaries

$$l_R \text{ or } l_V = S_R - T_R \text{ or } S_V - T_V \quad . \quad . \quad . \quad . \quad . \quad (7a)$$

§ 2. *Refraction lines in the spectrum of the limb of the solar disk.* Seen in the light of the dispersion theory, the widening of the Fraunhofer lines in the spectrum of the limb is due to the fact, that near the limb smaller values $\pm H'$ of $n-1$ are already sufficient for producing the same relative darkening, which in the central parts of the disk is only produced by the greater values $\pm H$. The H'-width, shown by the line at the limb, will be called B'. As a counterpart of (5) we now obtain the relation

$$H' = \frac{k}{B'} + \sqrt{\frac{k^2}{B'^2} + (n_0-1)^2}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and, in the case of two limb-lines, as counterparts of (7) and (7a):

$$l'_R \text{ or } l'_V = \frac{k}{\pm H'-(n_0-1)} + \sqrt{\frac{k^2}{[\pm H'-(n_0-1)]^2} + A^2} = S'_R + T'_R \text{ or } S'_V + T'_V \quad (9)$$

$$l'_R \text{ or } l'_V = S'_R - T'_R \text{ or } S'_V - T'^V \quad . \quad . \quad . \quad . \quad (9a)$$

§ 3. *Theoretical possibility of a general solution of our problem.* In principle, the formulae (5), (7), (8), and (9) embody a rather complete answer to the question how, on the basis of the dispersion

theory, the widening of the lines near the limb and the increase of their mutual influence must be connected. Indeed, if the distribution of the relative intensity were established both for an isolated refraction line of the centre type and for the corresponding limb line, it would now be possible to plot the curve giving the distribution of the light in a set of two neighbouring equal lines, and to examine how the asymmetry in it increases when passing from the centre to the limb of the disk. It would only be necessary to substitute for B and B' the values corresponding to relative intensities 0,9, 0,8, 0,7 etc., and then to calculate from (7) and (9) where the places of equal intensity ought to be found in the pair. [1])

For the present, however, the intensity curves are not sufficiently known; the observers of the solar spectrum provide us with the "visual widths" and the wave-lengths of the "centres of gravity" of the lines, quantities by no means free from subjectivity.

Nevertheless we can draw from such observations some useful inferences concerning our problem.

§ 4. *Limitation to what may be derived from already existing data.*

The average widening in the limb-spectrum of lines whose widths lie between 0,07 and 0,16 Å was found by FABRY and BUISSON [2]) to be 0,01 Å. Although we do not know the exact value of the relative intensity at the places where their interference method made them estimate the "boundaries" of the line, there was yet in this way assigned a definite width to each line. (We have some reason to think that in the ordinary visual estimates of the width of a dark line the relative intensity at the borders is about 0,8. This statement reposes on extrapolation of an empirical formula by which, in an earlier investigation, we were able to represent the visual boundaries of bright lines on the photographic plate. Cf. Ann. d. Phys., **71**, 59, 1923).

Whilst under the influence of a neighbouring line these boundaries shift asymmetrically, the central parts of the dispersion line, with

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[1]) If limb- and centre-lines have been photographed on one and the same plate (the centre spectra with shorter exposition so as to make the intensity of clear spaces equal in both centre and limbspectra) it is even possible to use the transparency values of the single lines directly for computing, by means of our formulae, the course of the transparency in pairs of lines occuring on the same plate. It is unnecessary then, first to translate the degrees of blackening into original intensities.

[2]) FABRY and BUISSON, C. R. **148**, 1741, (1909); Astroph. Journ. **31**, 97, (1910).

their greater darkness remain almost stationary. The point midway between the boundaries will, therefore, by its position depict all asymmetrical distortions of the dispersion line somewhat exaggerated in comparison with the "centre of gravity" instinctively used by the observer to identify the place of the line. If, therefore, we calculate the displacements of that *midway point*, we are sure to find *upper limits* for the displacements which, according to the dispersion theory, may be expected as the result of measurements.

§ 5. *The difference in mutual influence of refraction lines at the limb and in the centre of the disk.*

It is easily seen that the midway point M_H between the H-boundaries of the red-facing displaced refraction line is determined by the abscis

$l_M = {}^1/_2(l_R + l_V) = {}^1/_2(S_R + S_V + T_R + T_V)$ in the centre-spectrum,
and by (10)
$l'_M = {}^1/_2(l'_R + l_V) = {}^1/_2(S'_R + S'_V + T'_R + T'_V)$ in the limb-spectrum,

so that the amount of its displacement, when passing from centre to limb, is:

$$l'_M - l_M = {}^1/_2(S'_R + S'_V - S_R - S_V + T'_R + T'_V - T_R - T_V) . \quad (11)$$

This expression contains side by side all the various systematic displacements of Fraunhofer lines which the dispersion theory foresees as consequences of irregular ray-curving. The first two terms give the general displacement of limb-lines against arc-lines; the third and fourth the general displacement of centre-lines against arc-lines[1]); the fifth and sixth term show the apparent repulsion of neighbouring lines in the limb-spectrum; the seventh and eighth the apparent repulsion in the centre-spectrum.

At present we are especially interested in the *increase* which the apparent repulsions must undergo when passing from the centre to the limb, because we are in possession of a good many observational data concerning this phenomenon[2]).

For each component of a pair the said increase is represented by:

$${}^1/_2(T'_R + T'_V - T_R - T_V)$$

which expression, after substituting the quantities determined by (9), (8), (7) and (5), becomes

[1]) Here are, of course, not included those displacements which the core-lines or true absorption lines may perhaps be subjected to as a result of radial velocities, pressure, or fields of force. Such displacements will simply have to be added to the phenomena we are considering.

[2]) Cf.: W. H. Julius, Mutual Influence etc., Astroph. Journ. **54**, 92, 1921, and W. H. Julius and M. Minnaert, Ann. d Phys. **71**, 50, Kayser-Festheft, 1923.

$$\frac{B}{2}\left\{\sqrt{\frac{1}{\left[\frac{B}{B'}+\sqrt{\frac{B^2}{B'^2}+\frac{(n_0-1)^2B^2}{k^2}}-\frac{(n_0-1)B}{k}\right]^2}+\frac{A^2}{B^2}}+\right.$$

$$+\sqrt{\frac{1}{\left[\frac{B}{B'}+\sqrt{\frac{B^2}{B'^2}+\frac{(n_0-1)^2B^2}{k^2}}+\frac{(n_0-1)B}{k}\right]^2}+\frac{A^2}{B^2}}-$$

$$-\sqrt{\frac{1}{\left[1+\sqrt{1+\frac{(n_0-1)^2B^2}{k^2}}-\frac{(n_0-1)B}{k}\right]^2}+\frac{A^2}{B^2}}-$$

$$\left.-\sqrt{\frac{1}{\left[1+\sqrt{1+\frac{(n_0-1)^2B^2}{k^2}}+\frac{(n_0-1)B}{k}\right]^2}+\frac{A^2}{B^2}}\right\} \qquad (12)$$

In order to estimate the numerical value of this expression we base ourselves on the result of observations of FABRY and BUISSON, who found the widening at the limb to be approximately 0,010 Å with lines varying from 0,07 to 0,16 Å in width (mean width 0,11 Å). The mean width of those other lines, taken from the observational material of Mount Wilson and Kodaikanal concerning limb-centre displacements, for which the existence of mutual influence has been stated [1]) by us in the above mentioned papers, amounts to 0,09 Å. Taking these data into consideration, we have calculated the value of the expression (12) after substituting $B = 0{,}100$ Å, $B' - B = 0{,}010$ Å, and, in succession, $\frac{(n_0-1)B}{k} = \pm\infty$, ± 4, ± 1, and 0. The results have then been plotted as ordinates against abscisses $\frac{2A}{B}$ (which, therefore, represent distances of the lines expressed in their width as unit). We so obtained the full drawn curves of Fig. 7 (p. 346). They represent (for a refraction line) by how many thousandth parts of an Ångström unit the middle point M'_H between the boundaries of a limb-line is shifted in excess of the middle point M_H between the boundaries of the corresponding centre-line, in consequence of the presence of an equally strong neighbouring line, if this is situated at a distance equal to 3, 2, 1 times the estimated width of the lines. It will be seen that the repulsion is already perceptible at a

[1]) W. H. JULIUS, Astroph. Journ. **54**, 92, (1921); W. H. JULIUS and M. MINNAERT, Ann. d. Phys. **71**, 50, KAYSER-Festheft, 1923.

rather great distance, and increases slowly to 0,004 Å maximum. Obviously the value of n_0 has only little influence on the result.

As B' differs little from B, the radical quantities of (12) can be developed into rapidly converging series. It will then appear that as a first approximation the repulsion is proportional to the absolute value of the widening at the limb, i.e. to $B'-B$, Accordingly, our curves are also valid for lines differing in width from those here considered, provided their widening at the limb has the value found by Fabry and Buisson. They are therefore applicable to the case of lines having the average type of those for which mutual influence has been observed.

§ 6. *Diffusion-lines in the spectrum of the centre of the solar disk.* The distribution of the intensity in a pure molecular scattering line (in the absence of irregular gradients of optical density) depends on the manner in which the scattering coefficient (cf. p. 335):

$$h = C + \frac{32\pi^3}{3\lambda^4} \cdot \frac{(n_j-1)^2}{N_j}$$

varies with λ in the surrounding small part of the spectrum. And because even the variation of λ^4 may be neglected there, the distribution is entirely governed by the nature of

$$\frac{(n_j-1)^2}{N_j} = [f(\lambda)]^2,$$

a function, obviously symmetrical with respect to the position of the absorption line. On either side of the latter we may again mark a wave-length where (omitting the index j) $\frac{(n-1)^2}{N}$ equals a certain — provisionally arbitrary — quantity L^2. By these places in the spectrum we define the "L-boundaries", and by their distance the "L-width" of the diffusion line (Cf. Fig. 6, on p. 338).

We now introduce the dispersion formula (2) of p. 337 and confine our attention to the case that there is only one single absorption line, so that we may write

$$n-1 = \frac{k}{\lambda-\lambda_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Our two L-boundaries will be found by substituting in this equation $n-1 = \pm L\sqrt{N}$ and $\lambda = \lambda_R$ or $= \lambda_V$, which leads to

$$\lambda_R-\lambda_1 = \frac{k}{L\sqrt{N}} \quad \text{en} \quad \lambda_V-\lambda_1 = -\frac{k}{L\sqrt{N}}, \quad . \quad . \quad . \quad (14)$$

and makes the L-width of the diffusion line equal to

$$B = \lambda_R - \lambda_V = \frac{2k}{L\sqrt{N}},$$

from which follows

$$L = \frac{2k}{B\sqrt{N}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In case we are dealing with *two* neighbouring lines of equal strength $\left(\text{that is: equal value of } \frac{k^2}{N}\right)$ at distance $2A$ from each other, it is convenient to indicate all places in the spectrum (like we did on p. 338) by a new system of abscisses:

$$l = \lambda - \lambda_M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

λ_M representing the wave-length of the point midway between the absorption lines, where we place the zero of our scale of l-values. The abscisses of the two absorption lines are now $-A$ and $+A$.

According to the equation $h = \Sigma h_i$ of p. 335, and considering the smallness of the selected spectral region, the distribution of the light in it will entirely depend on the quantity $\frac{(n_1-1)^2}{N_1} + \frac{(n_2-1)^2}{N_2}$ as a function of λ or of l. Applying (13) and (16) we find

$$\frac{(n_1-1)^2}{N_1} + \frac{(n_2-1)^2}{N_2} = \frac{k_1^2}{N_1(\lambda-\lambda_1)^2} + \frac{k_2^2}{N_2(\lambda-\lambda_2)^2} = \frac{k_1^2}{N_1(l+A)^2} + \frac{k_2^2}{N_2(l-A)^2} \quad (17)$$

The L-boundaries of each of the components of the pair are obtained by making (17) equal to L^2 or, after (15), to $\frac{4k^2}{B^2N}$. Let us consider the *red*-facing component. Its L-boundaries are situated at $l = l_R$ and $l = l_V$, and can be deduced from (17). According as the $+$ or the $-$ sign is taken, we obtain

$$l_R \text{ or } l_V = \frac{B}{2}\sqrt{1 + \frac{4A^2}{B^2} \pm \sqrt{16\frac{A^2}{B^2} + 1}} \quad . \quad . \quad . \quad (18)$$

The two *negative* values of the same radical quantities represent l_R and l_V of the *violet*-facing component of the pair.

§ 7. *Diffusion lines in the spectrum of the limb of the solar disk.* At the limb a smaller value L'^2 of $\frac{(n-1)^2}{N}$ will suffice to bring about the same degree of darkening that L^2 gave in the centre. The "L'-boundaries" determine a width B' through the relation $L' = \frac{2k}{B'\sqrt{N}}$, in analogy with (15). We thus find for the borders

of the components of our pair of limb-lines

$$l'_R \text{ or } l'_V = \frac{B'}{2}\sqrt{1+\frac{4A^2}{B'^2}} \pm \sqrt{16\frac{A^2}{B'^2}+1} \quad . \quad . \quad (19)$$

§ 8. *The difference in mutual influence of diffusion lines at the limb and in the centre of the disk.*

In conformity with our procedure with the refraction lines, we are now going to determine also in the case of pure diffusion lines an upper limit for the apparent displacements which the components of a pair impart to each other. We therefore consider the point M_L midway between the L-boundaries of one of the lines, defined by the abscissa

$$l_M = {}^1/_2\,(l_R + l_V)$$

and will only have to compute how much this value differs from $\pm A$. But we are especially interested in the *difference* between the apparent displacements of a component in the limb-spectrum and of the same line in the centre-spectrum, i.e. in the quantity

$$l'_M - l_M = {}^1/_2\,(l'_R + l'_V - l_R - l_V).$$

for which we find, after substituting (18) and (19),

$$\left.\begin{aligned} l'_M - l_M = \frac{B}{4}\Bigg\{ &\sqrt{\frac{B'^2}{B^2}+4\frac{A^2}{B^2}}+\frac{B'}{B}\sqrt{16\frac{A^2}{B^2}+\frac{B'^2}{B^2}}+ \\ &+\sqrt{\frac{B'^2}{B^2}+4\frac{A^2}{B^2}}-\frac{B'}{B}\sqrt{16\frac{A^2}{B^2}+\frac{B'^2}{B^2}}- \\ &-\sqrt{1+4\frac{A^2}{B^2}}+\sqrt{16\frac{A^2}{B^2}+1}- \\ &-\sqrt{1+4\frac{A^2}{B^2}}-\sqrt{16\frac{A^2}{B^2}+1}\Bigg\} \end{aligned}\right\} \quad (20)$$

The numerical value of this expression has been calculated for four different widths of the lines, namely $B = 0{,}050$, $0{,}070$, $0{,}100$ and $0{,}200$ Å. We took B' always to be $= B + 0{,}010$ Å, and selected a number of distances A so as to have values of $\frac{2A}{B}$ (as abscisses) suitably situated for plotting curves.

The dotted curves in Fig. 7 show the result. All ordinates should be imagined *negative*, because in this case there proves to be an apparent *attraction* of the components. We notice that the effect is

less than 0,001 Å so long as the distance exceeds twice the width of a line. On closer approach the lines rapidly grow very asym-

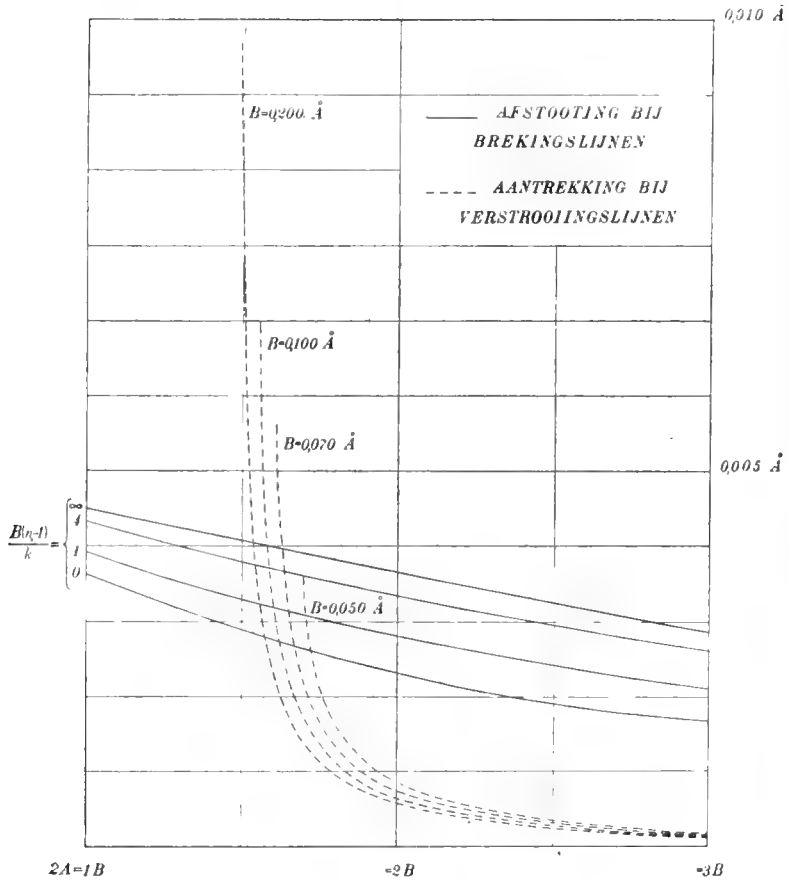

Fig. 7.

metric; at distances smaller than about 1,5 times the width, the second term of (20) becomes imaginary and the formula impracticable.

§ 9. *Comparison of the theory with the results of observations on Fraunhofer lines.*

In the foregoing we have supposed, for simplicity's sake, that the width of the true absorption lines could be neglected; but there are, of course, reasons for assigning a finite width to these cores of the Fraunhofer lines. Especially as far as very strong lines of the solar spectrum are concerned (which were not considered in the above), it would have been necessary, therefore, to base the calculations on a still closer approximation to the shape of the

dispersion curve. There is still another reason why strong lines — many of which lose their "wings" near the limb — require separate treatment, namely because for such lines, according as the limb is approached, it is indispensable to make due allowance for the spherical shape of the source of light when the consequences of diffusion, and particularly of irregular ray-curving, are inquired into. Indeed, looking almost tangentially towards the source, we are no longer allowed to assume that the darkness of the line increases with $\frac{a^2}{l}$ and with h, particularly not if $n-1$ has great values. Such considerations suggest that in a further development of the theory it will be necessary to reckon with a different set of conditions and circumstances for different lines, especially very near the limb, where the Fraunhofer spectrum passes gradually into the chromospheric spectrum.

The sharply differentiated structure visible in the chromosphere at times of excellent seeing indicates that, at least at a level only slightly outside the apparent edge of the disk, the gaseous medium must be highly transparent along the path of the nearly tangential rays, even for waves belonging to the very Fraunhofer lines. This proves that in those layers molecular scattering is unable to make the medium appear "foggy", in other words: that anomalous irregular refraction plays a greater part there in determining the distribution of the light, than anomalous molecular scattering.

We infer that probably with most Fraunhofer lines, also with the weaker ones, the darkness will depend to a greater extent on refraction than on molecular scattering — though it appears possible that the proportion between the respective influences differs from line to line.

All this has to be taken in consideration when comparing our theoretical results with observational data. Fig. 7 shows the upper limits of the effects of mutual influence to be expected in the cases we discussed, if the lines were *pure* refraction- or pure diffusion-lines. In Fraunhofer lines the two processes are probably intermingled and the respective displacements opposed; but refraction is likely to have the advantage.

We therefore may expect, e.g., if the distance between certain Fraunhofer lines lies between 1,5 and 3 times their width, that their mutual repulsion at the limb will exceed their repulsion in the centre by an amount certainly not greater than 0,002 Å.

Now, according to the above-mentioned observations of Mount Wilson and Kokaikanal, the examined effect has the average value

0,0017[5] Å, for pairs of lines whose average distance amounts to 1,7 times their mean width.[1]) This harmonizes, as regards order of magnitude, with the computed value.

In the publication just referred to we have shown that the mutual repulsions which two equal, symmetrical lines seem to exercise on each other as a mere consequence of systematic (photographical or psychological) errors of measurement, only become appreciable when the distance between the lines sinks below 1,5 times their width, and that, therefore, only a fraction of the mutual influence observed with Fraunhofer lines, can be ascribed to such errors.

The theoretical anticipation here advanced thus proves to be consistent with the observational material till now available; but for the present our conclusion cannot go beyond this, because the quantities involved in this investigation are near the limit of precision attainable with existing means for measurement in the solar spectrum.

Utrecht, April 1923. *Heliophysical Institute.*

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[1]) Cf. our article „Kritisches zu Deutungen des Sonnenspektrums", Ann. d. Phys. **71**, p. 50, 1923.

Botany. — "*Cytological investigations on Apogamy in some elementary species of Erophila verna*". By J. P. BANNIER. (Communicated by Prof. F. A. F. C. WENT).

(Communicated at the meeting of March 24, 1923).

After JORDAN, in 1823 [1]), made his well-known communications concerning the constancy of the elementary species, and in particular those of *Erophila verna*, this highly polymorphic species became not merely the classic type of absolute constancy of the elementary species, but also the subject of much experimental research. The best known work on this subject is that of ROSEN on the formation of new sub-species by cross-fertilization. According to this writer the hybrids do not conform to the laws of MENDEL, but, after having formed a very heterogeneous F_2 remain constant in the F_3 and following generations [2]). The explanation of this can only be found by cytological research, accompanied by repeated efforts at hybridization.

The investigations, the principal results of which so far obtained are given here below, were prompted by similar attempts at hybridization, carried out by Dr. J. P. LOTSY between two elementary species found near Bennebroek, and further cultivated constant by him, which, as they could not be identified with absolute certainty with any previously described sub-species, were christened *Erophila cochleoides* and *Erophila violaceo-petiolata*. These experiments, however, were unsuccessful in so far as no hybrids resulted from a cross-fertilization, but all the offspring were like the mother plant, and remained constant in following generations.

One plant only, at first regarded as a hybrid, was a very fine intermediary between the two aforesaid sub-species, but further cytological examination proved that it could not be a hybrid result of the applied cross-fertilization. The following generations of this

[1]) ALEXIS JORDAN. Remarques sur le fait de l'existence en société, à l'état sauvage, des espèces végétales affines et sur autres faits relatifs à la question de l'espèce. Bull. Ass. franc. Avanc. des Sciences Lyon 1873.

[2]) FELIX ROSEN. Die Entstehung der elementaren Arten von Erophila verna. Beitr. z. Biol. d. Pfl. 1911. Bnd. X. p. 379—421.

plant were perfectly constant. They all possessed quite the habitus of the intermediary form. That the plant in question cannot be a true hybrid, but had probably arisen from a seed of another elementary species which cannot be discussed here, was however, only demonstrated with certainty by the examination of the generative nuclei.

My thanks are due to Dr. Lotsy who, in the spring of 1921, gave me part of his material for the purpose of repeating the experiment of cross-fertilization, further cultivation of the plants, and cytological examination to ascertain the cause of the constancy.

My own experiments in cross-fertilization also yielded only plants which were the same as the mother plant. The cultures of *E. cochleoides* and of *E. violaceo-petiolata*, as well as those of the intermediate form which was first taken to be a hybrid, but which, since it appears that this is not the case, I will now term *Erophila confertifolia* on account of its extremely close roset of leaves, remained perfectly constant in the years 1922 and 1923 [1]). The results of the attempts at cross-fertilization soon suggested to Dr. Lotsy the possibility of apomixy. This would not agree with the results obtained by Rosen, but if correct it might explain why his *Erophila's* remained constant in the F_2.

The following notes upon the results I obtained will prove that the supposition of apomixy was correct and that apogamy [2]) played a part in the affair.

As regards the methods, it must be remarked that the best preparations were obtained by fixing with chloroform-alcohol-acetic acid after Carnoy. The sections, after being imbedded into paraffin, were made with a Reinhold-Giltay microtome to a thickness of 5 μ. The colouring was done with Heidenhain's haematoxylin.

Like all elementary *Erophila* species hitherto described, which were found together at the same place, the sub-species here treated exhibit, besides points of great difference, also a great similarity, which a very close systematic relation suggests. *E. cochleoides* is the smallest of the three, possesses short spatulate leaves, slightly narrower towards the base and only in the older stadia showing a shallow denticulation. The stalks are strong but not of great length. On the other hand *E. confertifolia* possesses longer and softer stalks

[1]) Although the plants have not yet flowered, the constancy can be proved with a fair degree of constancy from the young rosets.

[2]) „Apogamy" is employed here in the definition of Strasburger, i. e. development of an unfertilazed diploide ovule; according to Winkler this is a question of somatic parthenogenesis.

and its very close roset has larger leaves with a fairly broad base and which exhibit several deep dentata, while in *E. violaceo-petiolata* all three characteristics are much more pronounced. Also the flower differs in form in the three subspecies.

The cytological examination in the first place brought to light that the nuclei are extremely small; in young cells in rest they are but $2\frac{1}{2}$—$3\frac{1}{2}$ μ.

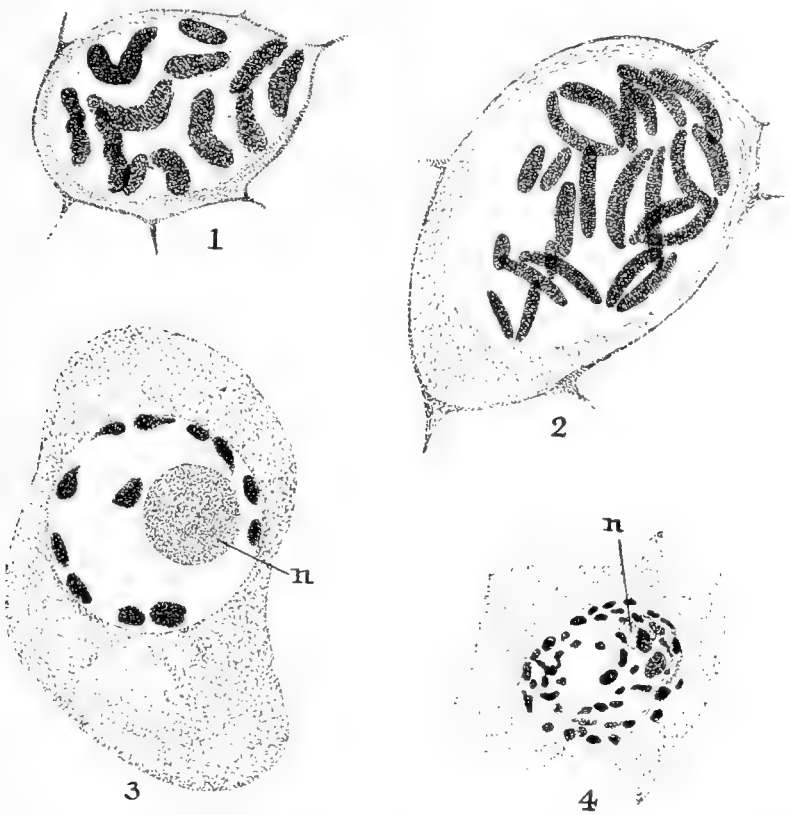

Fig. 1—4. **1** Vegetative equatorial-plate before the division of *Erophila cochleoides*. **2** Idem of *E. confertifolia*. **3** Vegetative prophase of *E. violacea-petiolata*; **4** Segmentation of the chromosomes in a vegetative cell of *E. violaceo-petiolata* *n.* = nucleolus (in all the figures). Magnification **1-2-3**: 2200×; id. **4**: 1100×.

Vegetative cell-divisions were studied in stem-tips, of which a cross-section is usually found in the sections through the entire inflorescence. No abnormalities are seen in the vegetative divisions of *E. cochleoides* and of *E. confertifolia*. *E. cochleoides* possesses 12 (Fig. 1), *E. confertifolia* 24 chromosomes (Fig. 2). They lie typically in pairs, a feature which recurs in all the divisions and in

nearly all the stages studied. The chromosome pairs differ appreciably in size. The vegetative cells of *E. violaceo-petiolata* exhibit a peculiarity which seems to belong only to this subspecies and occurs but very rarely in the vegetable kingdom. The normal number of chromosomes (diploid) is here 12 (Fig. 3). This number, however, was very seldom found. In almost every case the numbers found were higher and invariably different, up to 100 and probably still higher. Only in distinctly early prophases could the number 12 be found with certainty, and in very late telophases, shortly before the period of rest commences, this number is again nearly reached. In this last stage the counting is a matter of great difficulty, as the nuclei are very small and the outline of the chromosomes indistinct. Finally there is a third stage in which the normal number occurs, namely, the stage of splitting and seperation of the chromosomes. Occasionally, however, the number 12 was clearly seen. In all other stages of division the chromosomes divide up into numerous chromatic particles (Fig. 4). The longer the time is between the division stage and the resting stage, the larger is this number. How the transition from these stages and the metaphasic division-stage is accomplished could not be investigated.

The formation of the embryosac takes place in all three elementary species mainly in the same way. One large right-angled subepidermal cell immediately becomes an embryosac-mother-cell, without first forming a tapetal-cell. The embryosac grows considerably in size and the nucleus passes through a lengthy synapsis-stage. Finally it divides into two daughter-nuclei which do not divide again directly, but round off and like normal mitotic nuclei pass over into a resting-stage. A cell-wall is formed, and for a short time the two daughter-cells lie undivided. Then only does a second division take place in the two cells. Frequently the micropylar cell degenerates during this division; in other cases this takes place with the newformed products from it. This division of the micropylar daughter-cell very often takes place in a transverse direction, whereas that of the chalazal daughter-cell always takes about the same direction as the first division of the embryosac-mother-cell. One of the four grand-daughter- or tetrad-cells, that is situated nearest to the chalaza, increases and becomes primary embryosac-cell. The other three tetrad-cells have usually degenerated by now and meet closely over the embryosac-cell.

The development of the primary embryosac-cell to an embryosac probably takes place according to the normal plan; stages with 2 and 4 nuclei are frequently met with. The nuclei lying near the

micropyle in the latter stage form the egg-cell, synergidae and one of the polar nuclei. It was not possible to ascertain whether the division of the group lying towards the chalaza takes place in the normal way, as the antipodal cells degenerate very early, perhaps even during their formation. So much is certain, however, that one or more antipodal cells and a lower polar-nucleus are always formed, and the two polar nuclei speedily fuse together.

The formation of pollen did not exhibit any special features in the cases under examination, but very typical tetrads are formed from the pollen-mother-cells. It was immediately seen, however, that the pollengrains which were formed were largely sterile. No division of the nucleus of a pollengrain was clearly observed, and artificial cultures of pollen were unsuccessful, although a considerable quantity of pollen was usually found on the ripe stigmas. From here the pollentubes penetrated to any depth only in a very few stigmas. In one single case did the pollentube reach the cavity of the ovule. Although in this way the chance of fecundation was augmented here, the ends of the pollentubes were not found in this embryosac any more then in any of the other preparations. A male nucleus was never in a single case to be found in this embryosac; the egg-cell invariably remains lying alone and after some time begins to enlarge of itself. Finally it begins to divide, after which the first embryo- and suspensor-cells are formed. The further development of the young embryo is quite normal.

While this points to apogamy, it is only proved with absolute certainty from the behaviour of the nuclei in the embryosac-mother-cells. These commence to divide, like in so many other apogamous plants, according to the heterotypical scheme. Many synapsis- and spireme-stages are observed. Instead of real gemini of chromosomes which totally or for the greater part fuse together, merely pseudo-diakinese-pairs are observed. The chromosomes approach each other, but remain at some distance from each other. After this the division has a homoiotypical character. Fig. 5 represents a telophase-stage of the division of the embryosac-mother-cell of *E. cochleoides*. In the uppermost micropylar daughter-cell the chromosomes are present in diploid number (12). The same number can also be counted in the chalazal daughter-nucleus, though less distinctly. The knife of the microtome had touched this nucleus, so that a few ends of chromosomes are to be found in the adjoining section. The figure shows which fragments in the two cross-sections belong to each other. The telophase-stage of *E. confertifolia*, which possesses vegetatively 24 chromosomes, is a still clearer and stronger proof of the apogamy,

as is shown in Fig. 6. Here there are 24 chromosomes in both nuclei; they can be best counted in the micropylar nucleus. The fact, that the chromosomes after the division still lie so clearly in

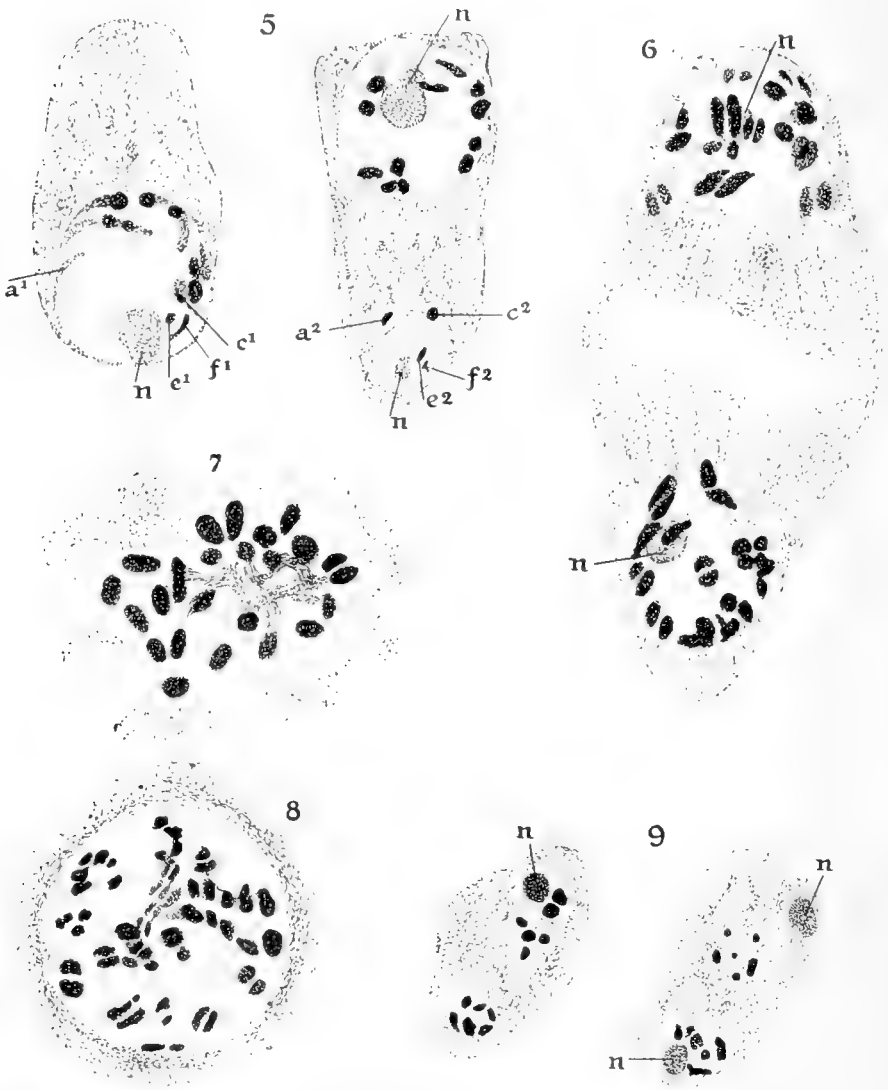

Fig. 5—9. **5.** Daughter nuclei of the embryosac-mother-cell of *E. cochleoides*, on the left the chalazal nucleus, on the right the micropylar nucleus; a^1—a^2, c^1—c^2 etc. fragments belonging to the same chromosome. **6.** Idem of *E. confertifolia*. **7.** Endosperm nucleus of *E. violaceo-petiolata*. **8** One of the three sections through a pollen-mother-cell of *E. violaceo-petiolata*. **9.** Formation of the tetrad nuclei in a reducing-division in a pollen-mother-cell of *E. cochleoides*. Magnification **5-6-8**: 2200 ×; id. **7**: 1450 ×; id **9**: 1100 ×.

pairs, points to a very strong affinity which cannot be broken by the individual splitting.

In the division of the embryosac-mother-cell and the pollen-mother-cell of the *E. violaceo-petiolata* we have the same phenomenon again as was also seen in vegetative cells, namely the segmentation of the chromosomes. It is remarkable, however, that here the chromatic particles lie in pairs, as we find all the chromosomes in the two other subspecies in pseudo-gemini. Here too very large numbers were found; approximately 50, 64, 70 and even as high as 130 or 140 were found. Fig. 8 represents such a stadium taken from a pollen-mother-cell, which had been cut into three sections, only one of which is shown here. Nevertheless about 60 chromosome particles can be counted. As the embryosac-mother-cell has exactly the same appearance and as here too the same phenomenon is seen directly after the division, it was impossible to find a pair of daughter nuclei with the diploid number to prove apogamy. Here, however, some very distinct endosperm-divisions lend assistance. As it was established that the polar-nuclei unite with each other in this apogamous plant also, the endosperm-nuclei must possess twice as many chromosomes as the embryosac-nuclei. Thus, to demonstrate apogamy this number would have to be 24, and that this is actually the case is shown by fig. 7, which illustrates a cross-section through the middle of one spindle, looking in the direction of one of the poles. The ends of the 24 chromosomes can be clearly distinguished, while the attraction of some chromosomes by the poles can also be observed.

Whereas in the divisions of the embryosac-mother-cell there is no reduction of the number of chromosomes, even though it has passed from the heterotypic phase to the homoiotypic very shortly before the division, the reducing division in the pollen-mother-cells occurs normally. During this division no peculiarity was observed in any of the cases examined other than the segmentation above-mentioned in *E. violaceo-petiolata*. Fig. 9 represents 2 sections of the tetrad nuclei of a pollen-mother-cell of *E. cochleoides*, all of which form the reduced number of chromosomes.

As has been said, however, the great majority of the pollen-grains produced from them are sterile. But even if there be fertile ones among them, they are not productive.

Thus the most important conclusion arrived at was that apogamy occurs in these three elementary species of *Erophila*, which explains the failure of the attempts at cross-fertilization. The experiments of Rosen have shown that not all subspecies are apogamous, or at

least they are not obligatory apogamous. The constancy of his new forms in the F_2 might find their explanation in apogamy. The intermediate hybrid formation in the F_1 and the singular appearance of the F_2 on the other hand, are not explained, and in respect to this a special theory would have to be applied to explain the sudden occurrence of apogamy.

Utrecht, March 1923. *Botanical Laboratory.*

Botany. — *"On the nature and origin of the cocos-pearl"*. By Dr. F. W. T. HUNGER. (Communicated by Prof. G. VAN ITERSON JR.).

(Communicated at the meeting of March 24, 1923).

In the endosperm cavity of the seed of Cocos nucifera a local calcareous formation is sometimes found to occur, to which the name of "cocos-pearl" has been given, and which must be looked upon as a highly remarkable and very rare phenomenon [1]). Such a cocoa-pearl has usually the form of a pear, or egg, sometimes it is almost spherical and has a smooth surface, as a rule of a milky-white colour. Its chemical composition corresponds somewhat to that of the oyster-pearl, from which it differs, however, in appearance by the lack of the pearly sheen.

RUMPHIUS was the first to describe this calcareous formation as "calappites" [2]), and for more than a century after him nothing was heard of this phenomenon, till at the Meeting of the Boston Society of Natural History on the 1st. of February 1860 [3]), Mr. FRED. T. BUSH presented a specimen of this cocos-pearl for chemical and microscopical examination. The research was entrusted to Dr. BACON, who submitted his report on the subject at the Meeting of the same Society on 16th. May 1860 [4]).

In 1866 Dr. RIEDEL, Ex-Resident of Menado, reported having found a pearl in a cocoanut he opened [5]). This was the first report by an eye-witness who had actually seen this phenomenon, apart from the many stories told by natives about it.

Contrary to the statement of BUSH to the effect that cocos-pearls "are said to be found free within the cavity of the cocoa-nut", SKEAT [6]) reported in 1900 that they are "usually, if not always, found in the open eye or orifice at the base of the cocoa-nut".

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[1]) F. W. T. HUNGER, Cocos nucifera, 2nd Ed. pp. 243—250, Pl. LXVII (1920).

[2]) E. RUMPHIUS, Herbarium Amboinense, Vol. I, pp. 21—23 (1741).
Idem, D'Amboinsche Rariteitkamer, pp. 291 - 292 (1741).

[3]) Proceedings of the Boston Soc. of Nat. Hist., Vol. VII, pp. 229 (1861).

[4]) Idem, Vol. VII, pp. 290—293 (1861).

[5]) Nature, Vol. XXXVI, pp. 157 (1887).

[6]) W. W. SKEAT, Malay Magic, being an introduction to the folk-lore and popular religion on the Malay Peninsula, pp. 196 (1900).

No other data regarding this remarkable phenomenon exist, and at the present day we are still completely in the dark as to the nature and origin of such a cocos-pearl.

On my last voyage to the East Indies for purposes of study, I resolved to endeavour to find out something further about the cocos-pearl and if possible solve the problem of its formation. At the same time I realised the utter futility of going to look for cocos-pearls in the Tropics on account of their extremely rare occurrence. In proof of this it may be mentioned that on a cocoa-nut estate, where approximately 3 million nuts have been opened annually for years, no such pearl has ever been found, although stories about them have led to their existence being suspected.

I therefore directed my research to gathering as many authentic data as possible.

On one of my voyages I met a native of British India who possessed a very fine cocos-pearl. According to his own account he had seen with his own eyes this specimen inside an opened cocoa-nut which had been brought to him from Madras. He assured me solemnly that his pearl had been attached to the kernel of the cocoa-nut and exactly at the place where, in germination, the cotyledon forms a haustorium.

Later on I also met with an Arab on whose cocoa-nut plantation in South Borneo a cocoa-nut had been gathered which, on being opened, proved to contain a pearl attached to the inside of it. He had dislodged the pearl from the kernel of the nut with his own hand. In this case also the pearl had been attached at exactly the same place as in the case first-mentioned.

These two corroborative declarations of eye-witnesses, who had both seen a cocos-pearl still attached inside an opened cocoa-nut, furnished me with a preliminary guiding-thread and led me to suppose that the spot which they indicated would probably be the normal point of attachment of such a cocos-pearl.

The normal germination process of the cocoa-nut begins by an enlargement of the embryo, whereby the cotyledon commences to grow inwards to an absorbing organ (haustorium), and thereby comes to protrude outside the endosperm and into the central cavity. Simultaneously with this, the plumule grows out and, breaking through the membranous operculum of the germinating pore, it pushes its way out through the hard shell.

Proceeding from the provisional determination of the place of

attachment of the cocos-pearl, the following hypothesis could now be formed. Given that the germination, being in progress, is stopped by some cause or other, thus preventing the further development of the haustorium, it is conceivable that the haustorium in this state might become encrusted by the influence of the cocoa-nut milk, and that from this the completely petrified cocos-pearl would gradually be formed.

It was now essential to find the reason for any such check in the process of germination and the accompanying solidification of the haustorium, and I wish now to submit the following remarks on this head.

At the side where the cocoa-nut has been attached to the stalk, three thin spots so-called germinating pores, or "eyes", can be seen in the hard inner shell of the fruit. As a rule one of these holes, the so-called "porus pervius", is closed by a membrane, whereas the two other, the so-called "pori caeci", are furnished with a hard tegument. In germination, the plumule pushes its way out through the porus pervius.

By way of exception there may be, instead of three, two germinating pores, viz. one porus pervius and one porus caecus, and only very rarely will there be only a porus pervius with both pori caeci entirely absent. Nevertheless a cocoa-nut of this description can germinate in the usual way.

It is a different case, however, when there is not even a porus pervius, the base of the inner shell showing no germinating pore at all, as occurs in extremely rare cases.

Such a cocoa-nut is known in the Malay language as a "kĕlapa boeta", or "klåpå boentĕt" in Javanese, which signifies a "blind cocoa-nut".

As remarked above a cocoa-nut without germinating pores is a very great rarity, for which reason they are regarded by the Mahommedans as sacred. The "kĕlapa boeta" is a talisman (*tjimat*) *par excellence*, and consequently it is very difficult to obtain a specimen.

This meeting with the *kĕlapa boeta* furnished me with an instance of the way in which a normal germination is rendered impossible by nature, and I did my utmost to procure some specimens.

I finally succeeded in collecting eight unopened "blind" cocoa-nuts from the East Indian Archipelago. Two of them came from South Borneo, one from Halmaheira, one from Ceram, one from the North of New Guinea, one from South New Guinea, one from the Aröe Islands and one from the Tanimber Islands, all of which I have collected personally from these several places.

Most of the specimens were very old nuts; some, according to their owners, had been preserved for scores of years as family heirlooms.

The first four "boetas" which I opened produced nothing, but in the fifth I found a really beautiful pearl still attached to the kernel; the two next produced negative results again, and the eighth specimen I have kept unopened.

The nut which had contained the pearl, as shown in Fig. 1, had been purchased from an old native at Ritabel (Larat), one of the Tanimber Islands in the Moluccas, who informed me that it had been gathered but a short time before. This proved to have been the case, because the endosperm in it was quite normal, whereas in the other nuts the kernel was either very much dried up or had even partly become a mass of brown powder.

The pearl was attached without the least trace of a stalk, being merely embedded in the endosperm (Fig. 2), and was quite easy to remove from the kernel. It lay exactly at the base of the nut, just under the spot where the germinating pores ought to have been, and thus agreed completely with the indications as given above.

This discovery, in my opinion, warrants the inference that the cocos-pearl actually represents a calcified haustorium, which has been retained in the nut after the primary germination was checked, owing to the plumule not being able to get through the shell on account of the porus pervius being lacking. As the inner shell of the *kelapa boeta* remains hermetically closed, the newly formed haustorium becomes encrusted under the influence of the cocoa-nut milk with calcium-salts, although it still remains unexplained why the cocos-pearl consists almost entirely of calcium carbonate, while neither the cocos-kernel nor the cocoa-nut milk contains any calcium carbonates.

The belief that a *kelapa boeta* invariably contains a cocos-pearl was sufficiently disproved by my experience that of seven specimens only *one* such formation was found in a "blind" cocoa-nut. On the other hand, it is probable, in my opinion, that it will be principally (or exclusively?) the *kelapa boeta* that contains the cocos-pearl.

The nature and origin of the cocos-pearl as a calcareous plant germ might botanically be considered as analogous to a phenomenon seen in human and animal pathology in the petrifaction or mummification of the embryo, and termed Lithopaedion or Lithoterion respectively.

Amsterdam, March 1923.

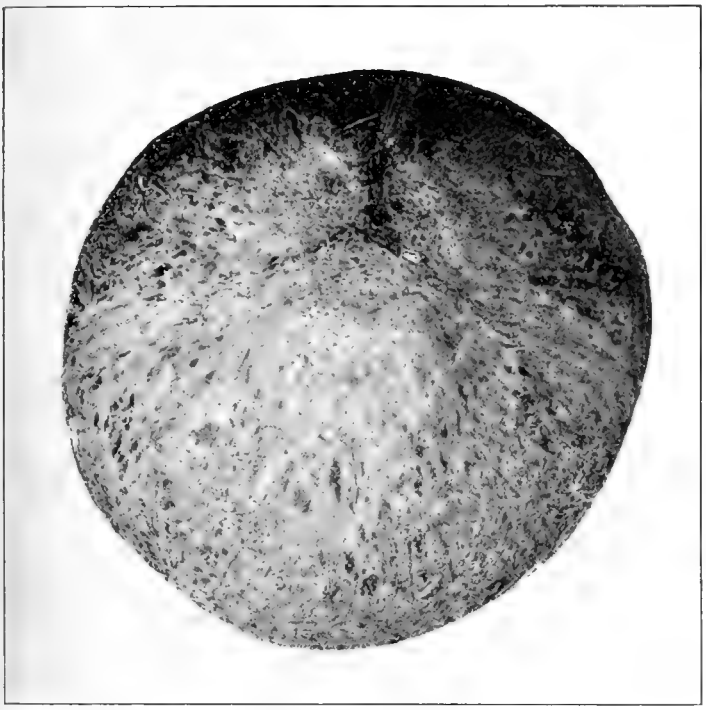


Fig. 1. *Kĕlapa boeta* Basis of a blind cocoanut, without germinating pores. $^{3}/_{5}$ nat. size.

Fig. 3.
Cocos-pearl from fig. 2.
nat. size.

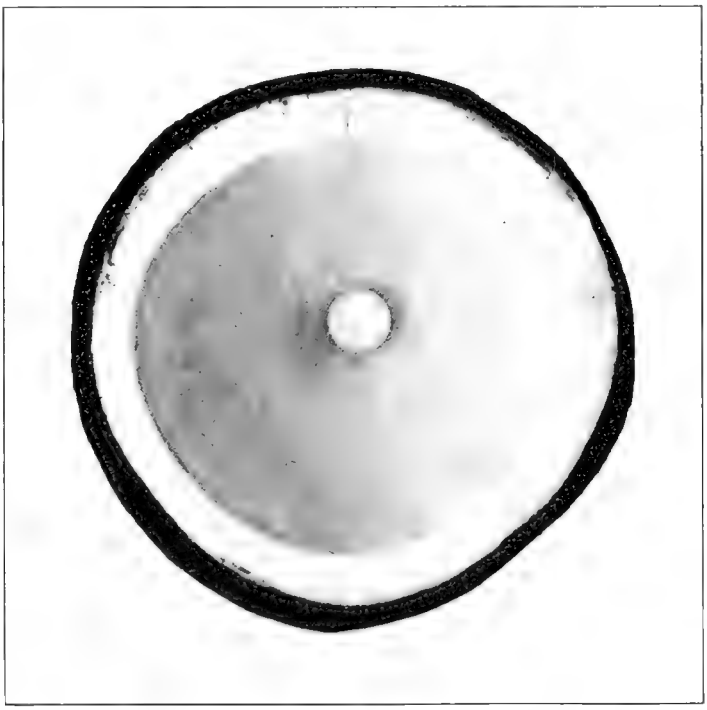


Fig. 2. *Kĕlapa boeta* Endosperm cavity with a cocos-pearl insite. $^{3}/_{5}$ nat. size.

Botany. — „*The genus Coptosapelta* KORTH". (*Rubiaceae*). By Dr. TH. VALETON. (Communicated by Prof. J. W. MOLL).

(Communicated at the meeting of April 28, 1923).

§ 1 In my paper on *Lindeniopsis*, a new sub-genus of *Coptosapelta* KORTH. (Proceedings of the Academy of Sciences of May 30, 1908) I gave a synopsis of the few species of the genus, known at that time. At my further study of the Rubiaceae of the Malay Archipelago and of New-Guinea, I again found a number of species not described at all or not in the right genus, in consequence of which this number has increased to 11. Besides it appeared from the research, that the existing diagnosis, already revised by me, could no more be applied to all species. For this reason I want to subject the chief characteristics of the genus of systematical interest to an investigation and subsequently to summarize the species known at the present time.

§ 2. *Historical review.* The genus was constituted by KORTHALS (1851) on some fruiting branches of a liane, gathered by his colleague Dr. MÜLLER on the sandy plains near Karrau (Southern and Eastern division of Borneo). He found them to belong to a new genus in the group of the Cinchoneae DECANDOLLE, of which there are but a few genera known in the Dutch Indies.

As chief characteristics he considered the liane-like habit, the fruit splitting up in two cells, each of them splitting up again and the peltate seeds provided with a fringed wing, a combination of characteristics, not yet found in any genus. In naming the genus he apparently referred to the seeds. At least I think to recognise the words κοπτω, in the meaning of „Chopping" or „Hewing" (because of the notched wings) and πελτη shield. The significance of the connecting syllabe „sa" is not clear to me. Probably the name originally ran: *Coptospelta*, a bad word-formation. As a specific name he used „flavescens", alluding to the yellowish tint the leaves get on drying.

KORTHALS's specimen is lacking in the Dutch and Dutch-Indian Herbaria. It is not apparent either, that MIQUEL knew it (1856). It was however known to HOOKER, when describing in 1876 a second species of the same genus, *C. Griffithii* HOOK f. in Icones plantarum

tab. 1089, in which he quoted KORTHALS's original and Borneo, Sumatra and Malacca as its native places. A short description of the species was afterwards given by HOOKER in HOOK. Flora indica III (1885) especially to distinguish this species from *C. Griffithii*. A little more detailed was KING in KING and GAMBLE, Flora of the Mal. Peninsula (1903).

The species however had not escaped the attention of either WALLICH or BLUME. The former published it in 1828 mistakenly as *Stylocoryne macrophylla* (= *Webera macrophylla* ROXB.), the latter took it for a new species of the same genus and gave a brief diagnosis of it in BLUME, Bijdragen (1826), as *Stylocoryne tomentosa*, while MIQUEL gave a somewhat fuller description of the same species, gathered by ZOLLINGER in Tjikoja in Java (number and date unknown), in 1856 in Fl. Ind. bat., as *Stylocoryne ovata* MIQUEL. A third species of this genus, in order of time of discovery, is the *Coptosapelta Hammii* (subgenus Lindeniopsis) I previously discussed. It was gathered by HAM in Billiton in 1907. At about the same time a fourth species was collected in the Philippine Islands and, by E. D. MERRIL, described as *Randia olaciformis* and classed with the right genus by ELMER in 1912 (in Philippine Leaflets). A fifth species, already gathered by H. O. FORBES in British New-Guinea in 1885—86, was described by WERNHAM in 1917 (in Journ. of Botany). He classed it however with the genus *Tarenna* GAERTN. (= *Stylocoryne* WIGHT et ARNOTT). Besides I found two Borneo species undescribed in the Herbaria at Leyden and Berlin and three of New-Guinea, while finally an eleventh species was discovered, gathered bij the army surgeon JANOWSKY at the "Geelvinkbaai" in 1910.

§ 3. *Habit.* Except the deviating species *C. Hammii*, above mentioned, a half-climbing shrub, all *Coptosapelta*-species hitherto known are lianes. To all of them the excellent description by ELMER of *C. olaciformis* (Phil. Leaflets V. p. 1856) is mainly applicable: "A looping treeclimber; stem two inches thick, very irregular, heavylooping, numerously branched toward the top and forming hanging masses; leaves coriaceous, descending, curved upon the upper deeper green surface, apex recurved; inflorescence from the longer samewhat drooping branches, *erect*.

Of the species, gathered in German New-Guinea by LEDERMANN, is twice given "Liane mit beindickem Stamm", once "Liane mit armdickem Stamm". For *C. Griffithii* from Malacca as well as for the oldest species *C. flavescens* is given "Liane", to which KING's native collector adds: "A handsome creeper, 30—50 ft. high". The

two species from Borneo first described here, were probably of a similar habit. Of JANOWSKY's species is only said: "10 Meters high"; the piece of branch or stem, about as thick as a finger, gathered by him, shows a soft whitish strongly-lobed wood-cylinder with large vessels.

§ 4. *Stem* and *buds*. The rod-shaped twigs, as occurring in the herbaria, are nearly cylindrical (only in some species e.g. *C. montana* the utmost twigs are square), the nodes swollen and provided with an annular groove. As a rule only the flowering lateral and terminal branches are gathered, consequently but a few terminal buds, all of young specimens of *C. flavescens* and *C. montana* are present. These are wanting bud scales; they are formed by the two youngest leaflets, pressed together with the flat upper-surfaces, and are enclosed by the two rather small stipules only at the base. With the young growing twigs these very young leaflets are lanceolate and they consist more than half of a broad "*Vorläuferspitze*" rounded at the tip and certainly dark-green when alive (see RACIBORSKI in Flora 1900), reminding us of *Dioscorea*-species. Where there are axillary-buds, they are but a couple of mms. long, ovate, covered with long and dense hair.

§ 5. *Indument*. All species have a coat consisting of single short appressed hairs, and long hairs lying flat but free at the top; the latter are soft, straight, colourless or rarely (in sicco) yellowish, usually thinly spread; on the young twigs and leaves, the inflorescences and generally also the petioles, they are closer together, forming a soft, thin "tomentum".

On the full-grown leaves they are almost or totally absent in *C. olaciformis*, *fuscescens* and *maluensis*, where the twigs also grow bare in course of time. *C. Griffithii*, *C. Beccarii* and a hairy type of *C. flavescens* have a soft hairy covering, consisting of long curved hairs not close together.

§ 6. *Leaves*: 1. Shape: In most species hitherto known, the almost exact elliptical shape of the lamina is characteristic for the average-leaf; i. e. a symmetry of the two halves with respect to the transverse as well as the longitudinal diameter of the leaf, apart from the frequently lengthened tip and wedge-shaped base.

HOOKER (1882) and KING (1903) refer to it in their descriptions of *C. Griffithii* and *C. flavescens*, MERRILL of *C. olaciformis*, WERNHAM of *C. hameliaeblasta*.

Of course the elliptical shape is not constant with any individual, but often passes into the ovate form or becomes oblong (in this case the symmetry is preserved), the leaf-base varies between rounded and wedge-shaped. *Young plants* of *C. flavescens* have lanceolate leaves. The few known leaves of *C. Janowskii* (a mountain-species) are likewise lanceolate and provided with a long dropping-point. *C. montana* (a mountain-species from Borneo) has on several twigs elliptical and oval leaves with rounded base, and lanceolate, acuminate leaves. *C. Hammii* (the xerophilous species above-mentioned) has the tip ending in a very short hard mucro. For the rest the leaves of all species have a clearly marked acumen, sometimes very short.

2. The consistency of the leaf of old plants and twigs is thin-leathery, the colour of the upper-surface is glossy dark-green, of the lower surface lighter green with dark-green veins, in a dry condition hard and in herbaria as a rule brittle. Of young plants (see above) they are much thinner, in sicco almost membranous (in vivo herbaceous). *C. Janowskii* (see above) has likewise thin ones. When drying the leaves always change their colour to yellow or yellow-green, more or less mixed with sepia-brown, the upper-surface is as a rule dark-brown or olive-brown 153—155 (Code des couleurs de Klincksiek et Valette). For *C. olaciformis* 183—188 or 193, or paler 217; for *C. Hammii* 202—217, for *C. flavescens* the colour of the upper-surface frequently 114, of the lower-surface 153.

3. With respect to the diagnosis of the genus as well as the species the nervature of the leaves, though showing common characteristics for all species, is of some importance. The nervature of the leaves is penniform, and the secondary or lateral veins never start from the median nerve opposite to each other at the same level, their number being as a rule rather small, 2 or 3 or 4 on each side. In many species the secondary veins next to the tip do not start above the middle of the median nerve, so that the upper half of the leaf is mainly supplied by tertiary veins. Besides they start at unequal distances from each other and are closest to each other at the leaf-base, the *lowest two* (or sometimes one) starting *close to or even from the leaf-base*; in consequence of this they resemble triplinerved and trinerved leaves (*Ficus, Cinnamomum, Viburnum*). There often starts from the leaf-base on one or both sides a secondary vein so thin, that it may be counted among the tertiary veins and may easily be overlooked; yet it follows in its course the stronger veins. After starting from the midrib these go upward in a wide curve till close to the edge, next about parallel with the edge towards the apex. The two foremost veins

end in the apex (acrodromous veins of ETTINGHAUSEN), the next run some way between the edge and the first pair and all or most of them end in the tertiary net without uniting.

The secondary veins thus run parallel to the margin for a great length and most of the basal veins partly embrace the higher ones. A definition answering exactly to this nervature, I do not find in ETTINGHAUSEN. It forms a mixture of the common *camptodromous*, (bogenläufige) with the *acrodromous* (= spitzenläufige) nervature; the term *amplexidromous* might be applied (see e. g. the figures of *Thibaudia* species (acrodromous) in v. E.'s work, besides *Nectandra* and other *Lauraceae*). The species with larger leaves *C. flavescens*, *olaciformis*, *Beccarii* have a somewhat greater number of veins (11—12), while the basal veins sometimes curve inward and unite with the preceding: schlingenläufige (brochidodromous) nervature.

The number of secondary veins of the deviating species *C. Hammii* amounts to 12; in the rather small leaves they are more crowded and fairly equally divided over the length of the leaf, joining with a curve. This is an instance of regular brochidodromous nervature, but the leaf-base is pointed and the veins are ascendent and embrace each other upward from the base, so that the character of the genus is not quite lost. The tertiary nervature is always clearly visible and equally spread over the whole leaf; the horizontal connecting veins are usually prominent and form a delicate lattice-like reticulation. Leaf-impressions made with carbon-paper usually show only this net-work.

4. Regarded biologically the leaves of *Coptosapelta flavescens* belong according to HANSGIRG (Phyllobiology, 1903, pag. 293) to the *Myrtus-* or *Lauraceae*-type with which he also classes the *Coffea*-species together with numerous other *Rubiaceae*, among which *Crossopteryx*, an african genus closely allied to *Coptosapelta*.

According to him these types are xerophilous. They belong to the periodically dry and moist regions along the Mediterranean from Spain to Palestine and also to tropical regions with similar climatological properties. As their characteristics he gives: "Strongly cutinized epidermis, rectilinear polygonal or sometimes undulated epidermis-cells, stomata sunk, very glossy lamina usually bare, sometimes grey- or white velvety, simple, narrow and entire or round, elliptical, oval and oblong, leathery and stiff", as protection against strong insolation, excessive evaporation, adhesion of water, winter-temperature, etc. Without doubt many of these properties belong to *C. flavescens*, occurring in the secondary woods of the first zone, a. o. in bamboo-woods between 200 and 500 meters, but only on

adult old plants, the leaves of which are indeed rather like those of *Coffea arabica*. Also the tomentose leaves of *C. Beccarii* and *C. Griffithii* belong to this type. On the other hand *C. Janowskii* and *C. montana* are both mountain-plants with narrower leaves and a long dropping-point, instances of HANSGIRG's "*ficus-type of the rain-woods*". To this type the young plants of the above-mentioned species also approach, in which the xerophilous habit does not much come to the fore.

Here it is not only the danger of too strong evaporation, brought along by the succession of the monsoons, but no less the risk of the damage, caused by strong rainfall which prevails.

Among the remaining species, of which *C. maluensis* does not grow higher than 200 meters above the sea-level, while the others occur at different levels in the mountains, various transitions between HANSGIRG's *Myrtus-* and *Ficus-*type are found.

An instance of real xerophilous habit is only given bij *C. Hammii* (Lindeniopsis) which as I previously mentioned should be classed with SCHIMPER's "Hartlaub formation".

§ 7. *Stipules.* The usual shape of the stipules is that of a small triangular scale, which has often been lost with the full-grown twigs in the herbaria. At the back-side and along the edges it is covered with hairs, turned to the front, often longer than the stipule and sometimes covering it entirely. The variations in shape are usually due to differences in the ratio of width and length, which depends on the width of the node. Sometimes however they may be of use in the determination of the species. This is for instance the case with *C. flavescens* and *C. olaciformis*, which show a great resemblance on superficial contemplation of leaves and flowers and were considered identical by MERRILL.

Here, in numerous specimens examined by us, the stipules are quite sufficient to distinguish between the two species. *C. flavescens* has linear-lanceolate ones, rather abruptly passing into the broad base. They vary in length between 4 and 8 mms. and strike the eye in the herbaria because, at least in the dry specimens, the back-side is absolutely bare and the broad hairy edges show clearly. *C. olaciformis* has smaller stipules, usually only 2 mms., slightly longer than broad, in old condition hairless and swollen at the base. This description has been taken from a specimen, distributed by MERRIL himself from Luzon (Ph. pl. 396) and classified as *C. flavescens*. It is also applicable to ELMER's original specimen (see below § 11. Synonymy and relationships).

§ 8. *Inflorescence.* In all species the inflorescence consists of axillary compound cymes or corymbs, starting from the leaf-axils near the top of the twigs. At the top they are closer together and often (by the reduction of the floral leaves) are combined to large terminal decussated panicles or thyrsi. Such terminal panicles also occur in other genera of the group of Cinchoneae, viz. on *Cinchona* and *Ferdinandusa*.

In the descriptions of the genus (HOOKER—SCHUMANN—VALETON in Ic. bog.) there is wrongly spoken of "thyrsi penduli". Undoubtedly the panicles are erect in all cases (see ELMERS' description above, § 3), but the ends of the long branching twigs are drooping and proper flowering-branches start sideways from these. In good herbaria it may sometimes be observed how the flowering-branches form an almost right angle with the leaf-twigs.

The extension and relative length of the axis determine the character of the inflorescences with respect to the species. First of all two types may be distinguished.

The simplest case is *C. Janowskii*, a New-Guinea-Mountain-liane, where the axilary inflorescences have been reduced to single flowers and the terminal thyrsus to a simple closed raceme. The pedicels are rather long and about midway provided with two bracts. It is highly probable that on more luxuriant branches these bracts are fertile, forming forked cymes (dichasia). *C. montana* likewise has isolated flowers (uniflorous cymes) in the axils of poor flowering-branches and at the top a raceme of 5 flowers. A more luxuriant terminal twig, consisting of 6 internodia, has in the lower axils long-stalked closed racemes, bearing 5 flowers, in the following three-flowered cymes, while the top again forms a closed raceme with linear bracts. The twig of *C. Hammii* also ends in a raceme of 5—7 flowers, but with very short internodes and pedicels, so that the flowers, provided with long corollatubes, are close together and take the shape of an umbel.

In the second type both the axillary and the terminal inflorescences are compound, and the latter have the shape of corymbi or depressed (almost umbelliform) thyrsi in consequence of the decrease of length towards the apex of the internodes and peduncles; the axillary ones too are more or less corymbiform. Especially the relative length of the peduncles of the partial inflorescences, the number and density of the flowers, the number of internodes of the terminal panicles, determine the character of these species.

C. olaciformis deviates most of the rest on account of the slight extension of the corymbi and the small number of flowers. The

axillary inflorescences are short-peduncled cymes with only 3—5 flowers, many times shorter than the leaves. The terminal thyrsi consist of but 2—3 internodes and cymes with few flowers and short peduncles, and are also shorter than the higher leaves.

In the remaining species of this second type both the axillary and the terminal inflorescences are multiflorous much branched, corymbous, with moderately long or very long stalks, while the terminal panicles may consist of 5 internodes.

§ 9. *Flower and Seed.* The *calyx* is now cup-shaped, only superficially emarginate with 4—5 very short pointed teeth, now divided into nearly free sepals down to or almost down to the base, in which case the limb is not sharply separated from the ovary; in a third more frequent case cleft to the middle or a little farther. To characterise the genus it is therefore of no value, but of great value to determine the species. For all species mention should be made of the "intestinal gland papillae", (Darmdrüsen papillen: Solereder), which are placed at the inside alternate with the lobes or teeth, and resemble those which the *Rubiaceae* always bear at the inside of the stipules and are sure to occur on their calyces more frequently than appears from literature

The *corolla* which is contorted in aestivation, but without externally visible torsion, is trumpet-shaped and reminds us of species of *Randia* and *Tarenna*, having a quinquepartite limb and as in the case of *Randia* the relative lengths of tube and limb, though not always constant in the same individual, is when the average is considered, a means of distinguishing the species.

The following average rations were found: Tube many times as long as the lobes (*Lindenia*-type), 3—6 cms. long: *C. Hammii.* Tube twice as long as the lobes: *C. Janowskii.* Tube about the same length as the lobes or a little shorter: most of the species. Tube about half the length of the lobes: *C. Griffithii, C. fuscescens* and *C. lutescens.* A peculiarity is, that the tube which is usually cylindrical and equally wide along its whole length, shows a sudden inflation above the middle in two species, *C. Griffithii* and *C. Janowskii*, which for the rest are farthest apart on account of the length of the corolla tube.

The internal hairy covering of the corolla tube is also of some interest. Only in 3 species *C. Hammi, C. olaciformis, C. flavescens*, the interior of the corolla tube and the filaments are glabrous. In the other species, where the filaments are covered in front with long furry hairs directed downwards, this hairy covering continues

as projecting ridges along the inside of the tube, down to the middle or till close to the base. Between these ridges the inside is covered with soft crisp hair; the descriptions of the genus however are wrong, where they say: "*Faux barbata*" for the hairy covering of the *faux* (regarded as orifice of the tube) is lacking everywhere.

When the limb is still closed, the corolla is externally entirely covered with thick-velvety or short silky hair.

The *stamina* have thin filiform filaments, which, as already observed, are congenitally attached to the corolla-tube, forming protuding ridges; the part projecting from the corolla is short and filiform, in some species hairless, in most of them covered with furry hair in front; the anthers are very narrow lanceolate and have a linear connective, coherent with the filament near the base at the backside; the long linear anthercells diverge more or less at the base, so that the base of the anther is retuse, or arrow-shaped as with *C. flavescens*, while the tip ends in a tapering point; the backside is covered with appressed hair, except in *C. Hammii*, where also the free filaments are almost lacking. The anthers hang more or less versatile from the corolla during the flowering and are curved up or contorted.

The *pistil* is highly characteristic for this genus. The stigma is wedge-shaped or cylindrical (in *Lindeniopsis* club-shaped) *not* divided into lobes, and proportionately long. The style is straight and smooth and compressed sideways, and about as long as the corolla-tube, so that the stigma overtops the corolla far. The papillary surface I generally found covered with pollen.

The *ovary*, covered with an annular disk, is regular, bilocular as in the whole group of Cinchoneae. Around a fleshy, cylindrical axis, nearly filling the two ovary-cells, are the numerous anatropous, flat, peltate, erect, imbricate ovules.

The *fruit* is globular or more or less oblong, compressed at right angles with the septum and has in a ripe condition a though, horny or thin parchment-like envelope, surrounded by a thin dry outer-integument. In very old fruits the outerlayer crumbles down and the horny valves come quite into view; in this respect there is some analogy with *Bikkia* (Condamineae). The splitting into valves is not perfectly regular. It begins with the separation of septum and axis, (loculicide dehiscence) at the top of the capsule, but next the septum itself splits, so that 4 cocci are formed open at the top and at the sides and connected at the base. This latter splitting however may fail to occur. During the splitting the fleshy placenta shrivels up, causing the numerous seeds to get gradually loose.

The *seeds* are flat, round or oblong with the hilum about in the middle (peltate) and surrounded by a membranous fringe-like notched wing, about as broad as the seed. For the distinction of species only differences in size are to be considered (except in *Lindeniopsis* where the edge of the wing is not fringed); *C. olaciformis* and *C. maluensis* have the smallest seeds; *C. Griffithii* the largest, as far as we know.

As to the process of pollination it may only be surmised. The contorted movable projecting anthers and the long protruding stigma point at the probability of wind-pollination, but the prominent flowers scenting of elder and orange-blossom may point at a connection with insects. The possibility of self- and inter-pollination is corroborated by the great mass of flowers and by the fact that (at least in the herbarium) the anthers are already open in the buds.

§ 10. The *station*: About the character of the locality in which the various species are found we only know as follows:

C. flavescens was gathered by KORTHALS on the barren sands along the river Karrau in Borneo; by KING's collector in bamboo-woods in Malacca 100—200 metres above the sea-level, by various collectors in Western Java at the foot of the mountains, on various spots in light secondary wood.

C. maluensis at 40—100 meters above the sea-level in passable primeval forest, about 20—25 meters, high; the ground covered with foliage („Galerie wald" SCHIMPER), with occasional low wood, mostly consisting of Pandanus and low feather-leaved palms (Camp Malu); idem with many tree-ferns and bamboo and Selaginella a metre high, as undergrowth (April-flusz): LEDERMANN.

C. fuscescens in "Buschwald" changing into mountain-wood up to 1500 metres above the sea-level, few large trees, many epiphytes and moss, many glades, ground often overgrown. On steep rocky slopes (Felsspitze): LEDERMANN.

C. lutescens in dense wood on hills, about 25 metres high, rather mossy; in the underwood many dwarf-fan-palms and lianes, *Freycinetia, Araceae, Agathis, Pandanus*: LEDERMANN.

§ 11. *Relationships and synonymy.* On account of the structure of ovary and fruit *Coptasapelta* belongs to the very natural tribe of *Cinchoneae* HOOKER (Genera plant. II p. 11) among which 44 genera are reckoned. This tribe is divided into two subtribes:

I. *Eucinchoneae* with a valvate aestivation.

II. *Hillieae* with an imbricate or twisted aestivation.

To the latter tribe *Coptasapelta* belongs, which genus in Genera plant. was placed among the former, a mistake already corrected by KING and by SCHUMANN.

The latter places (Pflanzenfam. IV, 4 p. 42 and 48) *Coptasapelta* immediately beside *Crossopteryx*, an African genus, I could not examine, to which only one species or group of species belongs, living on the barren Campos of Abyssinia — till lower Guinea. On comparing the detailed description OLIVER gives of this genus, I found, that nearly all more or less important characteristics given by O. are also applicable to *Coptosapelta*; only two are lacking, viz. Stigma clavatum bilobum and tubuscorollae gracilis, limbus parvus. The important characteristic of the length of the stigma however is present. *Lindeniopsis* however has a stigma clavatum and a tubuscorollae gracilis, so that only the bilobular stigma forms an important difference. This points to a close relation between these two genera, especially between *Crossopteryx* and *Lindeniopsis*, on account of the shrubby, xerophilous habit.

The leaf-nervature of *Crossopteryx* is not fully described, but the leaves have the same shape; they are larger than with most *Coptosapelta*-species, but equal to those of *C. flavescens*. The close relationship of the two genera cannot be doubted. I could not find any striking points of similarity with other genera of the tribe of *Cinchoneae*, of which but a small number of species occur in the old world. The most characteristic peculiarity, the structure of the stigma does not occur in any other genus of this tribe.

Remarkable however is the resemblance of pistil and corolla in species of two genera, belonging to the *bacciferous Rubiaceae* with many ovules, viz. *Tarenna* GAERTN. (syn. *Stylocoryne*, syn. *Webera*), which has given rise to a peculiar synonymy.

The name *Stylocoryna*, given in 1797 by CAVANILLES to a species from the Liu-tchiu-Archipelago, is formed from the words στυλος: pillar and κορυνη: club, briefly denoting the structure of the pistil of *Coptosapelta*, as described above. HOOKER referred this species to the genus *Randia* LINN., so that the characteristic generic name was lost. In 1834 WIGHT brought it up again in the form of *Stylocoryne* (independent of *Cavanilles*?) for a plant from Ceylon new to him, viz. *St. corymbosa* WIGHT, which again showed this peculiar shape of pistil. Neither could this name be kept, as the same species had previously been diagnosed by GAERTNER (in 1788) as *Tarenna zeylanica*, wich latter name of course enjoys the preference. The first generic name however had been accepted by various authors (ROXBURGH, BLUME, a.o.) and BLUME was the first to

apply it to *Coptosapelta flavescens* KORTH, discovered by v. HASSELT and himself in Java. He called it *Stylocoryna tomentosa*, while likewise WALLICH, MIQUEL and later MERRILL and WERNHAM classed species of *Coptosapelta* either with *Stylocoryne* or with *Randia* (see above p. 2).

Whether the great similarity in floral structure between two genera, belonging to different principal divisions of the family, also points to a natural relation, is still an *open* question.

§ 12. *New description of the genus. Calyx* cup-shaped, quinquepartite, quinquelobate or quinquedentate, perennial, with axillar glands.

Corolla, contorted in the bud, trumpet-shaped, tube varying in length, outside velvety or covered with sulky hair, inside bare or provided with furry ridges descending from the filaments, between those thinvelvety, straight or inflated above the middle, throat not bearded, lobes linear-oblong, obtuse.

Stamina 5, inserted on the throat, filaments filiform, short, the front furry or bare, anthers thin, linear-lanceolate, tapering at the top, at the base twice-pointed, obtuse or arrow-shaped, near the base dorsifix, on the backside provided with two rows of hairs directed upwards (in Lindeniopsis bare).

Disc small, annular.

Ovary bilocular, style anceps, hairless, stigma entire, cylindrical or club-shaped, long, far overtopping the corolla (in one species square with hairy angles); placentas coherent to the septum, ovules numerous, ascendent, imbricate.

Capsule more or less globular or oblong, bilocular, at the top loculicide bivalvular, later on quadripartite.

Seeds small, peltate, imbricate; membranous, winged all round with fringy notched (in Lindeniopsis undulate) wing; endosperm fleshy, germ straight, root straight, directed downwards.

Lianes or *Shrubs* (*Lindeniopsis*). Twigs velvety or bare, round or more or less square. Leaves opposite, thin-leathery, elliptical, lanceolate or oval, usually tapering with a rather abrupt acumen; usually hairy on the underside. Leaf-nervature more or less acrodromous. Stipules small, interpetiolar, triangular.

Flowers small or middle-sized, white or light yellow, in axillary closed racemes or trichotomous, branched cymes, united at the twig tops to many-flowered panicles.

§ 13. *Conspectus of the Species.*

I. Subgenus *Lindeniopsis*. Shrub. Seeds with a slightly crenate and undulated wing. Calyx-lobes longer than the ovary. Corolla tube long. Anthers hairless.

1. *C. Hammii*, Val. 1909.

Leaves elliptical with short, acute, hard point; secondary veins 5—7 on each side, arcuately anastomosing (brochidodromous). *Corolla* hairless inside. Twigs sharply squared. *Stipulae* very small. Plant grey velvety all over, later on bare. *Fruit* oblong, length up to 30 mms.

Distribution. Hitherto endemic in Billiton on sandy barren soil.

II. Subgenus *Eu-Coptosapelta*. Lianes Seeds with fringed wing. Calyx-lobes not longer than the ovary. Corolla tube not more than twice as long as the lobes. Backs of the anthers covered with long hair.

2. *C. olaciformis* (Merrill), Elmer 1913. *Randia olaciformis*, Merr. 1908. *C. flavescens*, Merr. (non Korth.) 1909.

Inside of corolla tube and filaments glabrous. Corolla lobes slightly longer than the tube. Inflorescences corymbose united to panicles at the tops of the twigs; cymes short-peduncled and few flowered. Flowers very small. Stipules small, triangular, no hairy edges. Leaves elliptical or oval, shortly acuminate, smaller than 100 mm. number of secondary veins 4—5 on each side, hairless when full-grown, colour in sicco pale greenish grey or olive grey. Width of fruit at most 6 mm., broader than long, calyx consisting of free oval lobes.

Distribution. Hitherto endemic in the Philippines, in the following places: Mindanao, lake Lanao, camp. Keithly, Mrs. Clemens n. 1220, 1907 (type); Mindanao, prov. of Agusan, in mt. Urdaneta, 700 M. above sea-level Elmer n. 13355?; Luzon, San Antonio, prov. Laguna, mt. Ramos Bur. of Science, Manila, n. 396!

3. *C. flavescens*, Korth. 1851. *Stylocoryna tomentosa* Bl., Bijdr. 1826; *Stylocoryne ovata*, Miq. 1856; *Stylocoryne* (*Webera*) *macrophylla*, Wall non Roxb.; *Coptosapelta macrophylla*, K. Schum.

Inside of corolla tube and filaments glabrous. Inflorescences corymbose long-peduncled and dense flowered, united at the twig-tops to large thyrsus-shaped panicles. Leaves elliptical or oval or oblong, shortly acuminate, base as a rule broad, rounded, length 80—125 mm., number of secundary veins 4—5 on each side, colour in sicco usually olive-brown, undersurface of leaves, especially along the veins thinly covered with accumbent or crisp hair. Young twigs and inflorescences coated with dense, soft hair. Fruit obovate, sepals free, oval, erect. Stipules linear-lanceolate with broad base, hairy edges.

Distribution: Malay peninsula, Burma, Western Java, Sumatra: Palembang, (Pretorius[1]), 1837, in Herb. L B; Borneo S. E. Division, on sandy plains on the river Karrau (Korthals).

4. *C. hameliaeblasta* (Wernh.) Val. nova comb. *Tarenna hameliaeblasta*

[1]) This species being rather widely spread, differs rather in habit according to the place where it is found. For instance the specimens from the Malay peninsula (Kings collector 10384 and 10393) have stronger flowering-twigs and considerably greater leaves and flowers than the specimens from Java and Sumatra. The latter are again distinguished from the Javanese form by smaller, narrower leaves, in sicco coloured darker brown, covered with crisp hair on their undersides. Similar leaves also occur in a specimen from Malacca (Maingay, 908).

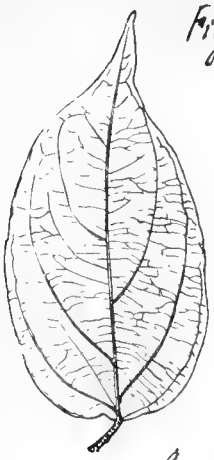


Fig. 1

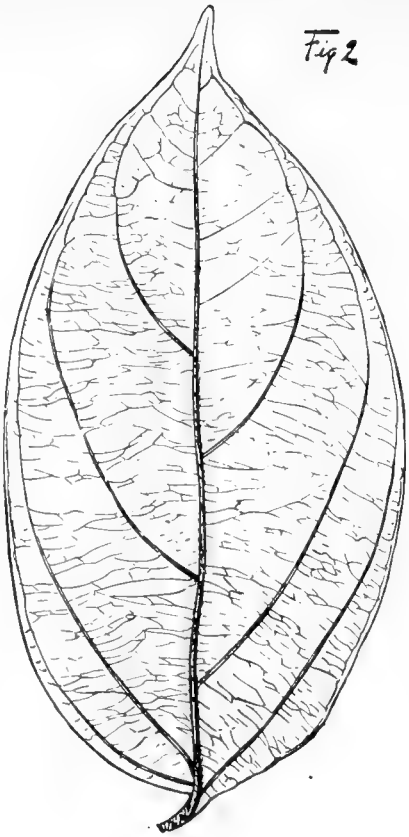
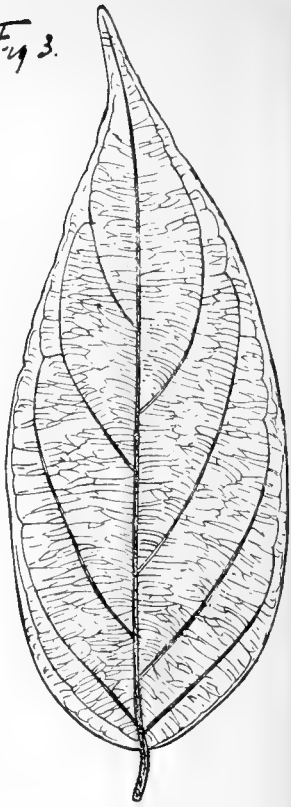


Fig 2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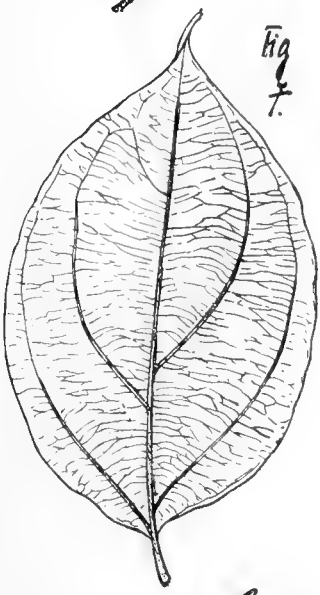
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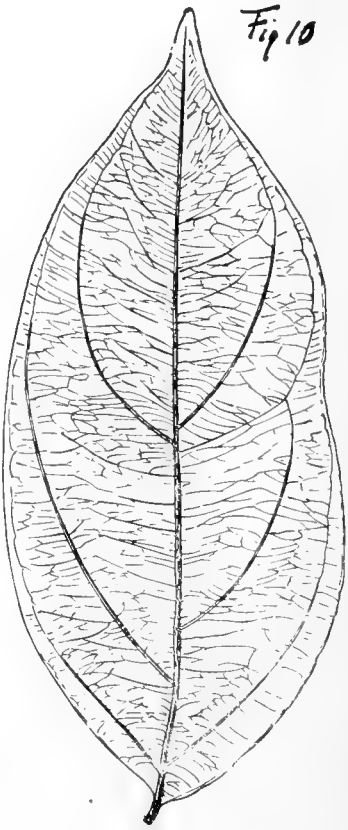


Fig 10

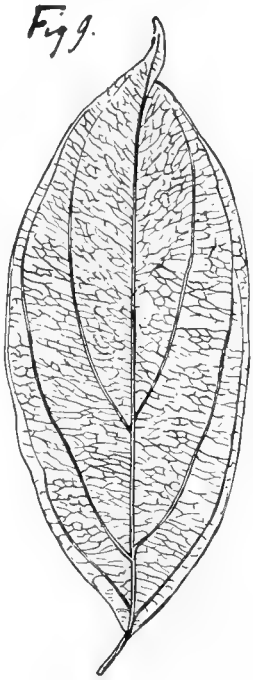
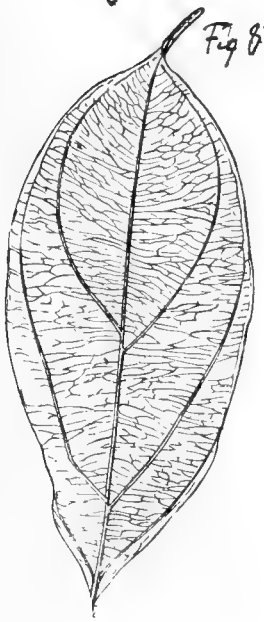


Fig 9.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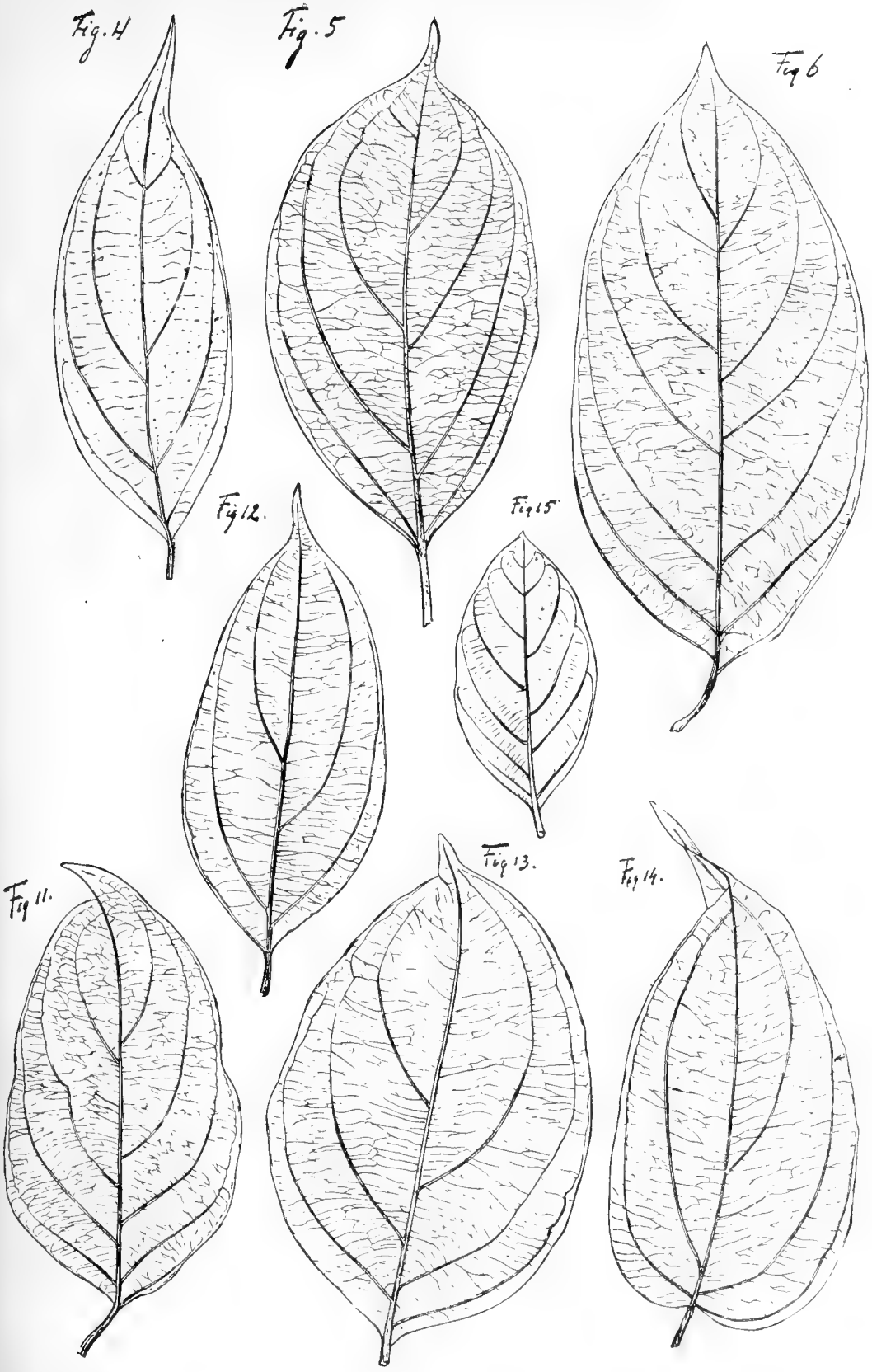


Fig. 4
Fig. 5
Fig 6
Fig 12.
Fig 15
Fig 11.
Fig 13.
Fig 14.

WERNH. Inside of the upper part of the corolla tube and the filaments densely hairy, the former not inflated. Corolla lobes about as long as the thin corolla tube. Axillary cymes longpeduncled and dense-flowered; terminal thyrsi many-flowered. Corolla tube (in sicco) covered with appressed whitish hairs. Calyx lobes about as long as the ovary, erect, curved outward. Leaves oblong or elliptical, with very short acumen. Secondary veins 3—4 on each side, sometimes with an additional thin basal vein; veins erect. Stipules very small, triangular, the edges covered with dense hair. Colour of the leaves in sicco yellow-olive-green. Stalks and inflorescences hirsute, leaf-veins at the backside with remote procumbent hairs.

Distribution: British New-Guinea, Sogeri-region, 950—1400 metres above sea level. (FORBES).

5. *C maluensis*, VAL. n. sp.

Upper part of the corolla tube not inflated, hairy as are the filaments. Corolla lobes a little shorter or of equal length as the corolla tube. Axillary inflorescences with long stalks; terminal thyrsi with abundance of flowers. Flowers the smallest of the genus. Outside of corolla covered with short appressed hair. Calyx-limb divided for half its length, lobes oval, erect. Leaves usually broad, elliptical with 3—4 rarely 2 secondary veins on each side (together 5—7), acrodromous. Fruit crowned by the very small calyx-lobes. Underside of leaves with a very thin hairy covering near the edge, for the rest bare. Stipules pointed, with thin indument.

Distribution: North-East New-Guinea, at 190—200 metres above sea level, in primeval wood. (LEDERMANN).

6. *C. Beccarii*, Val n. sp.

Upper part of corolla tube not inflated and at the inside covered with long and dense hairs, as are the filaments. Corolla grey velvety externally, lobes about as long as the corolla tube. Axillary inflorescences long-peduncled, thyrsus-shaped. Terminal thyrsi with abundance of flowers. Leaves broadly oblong, ending in a caudate acumen, large, with 3—4 secondary veins on each side. Petiole fairly long; underside of leaf covered with crisp soft hair.

Distribution: Borneo (BECCARI 2271).

7. *C. fuscescens* VAL n. sp. Upper part of the corolla tube not inflated, inside covered with dense hairs, as are the filaments. Corolla lobes twice as long as the tube. Axillary cymes long-stalked and repeatedly remotely branched; terminal thyrsi many-flowered, spreading. Outside of corolla tube covered with short silky hairs, lobes hairless. Calyx small, lobes detached nearly to the base. Leaves elliptical, glabrous Usually 3 secondary veins, or in a single specimen 2, on each side. Stipules very small, obtuse, triangular, hairy,

Distribution: Nord-East New-Guinea in mountain woods 600—1500 metres above sea level, in the Kani and Torricelli mountains (SCHLECHTER) on the Felsspitze at 1500 metres (LEDERMANN).

8. *C. lutescens*, VAL. n. sp.

Flowers as in *C. fuscescens*, but a little larger. Leaves with 2 secondary veins on each side, in sicco greenish-ochreous-yellow.

Distribution: North-East New-Guinea, on the Etappenberg at 850 m. in dense high wood (LEDERMANN).

9. *C. Griffithii*, HOOKER. f.

Upper part of corolla tube inflated, inside covered with long dense hairs, as are he filaments, lobes more than twice the length of the short wide tube. Axillary

cymes rather many flowered; terminal thyrsi densely. Outside of corolla grey velvety all over. Calyx-limb wide by cup-shaped, divided for half its length into broad lobes. Leaves elliptical, at the underside crisp hairs. Secondary veins 3—4 on each side.

Distribution. Gathered in numerous places in the Mal. peninsula, in the low lands.

10. *C. Janowskii*, VAL, n. sp.

Upper half of the corolla tube inflated, inside covered with long, dense hair, as are the filaments Corolla lobes half the length of the tube. Axillary flower-stalks with 1-3-5 flowers. Terminal inflorescences simple racemose. Flowers the largest in the genus. Outside of corolla-tube thin-velvety, lobes hairless. Calyx large, cup shaped, not incised, with short broad acute teeth. Leaves lanceolate, long-acuminate.

Distribution: Northern New-Guinea. Jabi mountains.

11. *C. montana*, KORTH. msc., in Herb. L. B.

Flowers unknown. Fruits in the leaf axils isolated or in peduncled cymes of 3-5-flowers, forming simple closed racemes at the twig-tops.

Calyx-lobes persistent on the fruit, only connected at the base, linear-subulate. Leaves lanceolate or elliptical, rather firm, with long tapering points and acute, obtuse or rounded base. Secondary veins 2-3 on each side Stipules small, triangular, having long hairs. Stems, inflorescences and under sides of leaf-nerves thin-velvety, in sicco ochreous yellow. Fruit obovate oblong.

Distribution. S.E. Borneo. Summit of the Sakoembang, 1000 metres above sea level.

EXPLANATION OF THE FIGURES.

Fig. 1 Coptosapelta montana; Leaf of an old plant.
Fig. 2 „ flavescens, flowering plant.
Fig. 3 „ „ very young plant.
Fig. 4 „ montana; young fruiting plant.
Fig. 5 „ hameliaeblasta.
Fig. 6 „ olaciformis.
Fig. 7 = Fig. 1.
Fig. 8 and Fig. 9 Coptosapelta fuscescens.
Fig. 10 „ flavescens, flowering plant
Fig. 11 = Fig. 6.
Fig. 12 „ lutescens.
Fig. 13 and Fig. 14 „ maluensis.
Fig. 15 „ Hammii.

The figures have been obtained by carbon-impressions according to the method of ELMER D. MERRILL. Fig. 4 is not retouched, only retraced with ink.

The others have all been worked up by the designer with the aid of the original print and of the leaf; the tertiary vein system is consequently a little too prominent!

Botany. — „*Dark growth-responses*". By D. TOLLENAAR. (Communicated by Prof. A. H. BLAAUW).

(Communicated at the meeting of April 28, 1923).

In our previous report on the light- and dark-adaptation of *Phycomyces nitens* (Proc. Vol. XXIV Nos. 1, 2 and 3, 1921) the existence of the so-called „dark-growth-response" was already proved in a great number of experiments. By dark-growth-response we understand the occurrence of a growth-response, when a sporangiophore of *Phycomyces nitens* adapted to light (by a four-sided illumination for hours at a stretch) is placed in the dark. It seemed worth while considering in how far this dark-growth-response of *Phycomyces-nitens* (the negative after-images of the human eye probably being in reality comparable) occurs in other organs.

In this communication the results are mentioned concerning the dark-growth-responses of the sporangiophore of *Phycomyces nitens*, the hypocotyledons of *Helianthus globosus*, the coleoptiles of *Avena sativa*, the roots of *Avena sativa* and the roots of *Sinapis alba*.

If possible the results have been compared with the light-growth-responses hitherto known.

Method and accuracy of the results.

In all experiments the preceding illumination was four-sided; the temperature being kept constant by means of the oil-thermostat, described in „Licht- und Wachstum I". In this way the temperature could be kept constant to 0.02° C. with moderate illuminations. It should be particularly kept in view, that the growth was as a rule only considered sufficiently constant, when it did not oscillate above 10%, i. o. w. with an average rate of growth of 100 no rates higher than 105 or lower than 95 occurred. This enables us to ascertain responses of growth more than 5% above or below the average; responses of growth therefore of an acceleration or retardation of 10% we can ascertain with some certainty.

We mention this in order to give the illustrations and reviews the value due to them, which could not be judged of without the full data — which we omit here with a view to space, but all of which will appear in the "Mededeelingen der Landbouw-hoogeschool" this year.

As long as on account of insufficient constancy of the outward circumstances or through inward causes, the growth already greatly oscillates before the change in light-conditions, it may be easily understood that a response of growth due to this one factor cannot be accurately ascertained.

As responses of growth of more than 50 % are but rare, they are not demonstrable when the growth shows such variations beforehand. With the data in literature however this repeatedly occurs. We repeat, *that for our reactions we only used organs, showing as a rule no oscillations of growth greater than* 10 %.

The figures subjoined all represent the response of individuals, approaching the average type as closely as possible. Only in the case of *Phycomyces nitens* a schematical figure of the process of reaction was given.

Just as in most of the previous curves published by BLAAUW such figures, in which the reaction-type of a definite experimental series is composed, are mainly based on the so-called *cardinal points*, to be found in the reactions of all individuals. These cardinal points are:

1. the average-point of time, at which the response of growth begins;

2. the average-time, at which the reaction reaches its first climax (either maximum or minimum of growth);

3. the average-rate of growth at that moment in percents of the original rate of growth; and next again the average time, at which eventually another maximum or minimum occurs and the average-rate of growth at that moment.

Dark- and light-growth-responses of Phycomyces nitens.

The light-growth-responses are known from the results of BLAAUW, published in "Licht u. Wachstum III" (Med. d. Landb. Hoogesch. 1918) p. 108. The cardinal points for some intensities follow:

TABLE I.

Light-intensity	First response after beginning of exposure	Maximum of response		Final rate of growth
		after beginning of exposure	in % of the rate of growth in dark	
1/8 MK.	8 Min.	9½ Min.	141 %	102 %
1 „	5½ „	9 „	148 %	103 %
8 „	5½ „	8½ „	152 %	111 %
64 „	4 „	8 „	174 %	112 %

At a temperature of about 17° C. some sporangiophores adapted to exposures to 1/512, 1/64, 8 and 64 M.K., were darkened, the growth-measuring being continued. The responses of growth, consisting in a retardation of growth were very characteristic.

The cardinal points, computed from sets of 5—6 experiments are given in the subjoined table.

TABLE II.

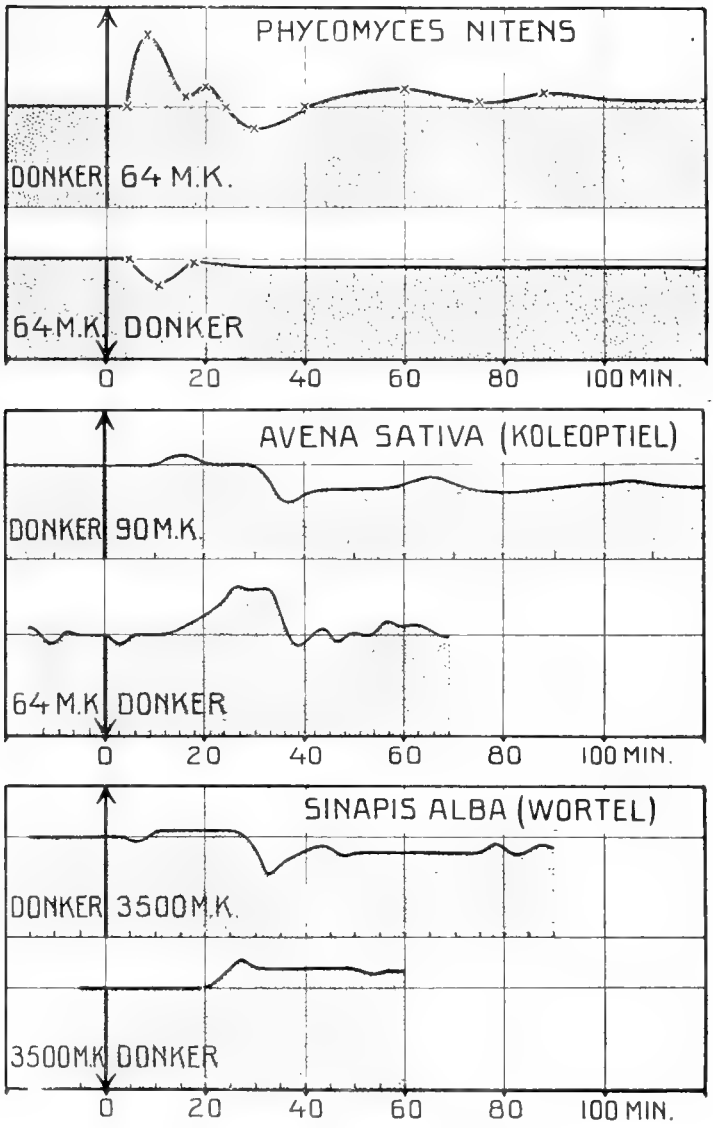
Adapted to	First response after beginning of darkening	Minimum of growth	
		after beginning of darkening	in % of the rate of growth in light
1/512 MK.	10½ Min.	12½ Min.	89 %
1/64 „	6½ „	12½ „	85 %
8 „	6 „	11 „	67 %
64 „	4½ „	10 „	73 %

The reaction at 64 MK was observed in a great number of observations (19). From the results obtained a maximum after about 17 min. could be derived with a rate of growth of about 98½% of the rate of growth in light; after that the oscillations get more and more indistinct and after 1¾—2 hours the equilibrium for the growth has externally been reached. The rate of growth appears to have become 93 % of the rate of growth in light, with a mean error of about 1 %.

From comparison of the above reports the contrary reactions, brought about by making light and dark, are clearly perceptible. (See the figure).

The dark- and light-growth-responses of hypocotyledons of Helianthus globosus.

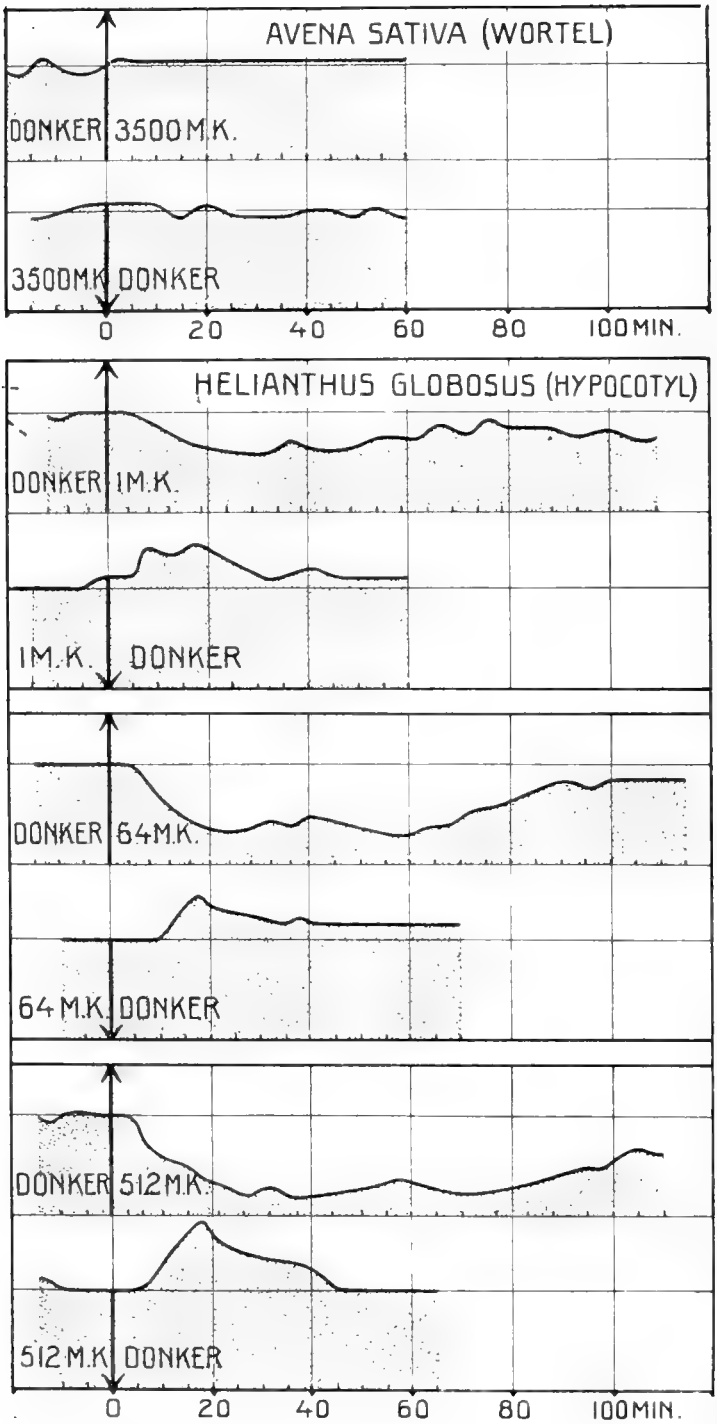
The light-growth-response of these organs is sufficiently known from "L.u.W. II". It consists in a retardation of growth, making its first influence felt, when exposed to 1 MK after 8 min.; the minimum of 74 % of the rate of growth in light appears after 27—38 minutes, after which the growth reverts to its previous rate, at least in a slight number of observations it is after 3 hours not perceptibly different from the rate before exposure.



EXPLANATION OF THE FIGURES.

These figures have been arranged in twos, in such a way, that above the process of growth has been represented when the organ after having been in dark, was permanently exposed to light (↑) below the organ made dark (↓) after having been exposed for hours. The height of the dotted space represents the rate of growth. In the case of *Phycomyces* the two growth-curves have been plotted according to the average progress of a number of individuals; the cardinal points are indicated by X For all the other organs the curves have been composed of the figures found for one of the individuals. The curve for the coleoptile of Avena sativa has been plotted after an individual reaction after KONINGSBERGER.

How these curves have been plotted will be further discussed and explained in the more detailed publication.


AVENA SATIVA (WORTEL)
DONKER 3500 M.K.
3500 M.K. DONKER
0 20 40 60 80 100 MIN.
HELIANTHUS GLOBOSUS (HYPOCOTYL)
DONKER 1M.K.
1M.K. DONKER
DONKER 64 M.K.
64 M.K. DONKER
DONKER 512 M.K.
512 M.K. DONKER
0 20 40 60 80 100 MIN.

At 64 MK the first reaction already appears after about 3½ min., the minimum amounting to 39 %, after 20—25 min., while after that the rate of growth gradually reverts to its initial value.

Finally at 512 MK the reaction-period is 3½ minutes, the minimum, now 21 %, appears after about half an hour and continues for a long time. Even some hours after the beginning of the exposure the rate of growth remains considerably below the rate in dark. Compare the figures subjoined.

What reaction takes place, when we darken after these hypocotyledons have been mainly adapted to a constant illumination for 5—7 *hours*?

The results of these experiments, made at about 20° C. have been briefly summarized in the subjoined table.

TABLE III.

Adapted to	Beginning of response after darkening	Maximum of response		Second maximum after
		After beginning of darkening	in % of the rate of growth in light	
1 MK.	7½ Min.	18½ Min.	128 %	40 Min.
64 „	8½ „	19 „	137 %	—
512 „	8½ „	18 „	157 %	—

After about an hour and a half the growth had become settled again. As to the rate, putting together all data of 1, 64 and 512 MK and comparing the rate of growth in light to the rate 1—2 hours after darkening, we find of the 14 results: a retardation of growth in 7, an acceleration in 6 and an unaltered rate in one, while an average acceleration of growth of 5 ± 21 % may be computed. Therefore the chances for the existence of a lasting acceleration of growth may be called slight.

On comparing the light- and dark-responses to each other (see figure!), we are again struck by the reverse process, though there is no perfect symmetry. In both cases the reaction is more marked for higher intensities (lower minima, resp. higher maxima).

Upon the whole the dark-response is not so strong as the light-response. The reaction-period is longer, the change in growth less intense, the external equilibrium of growth sooner restored.

The light- and dark-growth-response of coleoptiles of Avena sativa.

By means of the experiments of VOGT, SIERP and KONINGSBERGER a light-growth-response has been ascertained. On application of 90 MK on 3 sides KONINGSBERGER finds a minimum of about 55% 35—40 minutes after the beginning of the exposure — next a maximum after about 65—70 minutes (about 80% of the rate of growth in dark) — while after 90 minutes a second minimum occurs amounting to about 65—70% of the rate of growth in dark. The latter however continues oscillating irregularly for hours together. In the figure the curve of the light-growth-response is taken from an individual of table 9 from KONINGSBERGER's dissertation.

What reaction occurs, if we darken after the rate of growth has been in the main adapted to light for some hours?

With *Helianthus* and *Phycomyces* darkening appeared to cause less intense changes of growth, than "Light". If this should be the case with the coleoptiles of *Avena sativa*, there would be danger of this reaction finding no expression at all or but indistinctly, on account of the irregular growth of *Avena*, in consequence of occurring nutations.

We have therefore tried to eliminate or restrict these impeding movements. Not only were a great number of Oat-races observed in this respect, but also conditions of more or less moist and hot cultivation were tried. In this way we have succeeded in finding an Oat-race called "Zwarte President" which when cultivated in a very dry soil but very rarely nutates inconveniently. As long as the coleoptiles secrete little or no drops of moisture, the growth was extraordinarily constant and frequently remained within the limits fixed by us: no more than 10% variation of growth. The temperature at which the plants grew was about 22° C. In order to give a good idea of the results, obtained for this object, we decided to give the whole of its individual responses of growth in this communication. Our preceding illumination was 4-sided with 64 MK., which intensity deviates but little from that used by the above-mentioned investigators. The rate of growth has been given in microns per minute. [1])

N°. 1. *Exposed beforehand for* 4½ *hours at* 21°,9 *C. to* 64 *MK:*

[1]) The small figures denote the time of observation, by which the beginning of darkening is again put at the full hour (60).

45 12½ 50 12½ 55 12½ DARK! 0 12½ 5 12½ 10 12½ 15 12½ 20 13 25 **16** 30 13 35 12 40 10½ 45 9 50 10 55 10½ 1 hour 10½ 1.05 10½ 1 10

N°. 2. Exposed beforehand for 4 hours at 21°,9 C. to 64 MK.
51 20 54 20 57 20 DARK! 0 19 3 19 6 19 9 20 12 19 15 18 18 18 21 **22** 24 21 27 18 30 17 33 17 38 16 41 17 44 17 47 18 50 19 53 18 56 18 1 hour 18 1.03 18 1.06.

N°. 3. Exposed beforehand for 4 hours at 22°,2 C. to 64 MK.:
51 21 54 21 57 21 DARK! 0 22 6 21 9 22 12 24 15 28 18 **29** 21 27 24 26 27 **24** 30 24 33 25 36 24 39 22 42 23 45 24 48 25 51 27 54 27 57 28 1 hour 26 1.03 25 1.06 25 1.09 **25** 1.12.

N°. 4. Exposed beforehand for 8 hours at 21°,9 C. to 64 MK.:
50 9 55 8 DARK! 0 9 5 9 10 11 15 11 20 **12** 25 11 30 10½ 35 10 40 9½ 45 10 50 10 55 9½ 1 hour 9½ 1.05.

N°. 5. Exposed beforehand for 6 hours at 22°,0 C. to 64 MK.:
40 9 45 9 50 9 55 9 DARK! 0 8½ 5 9 10 11 15 12½ 20 **14** 25 12 30 **11** 35 10 40 10 45 10 50 9½ 55 9 1 hour 9 1.05 9 1.10 9 1.15 9 1.20.

N°. 6. Exposed beforehand for 4 hours at 21°,1 C. to 64 MK.:
45 26 48 24 51 25 54 25 57 25 DARK! 0 24 3 24 6 25 9 25 12 25 15 28 19 30 21 33 24 **36** 27 **36** 30 **36** 33 29 36 24 39 24 42 26 45 24 48 25 51 25 54 27 57 27 1 hour 26 1.03 27 1.10.

The occurring dark-growth-responses in the above cases yield the following averages for the cardinal points:

First response after beginning of darkening	Maximum of response		A minimum in the rate of growth (except in No. 5) after beginning of darkening
	After beginning of darkening	in % of the rate of growth in light	
16 Min.	± 23½ Min.	133 %	± 42 Min.

In some cases there is apparently a slight secondary maximum after 50—60 Min. (Nos. 2, 3, 4 and 6). Little may be concluded from these experiments with respect to the final rate of growth. It does not seem to deviate much from the rate in light.

The above shows a distinct response of growth, again contrary to the light-growth-response. Again it is less intense than the light-growth-response; the former gives a slighter change of growth: the undulatory movement is less vehement (undulation of shorter duration with slighter amplitude).

In the averages KONINGSBERGER's tables (4) of the light- and dark-growth-response a maximum occurring after darkening may indeed be found on pages 51, 52 and 53. It occurs after about 20—30 min., (circa 25 minutes), but also in connection with further experiments KONINGSBERGER does not consider these reactions as dark-growth-responses.

In the cases, in which VOGT observed the dark-growth-response, it lies averagely after 21—24 min., (averagely 22½ min.), which is in accordance with our results. SIERP finds his maximum averagely after 30½—35½ min. (averagely 33 min.). But we should bear in mind, that this investigator did not change the exposure to 320 M.K. to dark, but to a slighter illumination with 17.7 M.K. (pag. 699 and (following).

Accordingly in our experiments both after a previous exposure of 6 and 8 hours, and of 4 and 4½ hours, we found a dark-growth-response with the coleoptiles of *Avena*, contrary to the light-response of this organ.

The Light- and Dark-growth-responses of the root of Sinapis alba.

This organ being much less sensitive to light, I deemed it desirable to apply stronger illuminations, viz. of 3500 M.K. In spite of the insertion of a cooler with running water into the circuit, a gradual rise of temperature from 0°.5—1°.0 C. in the course of an hour could not be prevented. On darkening, a fall of temperature could be prevented by again putting the heating into operation. Then oscillations above 0°.05—0°.1 C. did not occur.

The roots were subjected to 4-sided illumination at 21°.5—22°.8 C. for 3—5 hours. First the light-growth-response was determined, yielding the following averages:

First response after	Minimum of growth		Rate of growth after some hours in light in % of the rate of growth in dark
	after making light	in % of the rate of growth in dark	
37½ Min.	39½ Min.	79 %	88 %

We observe a distinct response of growth. The retardation of growth is permanent in all cases also after the new external equilibrium of growth has been attained.

A subsequent darkening caused the following reaction:

First response after	Maximum of response		Rate of growth in dark in % of the rate of growth in light
	after darkening	in % of the rate of growth in light	
30½ Min.	35½ Min.	124 %	113 %

Here too the contrast between light- and dark-response is found. Both are fairly equally marked.

Meanwhile I have observed the dark-growth-response with an illuminating-power of 512 M.K. I found as an average of 7 experiments:

First response after	Maximum of response		Rate of growth in dark in % of the rate of growth in light
	after darkening	in % of the rate of growth in light	
27 Min.	36½ Min.	111 %	105 %

Here we already approach the limit of the reactions still perceptible, which also appeared from the fact, that a few plants *no* more gave a perceptible dark-growth-response. On subjecting these plants to an illumination of 512 M.K., there did not occur a light-growth-response either.

The sensitiveness to light, found by Blaauw ("L. u. W. III") for Sinapis alba was greater. At the time there was even found a marked response at 64 M.K. with a minimum of 81 % and a rate of growth after 2 hours of 91 %, an equally strong response, as the one found by us for 3500 M.K. To what causes this may be owing (older seed? other Sinapis alba race?) should be further investigated into and may become an indication for the deeper cause for sensitiveness to light.

The behaviour of the tap root of Avena Sativa, with respect to light and dark.

Blaauw did not find a perceptible response with illuminating-powers of 64—500 M.K. I exposed to 3500 M.K. Even then no reaction occurred, or so slight a reaction, that it might as well be attributed to the slight rise of temperature.

After a 3 hours exposure at a constant temperature of 20½°—22½° C. followed darkening. In not a single case there was a marked

response, i. e. the oscillations of growth remained of the size also occurring in constant circumstances (smaller than 10 %). With a reservation as to the existence of such an exceedingly slight reaction, we may observe, that the lack of a light-growth-response goes together with the lack of a dark-growth-response.

SUMMARY.

1. *With the organs observed the occurrence of a light-growth-response went together with a dark-growth-response, in the main contrary to the former.*

2. *The lack of a light-growth-response (Avena-root) goes together with the lack of a dark-growth-response. This going together seems to hold good also individually (Sinapis-root 500 MK.).*

3. *With Phycomyces, Avena coleoptile and Helianthus-hypocotyledon the dark-growth-response is less intensive than the light-growth-response: the waves have smaller amplitude and are of shorter duration so that externally a constant rate of growth is sooner attained.*

It remains to be investigated into, whether the inward equilibrium is likewise sooner restored than the externally observable light-growth-response. Equilibrium in the inward processes indeed does not coincide with the appearance of a constant rate of growth to be judged by the observer (7).

With regard to the word "dark-growth-response", used for convenience, sake, it should be borne in mind, that *dark as such does not cause the response: dark itself is not a stimulus, but the modification in energy-supply, either when suddenly occurring (light-growth-response), or suddenly ceasing (dark-growth-response), respect. increasing or decreasing.*

It may be easily understood, that stoppage of energy-supply causes a slighter and shorter reaction in an organ, i.e. it sooner settles down than when energy is supplied.

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Laboratory for Plant-physiological Research.

Wageningen, April 1923.

Mathematics. — "*Representation of a Tetrahedral Complex on the Points of Space.*" By Prof. JAN DE VRIES.

(Communicated at the meeting of April 28, 1923).

1. Let there be given a pencil of quadratic surfaces which has a twisted curve ϱ^4 as base curve. The polar planes of a point P with respect to these surfaces pass through a straight line p, which we shall call the *polar line* of P. Through P there pass two bisecants of ϱ^4; the straight line p joins the points of these bisecants which are harmonically separated from P by ϱ^4. If P lies in the vertex of one of the four cones belonging to the pencil, the polar line becomes indefinite; any straight line of the plane $\omega_k \equiv O_l\, O_m\, O_n$ may be considered in this case as a polar line.

The complex of rays T of the polar lines p is represented on the space of points $\{P\}$. The side $O_k\, O_l$ is represented in any of the points of the opposite side $O_m\, O_n$. If a straight line r is to belong to T, its polar lines r' and r'' with respect to the surfaces α^2 and β^2 of the pencil, must cut each other. If the straight line r describes a plane pencil, r' and r'' describe two projective plane pencils; the plane pencil (r) contains accordingly two rays for which r' and r'' cut each other. The complex T is therefore *quadratic* [1]) and has four cardinal points O_k and four cardinal planes ω_k; hence it is *tetrahedral*.

A point P of ϱ^4 is the image of the straight line p which touches ϱ^4 at P. The scroll of the tangents of ϱ^4 is therefore represented in the points of ϱ^4.

2. If P describes a straight line r, the polar planes of P with respect to α^2 and β^2 describe two projective pencils round the polar lines r' and r''. The polar line p describes accordingly a *quadratic scroll* $(p)^2$; the conjugated scroll consists of the polar lines of r with respect to the quadratic surfaces through ϱ^4. The points of intersection of r with the cardinal planes ω_k are the images of four

[1]) If the pencil is defined by $\sum_4 a_k x_k^2 = 0$ and $\sum_4 b_k x_k^2 = 0$, the polar planes of the point y have $a_k y_k$ and $b_k y_k$ for coordinates. The coordinates of p are in this case $p_{12} = (a_3 b_4 - a_4 b_3)\, y_3\, y_4$ etc. If we put $a_1 a_3 b_2 b_4 + a_2 a_4 b_1 b_3 = c_{13,24}$, T is represented by $c_{12,34}\, p_{12}\, p_{34} + c_{23,14}\, p_{23}\, p_{14} + c_{31,24}\, p_{31}\, p_{24} = 0$.

rays p, which pass through the cardinal points O_k. T contains evidently ∞^4 scrolls $(p)^3$.

If r is a ray of T, r' and r'' cut each other, so that the projective pencils of polar planes produce a quadratic cone which has the point $r'r''$ as vertex. From this follows that the *complex cones* of T are represented by the *point ranges* (P) lying on *complex rays*.

3. The rays of T which lie in a plane φ (and which accordingly envelop the *complex conic* φ^2), are represented by the points P of a twisted curve which passes through the cardinal points O_k. For the intersection of the planes φ and ω_k is a tangent of φ^2 and is represented in O_k. As ω_k can only contain the images points O_l, O_m, O_n, the image of the system of the tangents of φ^2 is a *twisted cubic* φ^3 circumscribed to the tetrahedron $O_1 O_2 O_3 O_4$.

4. The complex T cuts a *linear complex* A in a *congruence* (2,2) which has singular points in O_k, singular planes in ω_k. For O_k is the vertex of a plane pencil belonging to both complexes, hence to (2,2). The polar lines p' and p'' of the rays of this plane pencil with respect to α^3 and β^3 form two projective plane pencils in ω_k and these produce a conic circumscribed to $O_l O_m O_n$. The *image* of the *congruence* (2,2) is therefore a *quadratic surface* Ω^2 circumscribed to $O_1 O_2 O_3 O_4$.

As A does not generally contain any of the sides $O_k O_l$, Ω^2 will not generally contain any of these sides either.[1])

The ∞^5 surfages Ω^2 are the images of ∞^5 congruences (2,2) contained in T. To these belong ∞^4 *axial* (2,2) defined by the ∞^4 *axial linear complexes*.

5. The rays of T belonging to two complexes A_1 and A_2, form a scroll $(p)^4$ of the fourth order; this scroll belongs of course at the same time to all complexes A of the pencil defined by A_1 and A_2, hence also to both axial complexes of this pencil. Their axes are director lines of $(p)^4$ and moreover double director lines, for the complex cone of a point lying on one of these axes, is cut twice by the other axis.

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[1]) If Λ is defined by $\sum_6 d_{kl} p_{kl} = 0$, Ω^2 has for equation

$$\sum_6 d_{kl} c_{mn} y_m y_n = 0.$$

Inversely the surface $\sum_6 f_{kl} y_k y_l = 0$ is the image of the (2,2), which is defined by the complex $\sum_6 \frac{f_{kl}}{c_{kl}} p_{mn} = 0$.

The *scroll* $(p)^4$ is represented by the *twisted curve* σ^4 which is the intersection of the two surfaces Ω^2 that are the images of the congruences defined by A_1 and A_2.

If the axes r_1 and r_2 of two axial complexes cut each other, the congruence (2,2) which these complexes have in common with T, degenerates into the system of the complex rays p through the point $R \equiv r_1 r_2$ and the complex rays in the plane $\varrho \equiv r_1 r_2$. In connection with this the image surfaces Ω^2 defined by $r_1 r_2$ cut each other in the twisted curve ϱ^3 representing the complex rays in ϱ, and in the polar line r of R (the image of the complex cone of R); evidently r is one of the bisecants of ϱ^3.

If ϱ^3 is an arbitrary *twisted cubic* circumscribed to $O_1 O_2 O_3 O_4$, there pass ∞^2 surfaces Ω^2 through ϱ^3 of which any two have also in common a bisecant of ϱ^3; evidently they represent two axial complexes of which the axes cut each other, so that the corresponding (2,2) splits again up into a *complex cone* and a *complex conic*; the latter is represented by ϱ^3.

6. A *conic* $(P)^2$ has four points in common with the surface Ω^2 belonging to an axial complex A; it is accordingly the image of a *rational scroll* $(p)^4$. Any ray s of T lying in the plane of $(P)^2$, contains two points of $(P)^2$; the image S of s carries therefore two rays of $(p)^4$. Hence the curve $(S)^3$, representing the rays s, is the *double curve* of $(p)^4$.

If $(P)^2$ passes through O_1, it is the image of a cubic scroll $(p)^3$ of which the double director line passes through O_1; for the points of intersection of $(P)^2$ with ω_1 are the images of two rays p through O_1.

If $(P)^2$ passes through O_1 and through O_2, it is the image of a quadratic scroll $(p)^2$. Inversely a scroll $(p)^2$ has two rays in common with an axial complex; its image cuts accordingly the corresponding surface Ω^2 outside O_k in two points. Hence this image is either a *straight line* (§ 2) or a *conic* through two cardinal points O.

7. The points P of a plane φ represent the rays of a congruence $[p]$. The polar planes α and β of P with respect to two quadratic surfaces α^2 and β^2 of the given pencil form two projective sheaves of planes round the poles of φ. Their intersections with a plane ψ form two projective fields of rays, hence ψ contains three rays $p \equiv \alpha\beta$.

The planes α through a point Q form a pencil; one plane of the corresponding pencil (β) passes through Q, hence Q carries one ray p.

The *field of points* $[P]$ is therefore the image of a *congruence* (1,3). This consists of the chords of a *twisted cubic* σ^3 which passes through the points O; for the range of points (P) in ω_k is the image of the generatrices p of a quadratic cone which has O_k for vertex.

8. If the *twisted cubic* $(P)^3$ passes through three *cardinal points*, it is the image of a *cubic scroll* $(p)^3$. For an arbitrary surface Φ^2 representing an axial complex cuts $(P)^3$ in three more points; on the axis of this complex there rest therefore three lines of the scroll. One pencil (Φ^2) can be passed through $(P)^3$; for through any four points of $(P)^3$ ∞^1 Φ^2 can be passed, each of which contains seven points of $(P)^3$. The corresponding complexes A form also a pencil; the axes of both axial complexes belonging to this pencil, cut all rays of the scroll and are therefore the *director lines* of the *cubic scroll* $(p)^3$.

If $(P)^3$ passes through *two cardinal points*, it is the image of a *scroll* of the *fourth order*. In this case one Φ^2 passes through $(P)^3$: the scroll belongs to the congruence (2,2) which the corresponding complex A has in common with T; as it is *rational*, it has a *double cubic*.

9. A *surface* $[P]^n$ is the image of a *congruence* with *sheaf degree* n, for its intersections with a ray t of T are the images of n rays through the vertex of the complex cone represented by t. The *field degree* of the congruence is generally $3n$ for each point of intersection of $[P]^n$ with the cubic φ^3 representing the rays t lying in a plane φ, is the image of a ray of the congruence in φ. If $[P]^n$ passes s_k times through O_k, the field degree is evidently $3n - \sum^4 s_k$.

A *twisted curve* $(P)^n$ is the image of a *scroll* of the order $2n$, for the image surface $[P]^3$ of an axial complex cuts $(P)^n$ in $2n$ points, which are the images of as many rays t cutting the axis of the complex.

10. If the base of a pencil of quadratic surfaces consists of a cubic ϱ^3 and one of its chords, the polar lines of the points of space form a quadratic complex which is represented in the same way as the tetrahedral complex.

We can always represent this pencil by

$$\alpha\,(x_2^{\,2} - x_1\,x_3) + \beta\,(x_3^{\,2} - x_2\,x_4) = 0.$$

The polar planes of the point y relative to the cones $\alpha = 0$ and

$\beta = 0$ have for coordinates $y_3, -2y_2, y_1, 0$ and $0, y_4, -2y_3, y_2$. The polar line of y is therefore represented by

$$\frac{\pi_{12}}{y_3 y_4} = \frac{\pi_{23}}{4y_2 y_3 - y_1 y_4} = \frac{\pi_{31}}{2y_3^2} = \frac{\pi_{14}}{y_2 y_3} = \frac{\pi_{24}}{-2y_2^2} = \frac{\pi_{34}}{y_1 y_2}.$$

Hence

$$4p_{23}^2 = p_{12} p_{34}.$$

This complex has O_1 and O_4 as cardinal points, ω_2 and ω_3 as cardinal planes.

The complex cone of x touches $O_1 O_3$ at O_1, $O_4 O_2$ at O_4. The polar line of y lies in the plane ξ if the equation

$$\xi_1 (2y_2^2 x_2 - y_1 y_2 x_3) + \xi_2 y_2 y_3 x_2 + \xi_3 y_2 y_3 x_3 + \xi_4 (2y_3^2 x_3 - y_3 y_4 x_2) = 0$$

is satisfied by all values of x_2 and x_3. From this follows that the complex rays in ξ are represented by the points of the cubic which is defined by the cones

$$2\xi_1 y_2^2 + \xi_2 y_2 y_3 = \xi_4 y_3 y_4 \quad , \quad 2\xi_4 y_3^2 + \xi_3 y_2 y_3 = \xi_1 y_1 y_2.$$

(The chord $O_1 O_4$ does not belong to the image).

The congruence (2,2) which the complex has in common with the axial complex with directrix $a_x = 0$, $b_x = 0$, has for image the quadratic surface the equation of which is

$$(a_1 b_2) y_1 y_2 + [4(a_1 b_4) + (a_2 b_3)] y_2 y_3 - (a_1 b_4) y_1 y_4 + (a_3 b_4) y_3 y_4 + \\ + 2(a_1 b_3) y_2^2 + 2(a_2 b_4) y_3^2 = 0,$$

where $(a_k b_l) = a_k b_l - a_l b_k$.

Chemistry. — "*The Electromotive Behaviour of Magnesium*". II [1]). By Prof. A. SMITS. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of March 24, 1923).

Introduction. The fact that the rest potentials of magnesium and aluminium in aqueous solutions of their salts are too small negative has been the subject of frequent comment.

An apparently succesful explanation was that which assumed the presence of a film of oxide on the metal. This was however due to a not sufficiently careful examination of the consequences of such a premise.

This is especially true in the case of aluminium where it had been supposed, that the etched or even the polished metal was coated with a not porous film of oxide of molecular thickness.

Now a number of different investigations have proved with certainty that if an etched or polished aluminium electrode is immersed in mercury above which there is an aqueous solution of the aluminium salt, the aluminium immediately shows the potential of the mercury layer, whilst there was no indication of the penetration of a film of oxide [2]).

It follows from these investigations that either a film of oxide does not hinder the passage of the electrons or there is no film at all.

If the electrons only were going through an oxide layer we should expect the behaveour of a gas electrode. This is not in accordance with the fact. Consequently if the oxide film existed it would be penetrable for ions, but it is then manifest that we are dealing with a metal-electrode.

Now it is possible that under certain circumstances the liquid in the liquid bounding layer is saturated with respect to the hydroxide of the metal. This could easily be proved by the fact, that in the formula

$$E = -\frac{0.058}{r}\log.\frac{L_M}{(M^{\cdot\cdot}_L)} - 2.8. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

[1]) The considerations applied here are explicated in the book "Theory of Allotropy". Longmans, Green and Co. 1922.

The first Communication appeared These Proc. Vol. XXII, 876 (1920).

[2]) See Zeitschr. f. Electr. Chem. **27**, 523 (1921) and **28** (1922).

(M_L^{ν}) can be substituted by $\frac{L_M(OH)_\nu}{(OH_L')^\nu}$ so that

$$E = -\frac{0.058}{r}\log.\frac{L_M(OH')^\nu}{L_M(OH)_\nu} - 2.8 \quad . \quad . \quad . \quad (2)$$

or

$$E = -\frac{0.058}{r}\log.\frac{L_M}{L_M(OH)_\nu}\cdot\frac{K_W^\nu}{(H_L^{\cdot})^\nu} - 2.8 \quad . \quad . \quad . \quad (3)$$

From which it appears, that the electrode will behave as an oxygen or hydrogen one, but that the electromotive forces will show a constant difference.

These considerations however are no help to us, for expression (1) which always holds good, requires the potential of the metal to be very negative, because the concentration of the metal ions in a saturated solution of $Mg(OH)_2$ or $Al(OH)_3$ is very small. The exact converse is observed.

Ten years ago KISTIAKOWSKY [1]) calculated the normal potentials neglecting the temperature coefficient in the formula of GIBBS-HELMHOLTZ and found with Mn, Fe, Co, Cu and Cd differences between the calculated and experimentally found normal potentials of 10—60 m.V.; with Ni, Sn, Pb and Hg differences of 140—190 m.V.; with Ag he found diverences of 310 m.V., and with Tl of 360 m.V., whilst the difference with Al was 460 m.V. and with Mg 900 m.V.

As KISTIAKOWSKY found the electromotive force which the calculated for Mg and Al so much higher than that found experimentally, he simply assumed that at the two electrodes in the galvanic cell metal-electrolyte-hydrogen, the reactions

$$M_S \rightarrow M_L^{\cdot\cdot} + 2\theta_L$$

and

$$2\theta_L + 2H_L^{\cdot} \rightarrow H_{2G}$$

do not take place as in other cases, but the following:

$$M_S + 2\,OH_L' \rightarrow M(OH)_{2S} + 2\,\theta$$

and

$$2\,\theta + 2\,H_2O_L \rightarrow 2\,OH_L' + H_{2G}.$$

It should be noticed that the remarkable assumption is made there,

[1]) Z. f. phys. Chem. **70**, 206 (1910).

that a reaction which takes place at the hydrogen electrode is reversed when Mg is replaced by zinc.

KISTIAKOWSKY, however, rightly comes to the following conclusion: "Hieraus fogt unmittelbar, dass die Mg bzw. Al. Electroden die Eigenschaften von Gaselektrode besitzen mussen, d.h. ihr E_h von der Metallionen-konzentration unabhängig, dafür aber von der $H^{\cdot}$ und OH'-Konzentration abhängig sein muss; ausserdem muss es, wie bei Pt, von den reduzierenden Eigenschaften des Elektrolyten abhängen."

In this KISTAKOWSKY, however, quite overlooked that the behaviour of an hydrogen electrode will also be found with any other metal, if the boundary liquid consists of a saturated solution of the metal hydroxide.

KISTIAKOWSKY, instead of measuring the Mg and H_2 potentials in the same solution by changing the Mg concentration, dipped his Mg electrode, besides in a solution of $MgSO_4$ and in a solution of $MgCl_2$, in different other solutions, not containing Mg, and then obtained results, of course, from which no conclusions at all can be drawn. In his opinion, however, his results proved that the Mg-potential is independent of the $Mg^{\cdot\cdot}$-concentration.

BECK [1]) was the first to demonstrate in his Thesis for the Doctorate the invalidity of KISTIAKOWSKY's views; he has also shown experimentally that Mg *never* behaves as a hydrogen electrode. All the same electromotive behaviour of Mg in $MgSO_4$-solutions of *slight* ($H^{\cdot}_L$) was not yet cleared up, for it appeared to him that the difference in potential between the Mg and H electrodes in these solutions of small $H^{\cdot}$-concentration increases with the Mg-concentration. [2])

BECK found that the Mg electrode does not behave as a hydrogen electrode, but the Mg does not behave as a normal metal electrode either, for it was found that the Mg-electrode becomes more negative when the $MgSO_4$-concentration increases. It further appeared that on increase of the $H^{\cdot}$-concentration the Mg-potential becomes more negative, and that it reaches a maximum negative value for every $MgSO_4$-concentration at a definite $H^{\cdot}$-concentration. This maximum negative value varied with the $MgSO_4$-concentration, at least qualitatively, in a normal way.

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[1]) Rec. trav. chim. **41**, 353 (1922).

[2]) All the measurements were carried out by BECK in an atmosphere of very pure hydrogen, with vigorous stirring of the liquid, the Mg-electrode being at rest. If was found, that this way of stirring is much better than stirring by the electrode it self.

The maximum negative potentials are however no equilibrium potentials, that follows already from this, that the potential of Mg activated by amalgation in a solution of 1 gr. mol of $MgSO_4$ per litre, is more negative, i.e. — 1.856 Volt. instead of — 1.790 Volt, which value will also lie still below the real normal potential of equilibrium of Mg, as will be shown below.

Magnesium.

After this introduction we shall examine the metals Mg more closely.

The difficulties which are usually encountered in the study of the electromotive behaviour of magnesium and aluminium are owing to the fact that extraordinary phenomena appear when the usual methods of determining the equilibrium potential are applied to these strongly basic metals.

For example, suppose that the Mg potential is — 1.86 Volt. Since the Mg electrode develops hydrogen, this means that the above potential corresponds to the potential of the three phase equilibrium, magnesium (inner equilibrium) — hydrogen (by inner equilibrium) and the surrounded liquid layer.

The liberation of gaseous hydrogen takes place because hydrogen ions from outside diffuse into the surrounding liquid layer and combine with the electrons.

The above assumption holds for Fe and Zn because it can be shown by calculation that the surrounding liquid layer can coexist with metal and hydrogen, the two latter in inner equilibrium.

If however we take now strongly basic metals, we can see that the quotient $\frac{(M^{\cdot\cdot})}{(2H^{\cdot})}$ would be so large, that the electrolyte would become inconsistent.

The question now arises: "Can the above negative potential (—1.86 Volt) be the potential of magnesium and unary hydrogen (that is to say hydrogen in inner equilibrium) with respect to the surrounding liquid layer containing say 1 gr. ion Mg per litre."

Applying the formula $E = -\frac{0.058}{2} \log. L_{Mg} - 2 . 8$ [1]) and substituting for E the value — 1.86 we can calculate that

$$L_{Mg} = (Mg_L)(\theta_L)^2 = 10^{2 \times -16}.$$

If we consider that for hydrogen in inner equilibrium

[1]) The Theory of Allotropy p. 123.

$$L_{H_2} = (H_L^{\cdot})^2 (\theta_L)^2 = 10^{2\times -48}$$

it will be seen, that for the surrounding liquid layer which is in electromotive equilibrium with magnesium and hydrogen the following formula holds good:

$$\frac{(Mg_L^{\cdot\cdot})}{(H_L^{\cdot})^2} = \frac{L_{Mg}}{L_H} = 10^{2\times 32}.$$

It is evident that this ratio is not realizable.

If we chose $(Mg_L^{\cdot\cdot}) = 1$ then $(H_L^{\cdot})^2 = 10^{2\times 32}$ and since $(H_L^{\cdot})(OH_L^{'}) = 10^{-14}$ we have $(OH_L^{'}) = 10^{18}$.

If we take $(Mg_L^{\cdot\cdot}) = 10^{-4}$ then $(H_L^{\cdot}) = 10^{-24}$ or $(OH_L^{'}) = 10^{10}$.

From the above figures it is seen that if magnesium is in such a state that the solubility product is $10^{2\times -16}$ it can never coexist with unary hydrogen and liquid because the surrounding liquid layer, required for this coexistence, cannot exist.

A graphical representation of the above statement in E,X diagram (fig. 1), is given by the point C. C lies so near one axis that any stable aqueous solution lies to the right of it. If we assume that the

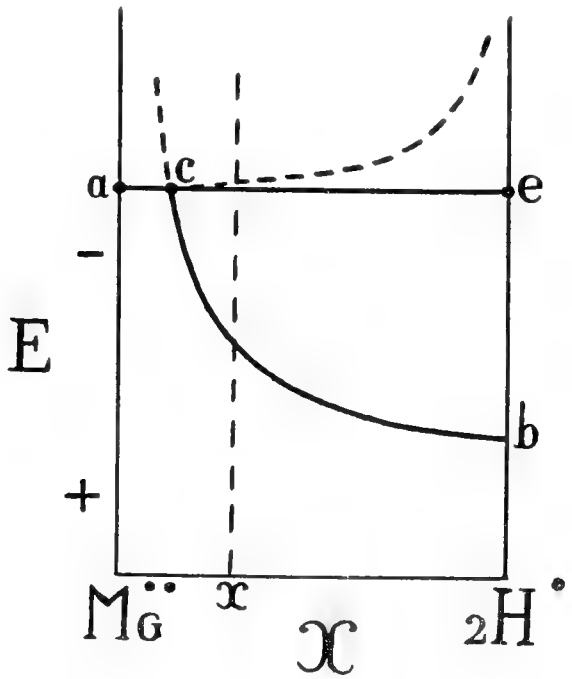

Fig. 1.

solution into which the Mg electrode is dipped has the composition X then there are two limiting possibilities for the coexistence of

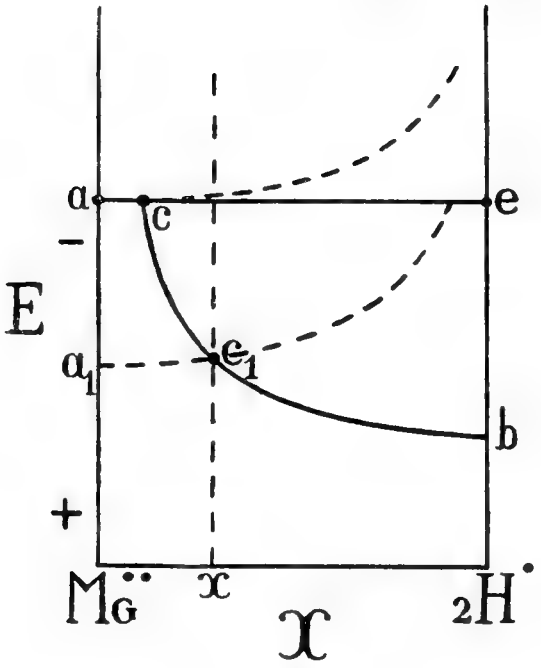

Fig. 2.

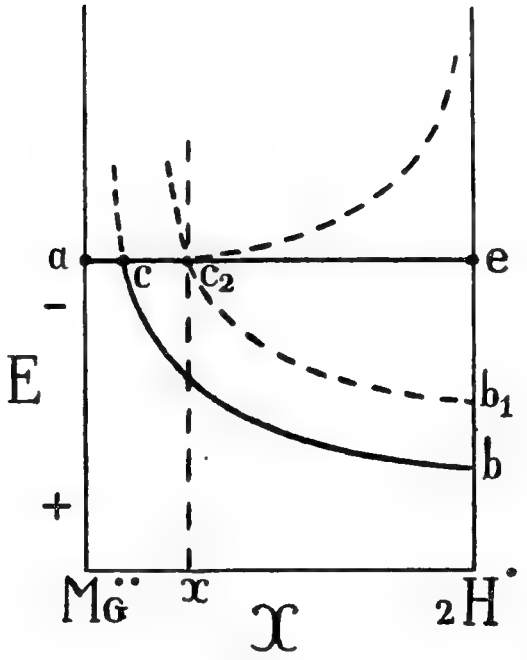

Fig. 3.

Mg, hydrogen and electrolyte. Between these limits the observable cases lie.

One limit is indicated in fig. 2. Here the hydrogen is in inner equilibrium but that of the Mg is displaced to such an extent that the potential line of this metal has the position $a_1 c_1$.

At the other limit the Mg remains in inner equilibrium but the liberating hydrogen is in a state of formation so that its potential line has the position $b_1 c_2$ in fig. 3.

In the latter case the observed potential of the three phase equilibrium ac_2e will practically correspond with the equilibrium potential of Mg [1]). The observed cases lie between these limits.

The above remarks concerning Mg with a potential of —1.86 V. also apply to Mg with a potential of —1.3 Volt. In this case $L_{Mg} = 10^{2 \times -26}$ and then $\frac{(Mg_L^{\bullet\bullet})}{(H_L^{\bullet})^2} = \frac{L_{Mg}}{L_{H_2}} = 10^{2 \times 22}$, so that if $(Mg_L^{\bullet\bullet}) = 1$, $(H^{\bullet}) = 10^{-22}$ or $(OH') = 10^{8}$.

Consequently when Mg of a potential of —1.3 V. was liberating hydrogen in inner equilibrium from a solution of a Mg salt in which $(Mg_L^{\bullet\bullet}) = 1$, then OH' in the surrounding liquid layer would be 10^8. This is practically also an impossibility.

From the above it follows that the hydrogen which coexists with magnesium and the surrounding liquid must be in such a condition that the value of L_{H_2} is much greater than that corresponding to the inner equilibrium.

This statement arouses a suspicion to the precipitations of Mg-hydroxide in the surrounding liquid layer, but if this occurred the coexisting hydroged would be formed in a stronger state of formation than even in the case that the surrounding liquid is no longer saturated with respect to $Mg(OH)_2$.

The solubility product of $Mg(OH)_2$ is about $10^{-10.5}$ since the value we choose for Mg_L is immaterial we will assume $(Mg_L^{\bullet\bullet}) = 1$. In this case $(OH_L') = 10^{-5.25}$.

If the Mg-electrode has the value $L_{Mg} = 10^{2 \times -26}$ we have already calculated that $(H_L^{\bullet}) = 10^{-22}$ or $(OH_L') = 10^{8}$ which is quite impossible for the solubility product of $Mg(OH)_2$ requires here $(OH') = 10^{-5.25}$.

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[1]) Here it must be remarked, that if hydrogen is being liberated the composition of the bounding liquid layer will always lie more to the lef than that of the liquid outside.

It is therefor evident that the apparent solubility product of hydrogen shows large deviations, from the value which would be expected when the hydrogen is in inner equilibrium. We will now calculate what the value of the solubility product of hydrogen must be in this case.

In the above we have manifestly employed a value for L_{H_2} which is $10^{2\times 13.25}$ times too small. The value of L'_{H_2} for the hydrogen which is being liberated, in the case under considerations, is therefore $10^{2\times -34.75}$ instead of $10^{3\times -46}$. In other words this hydrogen has become so much more basic, that in respect to its electromotive behaveour it somewhat resembles zinc.

If the OH-ion concentration in the surrounding liquid layer is lower than $10^{-5.25}$ then no precipitation of $Mg(OH)_2$ will take place. If $(OH'_L) = 10^{-10}$, then $L'_{H_2} = 10^{2\times -30}$ and electromotively the hydrogen is beginning to resemble manganese.

From the above considerations it follows that an approximation to the equilibrium potential of magnesium would only be possible if the hydrogen could appear in a stronger state of formation, for, as already has been demonstrated, an increase in the solubility product of magnesium will always be accompanied by an increase in the solubility product for the hydrogen which is being liberated. This is not necessary the case with less basic metals. It is clear that the foregoing conclusions will also hold for aluminium and we will now examine the conditions under which we can measure the most active potentials of these metals.

According to the theory of capillarity the change between two liquid phases or between a liquid and a gaseous phase is really an extremely sharp change in continuity. In the above case however we are dealing with random arrangements of particles in each phase. When we come to consider a metal and an electrolyte one has a definite structure and the other has not.

We are however sure, that in this case also in the bounding layer there will be a very sharp transition, though with a discontinuity, and that consequently the coexisting phases will only show quantitative differences with respect to compositions.

Now we make the assumption, that the parts, present in the metal bounding layer, in concentrations depending in the depth of the layer, in general will exert influence in the rapidity, with which the inner equilibrium is establishing in the bounding layer.

Oxygen, nitric acid, nitrates, etc., are already known to exert a retarding influence on the establishment of internal equilibrium in

metals and the electromotive behaveour of Mg and Al now shows that their oxids and hydroxides may exert a similiar influence.

In a solution of $MgSO_4$, to which no acid is added, some $Mg(OH)_2$ is in solution. If we dip a Mg-electrode into this solution then, besides other parts, present in the electrolyte, also $Mg(OH)_2$ will solve in the metal bounding layer. This does not mean to say, that the Mg will lose any of its characteristic properties such as the power to precipitate mercury from a solution but this small quantity of $Mg(OH)_2$ seems to exert a retarding influence on the velocity with which the internal metallic equilibrium is established.

A Mg-electrode under the above conditions dissolves slowly, evoluting hydrogen, and shows too low a potential owing to the disturbance of the inner equilibrium. Addition of sulphuric acid however decreases hydrolysis, and with this the $Mg(OH)_2$ concentration in the metallic surface and induces a change in the direction of the inner equilibrium of the metal, such that the potential becomes more strongly negative.

This effect of adding acid is however twofold. On the one hand the concentration of the negative catalyst in the metal $Mg(OH)_2$ is decreased, on the other hand direct attack at the metal is increased. This attack in cases where it is rapid, such as the one under considerations, always gives rise to disturbances and it might be expected, that the potential would first become more negative and finally would fall a little.

This was found to be the case by the author and the GRUYTER and also by BECK.

BECK's table XII p. 42 shows this quite plainly.

This table shows in addition that the differences between the Mg and the hydrogen potentials are not constant and that, whilst the hydrogen potential is becoming decreasingly negative, the magnesium potential changes in the opposite direction.

This means that Mg does not behave as a hydrogen electrode, which would be the case if the magnesium surface was unchanged and moreover was surrounded by a liquid layer saturated with respect to the $Mg(OH)_2$.

It is probable that this was the case with some solutions when the H-ion concentration was very low, merely with the vigorous stirring employed in these experiments. The certain conclusion from BECK's experiments is that, whether the surrounding liquid layer was saturated with respect to $Mg(OH)_2$ or not, the state of the Mg bounding layer was changing with the hydrogen concentration.

By increasing the hydrogen concentration the magnesium bounding

layer became more basic that is to say the normal inner equilibrium tended to be established.

Another phenomenon showed by Beck which has not yet been considered is that the potential of Mg in $MgSO_4$ solutions alone becomes more negative as the concentration increases.

Up to the present it has always been observed that a metal dipped into dilute solutions of the corresponding sulphates or chlorides were more early disturbed than in concentrated solutions of the same salts. This was particularly the case with sulphates but also with chlorides; the phenomenon was namely with chlorides also very distinct, though not so strong as in the case of sulphates.

This was always ascribed to the strong catalytic effect of Cl ions and the less one of the SO_4. Beck's measurements now show us that with magnesium not in inner equilibrium, SO_4 ions has also a powerful effect.

Although the highest potential shown in the last table (—1.816 V) is that of an active state of magnesium and the coexisting hydrogen must have been in a strong state of formation (strong overvoltage) yet this potential of Mg does not correspond with the inner equilibrium, for Mg containing small quantition of mercury shows a still higher negative voltage. This value was a maximum for 2 at $2\frac{1}{2}$ % Hg.

Now Beck found that the compound between Mg and Hg richest in the latter is Hg_3Hg and that the electrolytes in equilibrium with the various amalgams are practically free from mercury.

The influence therefore of the small quantity of mercury, under discussion on the Mg electrode can only he an activating one for the E—X fig. on the Mg side must be as follows (fig. 4). From this will be seen that if the influence of small quantition of mercury has not an activating one, then the potential of the amalgamated magnesium would have been less negative than that of the pure metal.

Thus activating by small quantities of mercury causes the true inner equilibrium to be approached more closely.

Magnesium which has been activated by mercury showed a potential of —1.856 Volts when placed in a solution containing 1 gr. mol. of $MgSO_4$ per liter.

Even this potential is below the equilibrium value owing to the disturbing effect, due to corrosion, but it is probable this is near the true equilibrium potential.

It is evident, that the potential of pure magnesium in true inner equilibrium must be more negative than that of the not disturbed amalgam, containing 2 at % Hg., because the E—X diagram

(fig. 4) shows us, that such potential is rendered less negative by increasing mercury content.

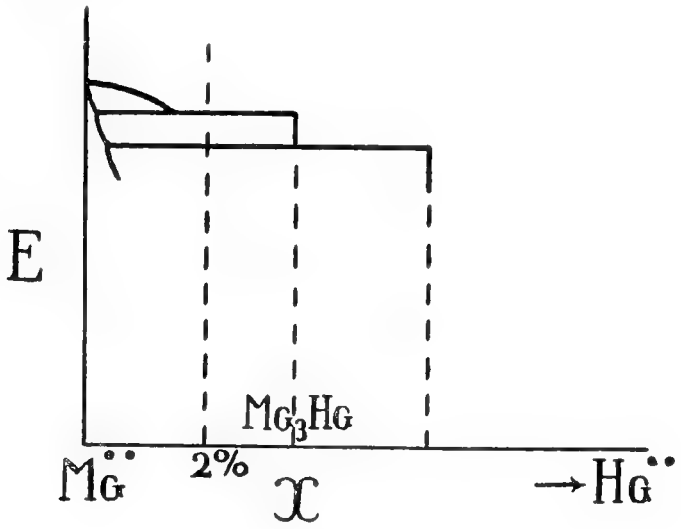

Fig. 4.

Finally we must consider a remarkable phenomenon to which brief reference has already been made.

If we add a little $HgCl_2$ in an aqueous solution of $MgSO_4$ or of $MgCl_2$ in which there is a magnesium electrode, there is an immodiate fine deposit of metallic mercury on the electrode whose potential becomes *less* negative.

As follows from the formula

$$\begin{array}{ccccc} Mg_s & \rightleftarrows & Mg_s^{\cdot\cdot} & + & 2\theta_s \\ \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow \\ Mg_L & \rightleftarrows & Mg_L^{\cdot\cdot} & + & 2\theta_L \end{array} \qquad \text{and } 2\,\theta_L + Hg_L^{\cdot\cdot} \rightarrow Hg_L$$

The precipitations of mercury proves that mercury ions penetrate the surrounding liquid layer and that these on arrival combine with electrons, thereby disturbing the heterogenous equilibrium with the result that electrons and magnesium ions enter the solution.

It must also be observed that in consequence of the hydrolysis in the magnesium salt solution to which no acid has been added the magnesium electrode will contain dissolved $Mg(OH)_2$ and will consequenty behave inertly, so that by sending ions and electrons into solution, the potential of magnesium will be altered in the direction of that of the noble metals.

The experiment mentioned here is very important; it shows in the first place that the magnesium electrode notwithstanding its non

equilibrium state and the dissolved $Mg(OH)_2$ has still retained its metallic properties. Still its properties have altered, for the precipitated mercury is not able to activate it at once. An apparent explanation, namely that the precipitated mercury does not dissolve in the magnesium, is not correct.

For if we remove the magnesium electrode covered with fine mercury, prepared as above, wash it with distilled water and than dip it into pure $MgSO_4$ solution, the potential is at first less negative, but it becomes increasingly negative, so that after a few minutes it is stronger than that which attained before the negative electrode was coated with mercury.

This is shown in the following table

Solution	Mg-potential in relation to a 1-N-calomel electrode
0,1 gr. mol. $MgSO_4$ p. liter	— 1.902 V.
to 150 ccm. of the above mentioned solution is added 5 ccm. of a saturated solution of $HgCl_2$	— 1.740 V.
The magnesium electrode was then washed with distilled water and dipped into a pure solution of $MgSO_4$.	
0 1 gr. mol. $MgSO_4$ p. Liter	— 1.898 V.
	— 1.956 V. after 5 minutes

The above data show that the magnesium electrode, though its surface is strongly disturbed by corrosion, has dissolved some mercury.

When we consider the great change brought about in a magnesium electrode by corrosion it is no wonder that its other properties, such as the power of dissolving mercury, are modified.

The explanation of the results in the above table now is clear. The activating influence exerted by the small quantity of dissolved mercury is not sufficient to decrease the retardation, exercised by the $Mg(OH)_2$ in such a way, that the electrode becomes insensible to the corrosive action of water and sublimate. When this solution has been substituted by one of pure $MgSO_4$ the influence of the sublimate disappears and that of the small quantity of mercury becomes manifest.

It might be supposed at first sight that in the experiment under consideration solid $Mg(OH)_2$ depositing on the magnesium electrode might diminish the contact between the magnesium and the mercury,

the fact is however that the contact between the magnesium and the electrolyte is so good that mercury is separated over the whole surface in a finely divided state, even whilst hydrogen is being given off.

At the same time it is clear, that if we wish to get a magnesium into as highly an active state as possible, it is desirable to make its surface as poor as possible in $Mg(OH)_2$ by first immersing it in an acid solution and then amalgamating it.

Magnesium, activated in this way, contains more dissolved mercury and even remains active in normal KOH, giving the high negative potential of -1.97 volts in relation to the hydrogen electrode on account of the low Mg-ionic concentration.

In a solution containing 1 gr. mol. $MgSO_4$ per liter this electrode gave a potential of -1.85 V. in relation to the hydrogen electrode.

Amalgamation experiments have also been studied in detail for Al and will be the subject of a next paper.

Amsterdam, Febr. 1923.

Laboratory for General and anorganic Chemistry of the University.

Botany. — "*A method of simultaneously studying the absorption of O_2 and the discharge of CO_2 in respiration.*" By D. S. FERNANDES. (Communicated by Prof. F. A. F. C. WENT.)

(Communicated at the meeting of May 26, 1923).

Before entering into details, writer will briefly indicate, how the apparatus works and what precautions should be taken, illustrated by a simple diagram. (fig. 1).

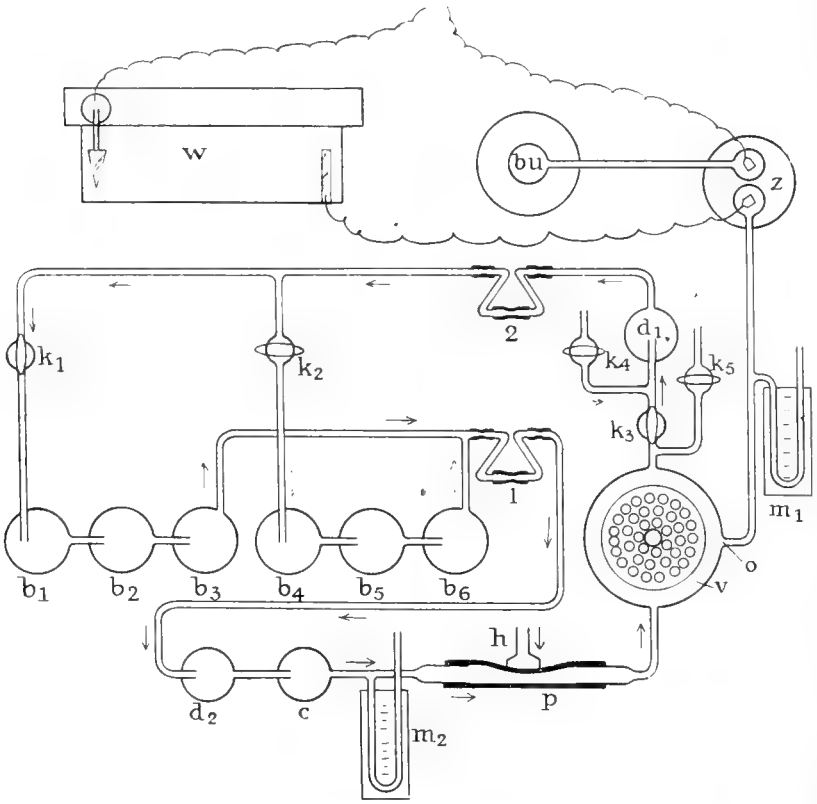

Fig. 1.

From p, a rubber sucking- and forcing pump, the air is pumped as the arrows indicate. The air enters the respiratory vessel v at the top, leaves it at the bottom and is dried in the wash-flask d_1, which contains concentrated sulfuric acid. From d_1 passing through

the glass cock k_1 (k_3 is then closed) it reaches the absorptiontubes b_1, b_2 and b_3, containing baryta-water. On its way back the air passes through the wash-flask d_2, containing sulfuric acid like d_1 and the control-baryta-tube c, after which it returns to p and recommences its circular course.

In a subsequent observation k_1 is closed and k_2 opened, causing the CO_2 absorption to take place in the tubes b_4, b_5 and b_6. The 6 absorption-tubes are fixed to a copper frame with clips. In order to enable us to take more than two observations, without bringing too many tubes in the glass vessel filled with water, which serves as a thermostat, we should have two of these frames at our disposal. If one has served its purpose, the connecting parts 1 and 2 are turned up and rise above the water, where they may be loosened. The whole frame with the 6 baryta-tubes is raised out of the vessel and the other (the tubes of which are meanwhile cleaned and filled each with 100 c.c. baryta-water) is put in. This exchange of frames is brought about in less than a minute, but before taking further observations with the newly-inserted baryta-tubes, we should wait (according to the temperature in the thermostat) 10—15 mins. that the tubes and their contents may adopt the temperature of the thermostat. The apparatus works ventilating during this time in the following way: Cock k_3 is closed, while k_4 and k_5 are opened. If next the pump is set working, the air, leaving the vessel, can only pass through k_5, while at k_4 air is sucked in, after having first been rid of CO_2 by means of wash-flasks containing strong KOH-solutions (not represented in the fig.). There is another advantage in the ventilating action of the apparatus. When in experiments of long duration the observations are stopped in the evening, the apparatus can continue to work ventilating the whole night. Consequently the objects are not subject to oscillations of temperature and the next morning the experiment may at once be continued by opening k_3 and closing k_4 and k_5. In experiments, lasting 10—12 hours, it saves a great deal of time, to put the plants into the apparatus the previous night, so that early in the morning the experiments can begin at once. After the ventilation during the night all CO_2 has been driven from the apparatus which may be demonstrated by blind experiments.

When the outer-air is shut from the apparatus, and the pump is set working, there is immediately produced an effective pressure on the vessel, while the manometer m_2, indicates a reduction of pressure. If next k_5 is opened, the air pressed in the vessel is blown off. On subsequent gradual closure of this cock, the pressure in the vessel

$= 1$. In the manometer m_1 the liquid is equally high in both limbs, whereas m_2 indicates a greater negative pressure than before. The broken equilibrium, generated by the action of the sucking-and forcing pump in the closed system is apparently shifted by the opening and closing of k_5 in such a way, that in the respiratory-vessel (accordingly on the plants) no effective pressure can arise. As soon as there disappears O_2 from the closed system through respiration, m_1 will indicate it at once. When however an equal quantity of O_2 is added at the same time, m_1 will remain at zero and the atmospheric pressure is preserved in the vessel. At O the oxygen, electrolytically produced in Z, enters the vessel. With the aid of the resistance w the O_2-development can be increased from a minimum to a definite maximum. The intensity of the electrolytic process may be thus regulated, that the O_2-production keeps pace with the O_2-consumption.

By increasing or reducing the resistance this equilibrium is soon found and the manometer m_1 indicates whether this condition is preserved. It may happen (for instance by rise or fall of the respiration-intensity), that for a moment there is a somewhat greater or smaller supply of O_2 to the apparatus. In this case the height of the manometer m_1, indicating as slight a difference as 0.1 cc., may at once be restored by means of the resistance, so that irregularities in the O_2 supply, amounting to more than 0,1 cc. need not occur.

The hydrogen simultaneously produced by the electrolysis in Z is collected in the burette bu. After necessary corrections (in height of barometer, temperature, water-vapour tension and pressure of the water-column in the burette) the quantity of hydrogen received, divided by 2, denotes the volume O_2, brought into the apparatus during the observation.

The manometer m_1 renders some other services. When a solution of kalium-jodide (with some soluble amylum) is used, m_1 is a sensitive test for the existence of spores of ozon. In the presence of this gas for instance the germ-plants of Pisum sativum do not develop normally, so that it is desirable to prevent ozon from entering the respiratory-apparatus.

Finally we have in the manometer m_1 a suitable test whether the desired temperature has been completely adopted by the whole apparatus as well as by the objects. If the observations are started before the whole has attained the desired temperature, the fluid will at once rise in the open limb of m_1, which signifies, that extension still takes place, while in consequence of the respiration (O_2 absorption) an immediate decrease of volume should appear.

For determining the period of preheating therefore m_1 is of practical interest.

The watervapour carried along from the vessel is combined in d_1 so that dry air enters the baryta-tubes. The watervapour taken from the lye is absorbed in d_2. By measuring the increase of volume in d_2 it may be found, how much water disappears from the lye and the titration standard may be corrected accordingly. This evaporation from the baryta-tubes is very slight and amounted to circa 2 cc. in experiments lasting 3 days, so that the correction may be left out without scruple.

The manometer m_2 is filled with mercury and serves to indicate the pressure, to be surmounted by the sucking and forcing pump, needed to drive the air through the various liquids. A drop of paraffine-oil on the mercury in the closed limb, prevents the originating of damaging mercury-vapours.

On the rubber-pump p taps a flat hammer h, moved vertically by an electro-motor (not represented in the figure). This hammer may be mounted higher or lower in order to regulate the capacity of the pump and consequently the size of the bubbles. The speed of the motor may be increased or decreased by means of a resistance, with which the regulation of the number of bubbles is possible. Size and number of bubbles are of course material to a good CO_2-absorption.

For an equable distribution of the air, entering the vessel, the ebonite plates on which the plants lie, are brought into a slow rotary movement by an axis. Accumulation of CO_2 in the vessel (see further on) is excluded in this way.

The suction of the air into and from the vessel, causes the liquid in m_1 to move up and down, which is not to be prevented. At an effective regulation of the pump this movement may be kept so slight, that it is no impediment. Indeed the motor may be stopped at any moment, to convince oneself whether the manometer is really at zero.

The whole apparatus is fixed to the inside of a copper frame and fits exactly in a glass vessel (contents about 45 L.), serving as a water-thermostat. Electrical heating enables us to keep the temperature of the water constant to 0.03° C. The oscillations of temperature in the apparatus itself are slighter than those in the thermostat, so that corrections relating to this, may be omitted.

If the apparatus is immersed in the water of the thermostat, it may be easily tested with respect to air-tightness. For this purpose air is pumped into the apparatus through k_4 and one watches whether

any bubbles rise from the water. When the connections are made with vacuum rubber-tube and glass to glass, leakages do not occur.

II.

Descriptions of the parts.

a. Sucking- and forcing-pump (fig. 2).

An air-tight pump, working for a long period without failing and having a sufficient capacity, is easily constructed.

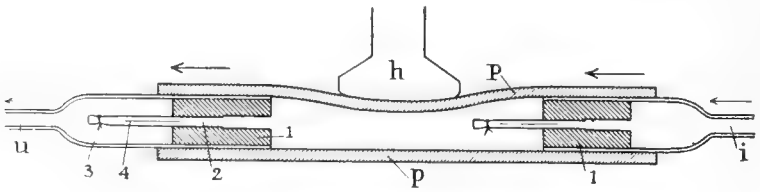

Fig. 2.

The glass tubes *i* and *u* are connected by a piece of strong rubber-tube *p* (about 15 cms. long and $2^1/_2$ cms. wide). Each of the tubes *i* and *u* is provided with a valve, consisting of a piece of vacuum-tube (1 cm. long) 1, to which the end of a piece of valve-tube 2 (about 3 cms. long) is glued on with solution. The other end of the valve-tube is tightly tied with a string at 3; in the valve-tube a straight lengthwise cut 4 is made, the two edges of which meet, when the pump does not work. To prevent these edges from sticking together afterwards, they have been rubbed in with talcum powder. The glass tubes *i* and *u* fit in the rubber-tube *p*, while the vacuumpieces 1 must also fit perfectly. How the pump works, when the hammer *h* taps on it, is clear from the fig. 2.

b. The respiratory-vessel (fig. 3).

As in KUYPER's research[1]) here too is made use of a copper cylinder 1. The experimental objects are on the ebonite plates *t*, fixed to an axis a_2. In each of the plates *t* 25 round holes are made in such a way, that germinating seeds of Pisum sativum cannot fall through. On the plates t_1 moist cotton-wool is put, on which the roots rest, in consequence of which there cannot occur a deficiency of water. The axis a_2 is enlarged at the top, provided

[1]) KUYPER J: Recueil des Travaux Botaniques Néerlandais. Vol. VII. 1910, pag. 1.

with 4 teeth ta, just fitting into the four teeth ta_1 belonging to a simular enlargement at the base of the axis a_1. This steel axis a_1 passes through a copper case k (soldered to the cover), in which it

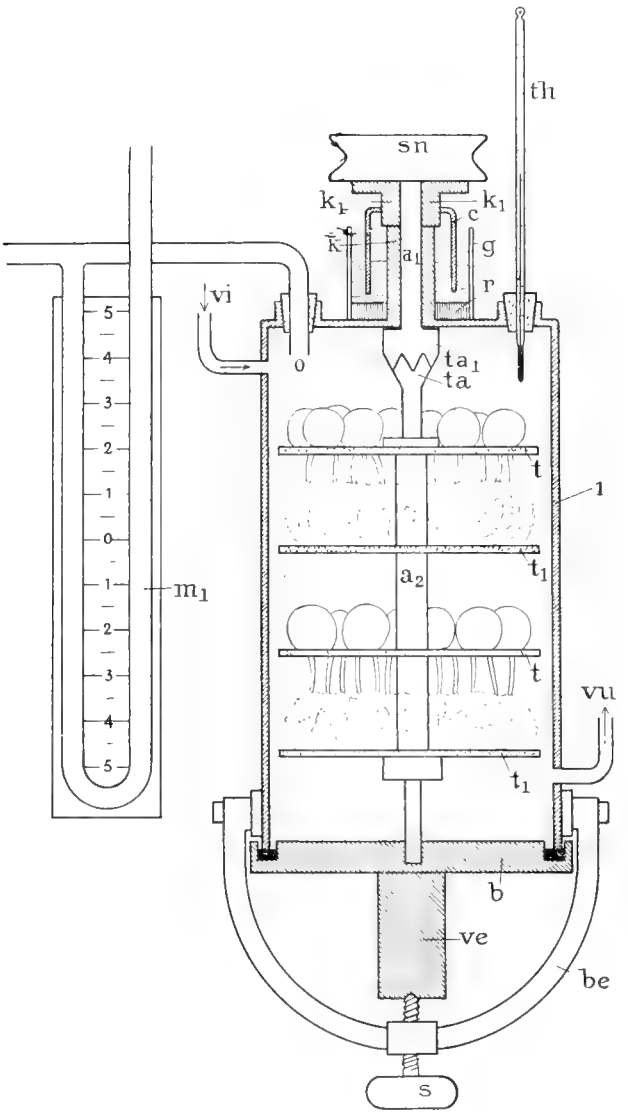

Fig. 3.

fits exactly, but may be easily rotated. Round k there is a glass cylinder g, closed at the bottom by the india-rubber-ring r. The axis a_1 is at the top tightly clasped in a copper tube k_1, at the bottom of which the hollow metal cylinder c is fastened, and at the top the grooved wheel sn. By the oil in g the axis is closed

off air-tight and leakage is impossible, because there never arise great differences of pressure in the vessel. In the middle of the loose part *b* there is a cavity, in which a_2 can rotate freely. When *sn* is slowly rotated by a motor, a_1 will transmit this movement by means of the teeth ta_1 and *ta* to a_2, which causes the circulating air to be equably distributed over the whole vessel, in consequence of which the germplants are constantly surrounded by fresh air. The necessity of ventilation in a cylindrical respiratory vessel (diameter 15 cms., height 20 cms.) was immediately apparent from one of the many test experiments. At a constant temperature of 20° C. the O_2-absorption caused in 50 mins. a height of 4 cms. on the manometer m_1. Next a quicker circulation of 10 mins. duration followed, causing an equal rise of the manometer as before in 50 mins. No other explanation of this could be found, but the occurrence of a CO_2-accumulation in the vessel. This was supposed to be due to the fact, that the air entering at *vi* passed by the easiest route through the vessel to the exit *vu*, taking with it only part of the CO_2. When in consequence of a more rapid circulation part of the accumulated CO_2 disappeared, this explained a sudden greater rise of the manometer. As soon as the rotary movement of the respiring objects, prevented all CO_2-accumulation in the vessel, there was indeed no abnormal rise of the manometer to be noticed. It needs no argument, that not only with a view to oxygen-supply and measurement, but also for other reasons, the CO_2 due to respiration, should be directly removed. With a CO_2-accumulation in the vessel, a volumetric determination of the vanished quantity of O_2 is no more possible. Besides in this case part of the plants gets into an atmosphere full of CO_2 and deficiency of O_2 will soon cause intramolecular respiration.

It seems to me, that in the respiratory apparatus after the model given by PFEFFER and DETMER and used e.g. by KUYPER, little or no attention has been paid to the error which may be committed, when in a respiratory vessel as described in this paper, no perfect ventilation is provided for.

The loose bottom *b* is provided with a marginal groove, containing a rubber-ring. The handle *be* bears in its middle a screw *s*, which, when turned up, presses on *ve* and by doing so presses the lower edge of the vessel tightly in the groove with rubber-ring.

In the cover of the vessel is, besides the aperture *o* to admit oxygen, also a pierced rubber-cork through which a thermometer *th* passes.

c. Fig. 4 gives a representation of the *drying-tubes* and the

control-tube. Cock 1 serves for filling, cock 2 for emptying and cleaning.

d. The *absorption-tubes* are fastened to a copper frame (fig. 5). As with a view to preversing a constant temperature the size of the thermostat cannot be chosen at will, straight absorption-tubes (length 25 cms., width 3 cms.) are more suitable than PETTENKOFER or WINKLER-tubes. When baryta-water is chosen for combining with CO_2 (21 grammes of bariumhydroxyde + 3 grammes of barium-chloride in 1 L. of water), the absorption is only complete, when the air passes through 3 of those tubes (each containing 100 cc. lye). Each frame of 6 tubes therefore can only serve for two observations. The tubes end at the base in thin open pieces, which may be plugged by rubber stoppers. At the top they are closed by rubber-corks 3 cms. thick. In each cork there are three holes, two of which serve for the inlet- and exhaust-tubes, while the third, which serves for filling can be plugged by a little massive glass bar. The tubes are connected with vacuum-rubber

Fig. 4.

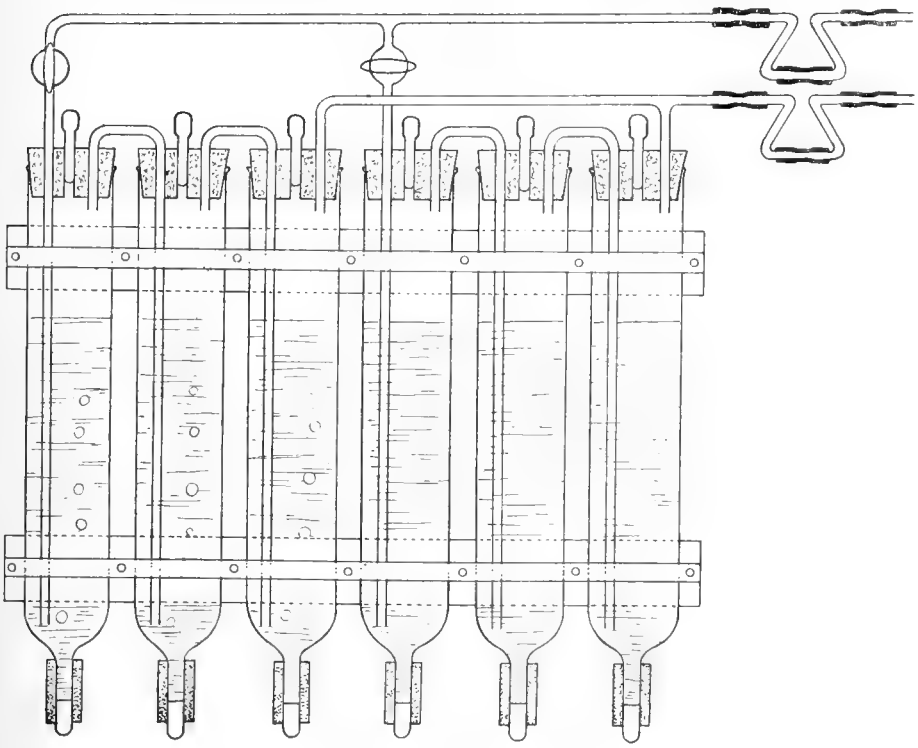
Fig. 5.

tube, just as all other connections in the apparatus are made. There was no sign of any CO_2 diffusion inward from the water of the thermostat through the rubber-connections and corks, nor of an O_2-absorption through the rubber. Blind experiments, lasting 24 hours gave no measurable change of titration standard of the lye at temperatures between 20° and 30° C., while the manometer m_1 remained at zero throughout that time.

e. The oxygen-supply and measurement.

In order to prevent ozon-formation, a 10 % natronsolution is to be preferred to diluted sulfuric acid for the electrolysis.

In fig. 6 *C* is a glass cylinder with natron-lye in which the platina-electrodes p_1 and p_2 are placed. By means of thin platina-wire these electrodes are fastened by melting in the glass-tubes 1 and 2 respectively. The tubes 1 and 2 pass through caoutchouc-

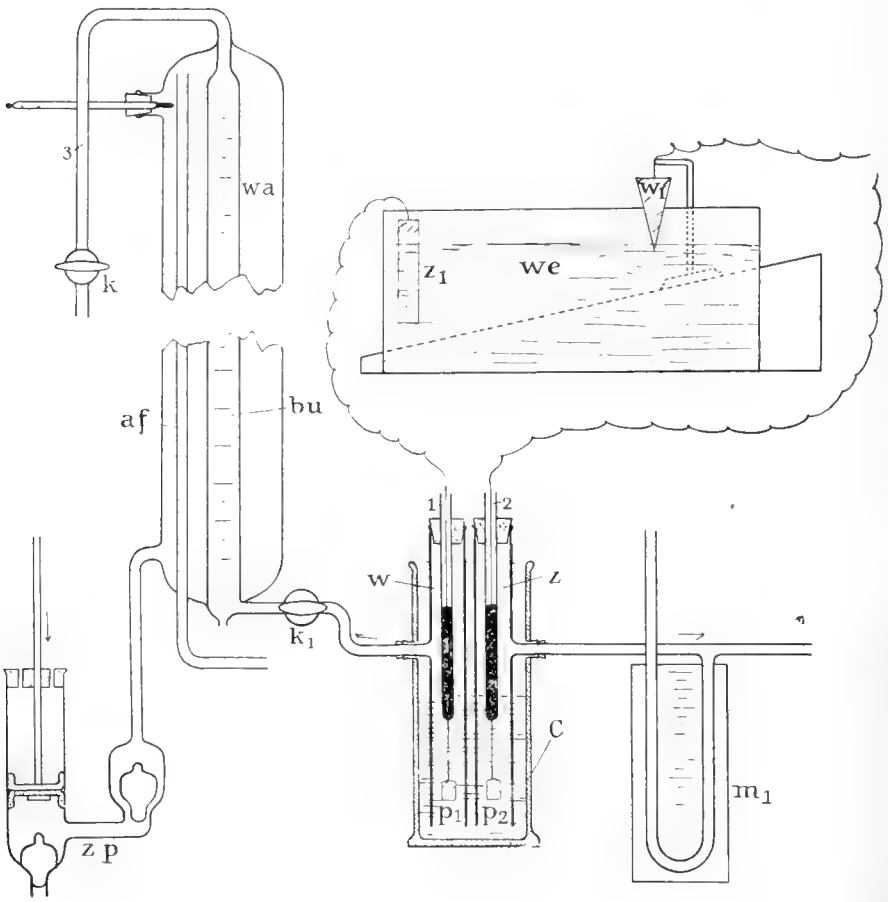

Fig. 6.

corks, fitting exactly in the wider tubes *w* and *z* (open at the bottom) and are filled with some mercury. By means of a resistance *we* the intensity of the current can thus be regulated, that the amount of the electrolysis can reach the desired extent. Thus it is possible to keep the oxygen-development, occurring in the tube *z* at the electrode p_2, in balance with the O_2-consumption of the respiration. As a resistance (*we*) a glass basin with water, in which the electrodes w_1 and z_1, is quite satisfactory for this purpose. By moving w_1, which is fastened to a stand, along a sloping board, not only the distance w_1—z_1 is made smaller or larger, but this electrode also goes more or less deep in the water.

The O_2 formed in *z* is in open connection with the manometer m_1 and the respiratory-vessel. The tube *z* really is likewise a manometer, in which the lye will be equally high as in *c*, when the quantity of O_2 developed is equal to the quantity disappearing in the apparatus; m_1 however, as already mentioned, is necessary to control the ozon-formation.

For receiving the hydrogen, formed at the electrode p_1 in the tube *w*, the burette *bu* serves, which gives accurate readings to 0.1 cc. This burette ends at the top in a bent glass tube 3, provided with a glass cock *k*. At the bottom the burette has a narrow aperture, while not far from this a lateral tube has been fitted on, forming a connection with the tube *w*. When the burette is placed in such a way, that the bottom aperture lies just below the water-level in the thermostat, it is impossible, that while water is flowing out, air is ascending in the burette at the same time. Filling the burette with water from the thermostat is done by closing k_1, opening *k* and sucking at the tube 3. When after filling *k* is closed and k_1 open, the only reason why water should flow from the burette, is the formation of hydrogen in *w*, which rises in the full burette as bubbles. The formation of the first hydrogen-bubbles in the burette requires a little effective pressure, which is shown by the fall of the fluid in the tube *w*. This effective pressure, which remains constant during the emptying of the burette, should exist before the observations begin, lest the first reading should give a too small figure. This error is prevented, when some minutes before the experiment commences — when the apparatus still works ventilating — the electrolysis is made to take place, till the first bubbles rise in the burette. In case that, during one and the same observation, the burette is filled several times, the sucking up of the water should occur very slowly and equally, lest the hydrogen, which is in the connective-tube between k_1 and the burette, should be sucked in

with it. If the water is sucked cautiously into the burette, the effective pressure once made is preserved in *w*.

Another error arises, when the burette is exposed to oscillations of temperature in the laboratory. In that case not only in *w*, but also in *z* and m_1 falls and rises occur, which are not due to absorption of oxygen. This may be prevented by keeping the burette likewise at a constant temperature, which may be attained as follows.

By means of a metal sucking- and forcing-pump *zp* (likewise fastened to the copper frame, to which the whole apparatus is fastened) water from the thermostat is pumped up with great rapidity into a wide glass cylinder *wa*, which contains the burette. The water enters *wa* at the bottom and is led back to the thermostat at the top through the tube *af*. Even at high temperature (50°, 55° C.), the temperature in the burette is kept equal to that of the water in the thermostat in this way.

f. The regulation of the temperature principally corresponds to the one described by RUTGERS [1]) and COHEN STUART [2]) and is an imitation of apparatus, used in the VAN 'T HOFF-laboratory at Utrecht.

The heating-apparatus *v* (fig. 7) consists of a copper case, surmounted by a metal tube, rising above water. In *v* is paraffine-oil, electrically heated by a nickel-chrome-wire, wrapped round a piece of mica.

Thermoregulator *t*, stirring-apparatus *r* and *v*, are close together in an open glass cylinder *c*, resting on legs in the centre of the thermostat *g*. To prevent all influence of vibration in the height of the mercury, the thermoregulator is hung from the ceiling on a steel spiral-spring, according to the method MOLL.

The method described above gives no new principle, with respect to the CO_2-determination. We have chosen the simple and always trustworthy baryta-method, which need not be further described here. On account of the insertion into a closed system, the various parts were subjected to some alterations in shape, which however have nothing to do with the principle of the baryta-method.

The problem of oxygen-supply, ever yielding many difficulties, could be satisfactorily solved. Compared with the methods [3]) already existing, the following advantages and simplifications are achieved:

[1]) RUTGERS, A. A. L., Recueil des Travaux Botaniques Néerlandais. Vol. IX, 1912, pag. 1.

[2]) COHEN STUART, Recueil des Travaux Botaniques Néerlandais. Vol. XIX, Livraison 2. 1922.

[3]) Cf. KROGH: "The respiration exchange of animals and man. LONGMANS, GREEN and Co., London 1916".

a. the decrease of pressure and oxygen-content in the apparatus is reduced to a minimum.

b. the place of the consumed O_2 is at once taken by pure O_2, without first passing a stop-valve, and may directly be controled.

c. an oxygen-bomb or other reservoir may be omitted.

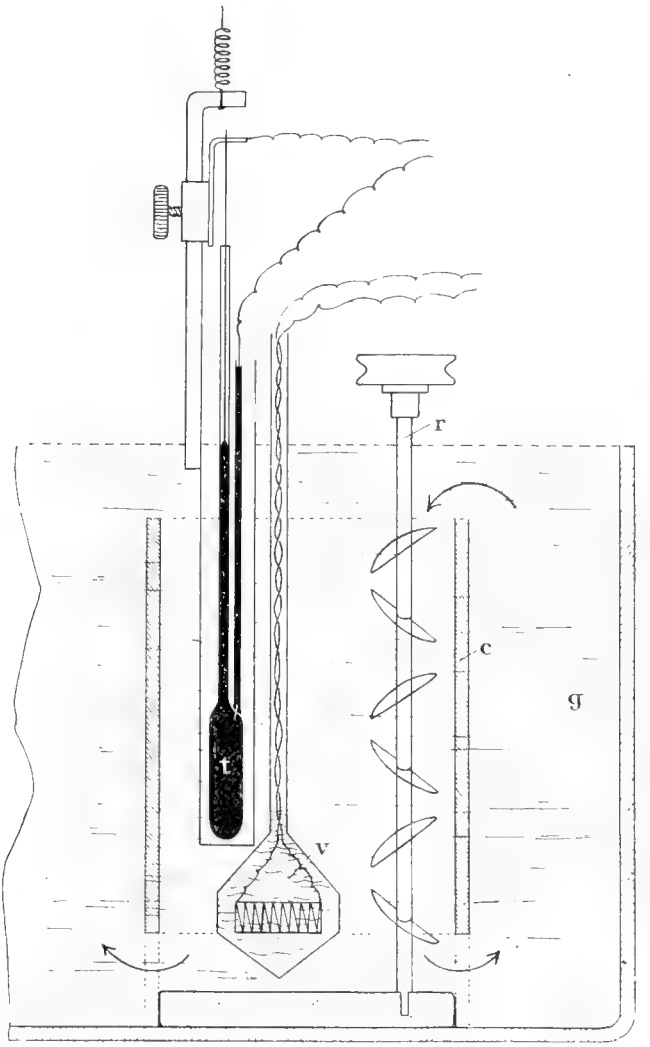

Fig. 7.

The apparatus has been constructed by Mr. P. A. DE BOUTER, amanuensis at the Botanical Laboratory at Utrecht. I am greatly indebted to him, not only for the way, in which he performed his task, but also for introducing some clever improvements.

Utrecht, May 1923. *Botanical Laboratory.*

Physiology. — "*A new form of correlation between organs.*" By Prof. H. J. HAMBURGER.

(Communicated at the meeting of May 26, 1923).

Thus far we were acquainted with two forms of cooperation between organs. As to the eldest known form, here the central nervous system plays an important rôle. If any one pricks my finger unexpectedly with a needle, I immediately withdraw my arm; a cooperation has taken place between the skin of the finger and the muscles of the arm, and well by means of the spinal chord. Here we have to deal with a reflex.

Some years ago we got acquainted with a second form of correlation between organs; this one is not effected by means of nerves, but here the bloodcurrent is the mediator of the cooperation. For instance, the glandula thyroidea produces substances, which are carried through the body by the bloodcurrent and influence the metabolism and growth of distant organs.

That nerves here don't play an essential rôle appears from the fact, that the glandula thyroidea still exerts its influence, even when it is detached from its nerves and transplanted to another part of the body.

Now, in the last years experiments, performed in our laboratory, have clearly demonstrated *a third new form*[1]) *of correlation between organs.* The starting point of these researches, carried out by Dr. R. BRINKMAN, Miss E. VAN DAM and Dr. L. JENDRASSIK, was the following experiment of O. LOEWI in Graz. The vagus nerve of an isolated frog's heart, which is filled with a salt solution, is for some time stimulated so that the heart stops its beat. Then the content of the heart is removed and transferred into another frog's heart, which was isolated in the same way. Then the well-known pharma-

[1]) See my lecture at the opening of the Biological Buildings of Mc. Gill's University in Montreal (Canada) in September 1922. See also: H. J. HAMBURGER. The increasing significance of permeability problems for the biological and medical sciences; the Charles E. Dohme Memorial Lectures. First Course, 10, 11, 12 October 1922, delivered in Baltimore; printed in: Bulletin of the Johns Hopkins Hospital, June 1923.

cologist saw, that the second heart often showed slower contractions. Experiments with the sympathetic nerve gave analogous results.

Now the purpose of our experiments was in the first place to control the results of LOEWI's researches under more physiological conditions.

In the vena cava of a frog A a glass tube is inserted and in this way a suitable saltsolution is conducted through the heart. A similar small tube is introduced into the aorta. Then we see, that the saltsolution will leave the heart in a rhythmical manner. If then the fluid, leaving the heart, is led to the vena cava of another frog B, the fluid will run through the heart B, and after leaving it by the aorta of this second frog, it may be taken up again by the vena cava of the first frog A. Thus we obtain a circulation of saltsolution through both frog's hearts. This method of socalled "crossing circulation" was first introduced by Prof. J. C. HEMMETER.

Now, if the sympathetic of the first frog A be stimulated electrically, causing acceleration of the heart beat of this frog, it can be observed that already after a few seconds, the heart rate of the second frog B is also quickened, *although the sympathetic of this animal has not been stimulated.* How to account for the acceleration of the second heart? Evidently in no other way than by assuming that in the first heart A, in virtue of permeability of course, substances were liberated which had a similar effect upon the second heart as if this had been directly stimulated. I shall presently come back to the probable nature of these substances.

How it is possible that substances, liberated by a physiological action of an organ, here the heart of the frog A, may also stimulate the same organ of the second animal B, I shall not discuss here. It is sufficient to say, that there is an analogy between this case and the secretion of saliva. If we allow a salt solution to percolate through the salivary gland, as J. DEMOOR has demonstrated some years ago, no saliva is secreted. However it does occur if a small quantity of saliva is added to the saltsolution. The product formed during the activity of the salivary gland is, it seems, a stimulus again to further secretion of saliva. The substances, formed in the stomach during conversion of protein, excite gastric secretion. It is therefore not strange that the substances, liberated in the first heart during stimulation of the sympathetic, should have a stimulating action on the second heart.

Dr. BRINKMAN and Miss VAN DAM made yet another experiment that in a still more convincing and striking manner demonstrates, that the transmission of stimuli can take place by means of fluids,

in other words that there exists a humoral transmission[1]), I say "in a still more convincing manner", for by the just mentioned experiment the remark could be made, that with the movement of the second heart hydrodynamic influences might have played a rôle.

For this reason for the second organ not the heart of the frog B was taken, but the *stomach* of this animal.

It is well known that stimulation of the sympathetic nerve is followed not only by an acceleration of the heart beat, but also it slows, even inhibits the spontaneous movements of the stomach. Now the question arose: if the fluid of the stimulated heart of frog A is transferred into the arteria gastrica of the frog B, will it then cause the spontaneous movements of the stomach of this last frog to grow slower, even to stop? This proved to be the case, as the experiments of Dr. BRINKMAN and Miss VAN DAM showed us. *In other words, on sympathetic stimulaiton of the first heart substances were liberated which influenced the movements of the stomach in an inhibitive way.*

Analogical phenomena as occur in stimulating the sympathetic nerve could be observed by stimulation of the vagus nerve.

As it is well known, stimulation of this nerve affects the rate of the heart beat and also influences the strength of the contractions of the stomach, but in an antagonistic sense. Stimulation of the vagus slows the heart, but causes the contractions of the stomach to become more powerful, contrary to what happens when the sympathetic nerve is stimulated. Now the experiment was repeated by crossing the circulation of the heart of the first frog with that of the stomach of the second frog; in other words, the salt solution coming from the heart of the first frog, is conducted to the stomach-circulation of the second frog. On stimulating the vagus of the first frog, the heart slows its beat and when the solution has passed through this heart and reached the stomach of the second frog, this organ shows typical vagal contractions, though the vagus of frog B has not been stimulated electrically. From this we may infer that stimulation of the vagus of the first frog sets free in its heart vagus-substances, which may cause the stomach of the second frog to contract, as if its own vagus nerve had been directly stimulated.

We are therefore in presence of two kinds of substances liberated by the vagus and sympathetic nerve respectively, which may be called vagus- and sympathetic substances.

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[1]) R. BRINKMAN und Frl. E. v. DAM, Pflüger's Archiv. Bd. 196, S. 166, 1922.

That really such substances exist, could be directly proved by the fact that the salt solution, leaving the heart after stimulation of the vagus, contains substances, which lower the surface-tension of the original salt solution, socalled capillary-active substances. On the other hand we find that the surface-tension of the salt solution, coming from the heart after the sympathetic nerve has been stimulated, is slightly increased[1]). Further it appeared that the vagus- and sympathicus-substances were able to neutralize each other in capillary-active sense, i. o. w. they were able to neutralize each other's influence on the surface-tension.

I shall not enter here into further particulars. It is an established fact now, that as an effect of stimulation of the vagus nerve, a liberation of vagus-substances takes place, and that on stimulating the sympathetic nerve, sympathetic-substances are set free. However the nature of these substances has not yet been determined; perhaps, at least with the vagus-stimulation, we have to do with cholin-compounds, which cooperate with the potassium.

As for the method to determine the surface-tension of very small quantities of fluids, we refer to two articles, which appeared last year[2]). There it is shown that a very simple apparatus will do for this purpose. By means of a torsion balance, well-known to the clinicians, the force is determined which is necessary to pull off a small platina-ring from the surface of the fluid which is to be examined.

The experiments discussed here, give rise to many questions. So the clinician will think of the bearing of these results on the nature of vagotonia and sympathicotonia and will ask himself under which conditions an excess of vagus- and sympathicus-substances will exist in the circulation and influence different organs; and also he will put himself the question how it will be possible to make this surplus harmless for the body.

The physiologist will ask himself whether the latent period and the after-effect in vagus-stimulation can be explained by the time, which is necessary for the liberating and the disappearing of the vagus-substances; further he wants to know whether the vagus-substances are specific for one and the same animal. And what will be of interest both for the physiologist and the clinician is the question: can we observe the same phenomena, seen in the frog,

[1]) See the article of Dr. BRINKMAN and Miss VAN DAM, in the Journal of Physiol., still to appear.

[2]) R. BRINKMAN und Frl. E. VAN DAM. Münch. Med. Wochenschr. 1921. S. 1550.
R. BRINKMAN, Arch. Néerl. d. Physiol. VII 1922, p. 258.
R. BRINKMAN und Frl. E. VAN DAM VIII, 1923, p. 29.

also *in warmblooded animals?* With this question Dr. L. JENDRASSIK has occupied himself very recently. The results obtained untill yet, can be summarized in a few words. If the surviving heart of a rabbit is perfused with a suitable salt solution, and we stimulate the vagus nerve, then the liquid, leaving the stimulated heart is able to accelerate in a high degree the contractions of an isolated piece of gut, taken from the same animal.

I cannot enter into these researches on this place. Dr. JENDRASSIK will describe them in a short time in the Biochemische Zeitschrift. Here we will only point out that the experiments proved, that on stimulation of the vagus nerve not only in the heart of coldblooded animals but also in those of warmblooded animals substances are produced, which are able to influence other organs in the very same way, as if the vagus of those organs were stimulated by an electrical current. Here the gut proved to be the most suitable object for the researches.

Further I might draw the attention of the readers to three remarkable facts. In the first place it appeared that an extract of the atrium of a rabbit's heart in saltsolution was also able to accelerate the contractions of the isolated piece of gut. This experiment was made in considering that it would be very probable, that the atrium still contained vagus-substances, which were formed there during the life of the animal. Secondly it appeared that if atropine, which, as is well known, inhibits the influence of vagus-stimulation, was added to the active extract, this was turned into an unactive one, i.o.w. then it had no more influence on the movements of the gut. In the third place it was found, that the extract of the ventricle-muscle of the heart has a sympathetic effect on the movements of the gut instead of a vagus-influence.

The experiments on warmblooded animals described above, were all performed in a room of body temperature.

SUMMARY.

Thus far we have been acquainted with only two forms of correlation between organs, one, the eldest, established through interference of the central nervous system in cases where a quick response is needed (reflexes). The second form comes into play where slow processes are concerned; it may be exemplified by the influence of the glandula thyroidea on metabolism and growth. For the formation of hormons the influence of the nervous system is not needed, neither for the transport by the bloodcurrent. In *the third new form*

of correlation the action is neither quick nor slow; it is to be seen at work where functions, holding the medium between these two, are concerned. The essential thing here is, that *by nervous stimulation* substances are set free, which are conducted to other parts of the body.

There is much evidence to lead to the belief that the three forms may finally by reduced to one, but I cannot enter into this here. I have spoken about this possibility already in one of my HERTER-LECTURES, delivered in New-York in October 1922.

It may be of importance to lay stress on the fact that the formation of vagus- and sympathicus-substances is not only postulated, but *that it is proved directly in a physico-chemical way.*

There is no doubt that an analogous correlation between organs as described here for heart and stomach and for heart and gut will be established also between other organs[1]). We face here a wide field of new researches; we are only in the beginning.

May 1923. *The Physiological Laboratory of the University of Groningen.*

[1]) So it appeared very recently in our laboratory, that when stimulating the nervus vagus and the nervus sympathicus of the heart, substances are set free, which influence the *lumen of the small arteries* of another animal. (Note after the correction).

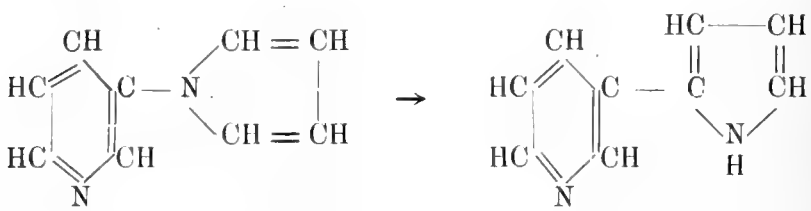
Chemistry. — "*The Synthesis of some Pyridylpyrroles.*" By Dr. J. P. WIBAUT and Miss ELISABETH DINGEMANSE. (Communicated by Prof. P. VAN ROMBURGH.)

(Communicated at the meeting of March 24, 1923.)

In the course of the researches on the structure of the natural alkaloids, several of these vegetable bases have been prepared by synthesis. In other groups of vegetable substances, investigators have not only succeeded in building up the substances occurring in nature, but also closely allied bodies were obtained synthetically. In the group of the sugars, e.g., a number of monoses have been obtained which do not occur in the vegetable kingdom, but which are isomeric with or closely related to the sugars found in nature. Our knowledge of the chemical and biochemical properties of the monoses has been greatly improved by these synthetic researches. It seems not devoid of interest to try and build up an isomer of a natural alkaloid, in order to examine afterwards in what respect the isomeric substance is distinguished from the natural alkaloid, especially with regard of physiological and biochemical properties.

Keeping this end in view we will try to build up an isomer of nicotine.

In his synthesis of nicotine PICTET started from β-amino-pyridine; this substance was heated with mucic acid, through which N (β-pyridyl)-pyrrole (I) was obtained. At high temperature N (β-pyridyl) pyrrole undergoes an isomerisation, in which C (β-pyridyl)-pyrrole (II) is formed:



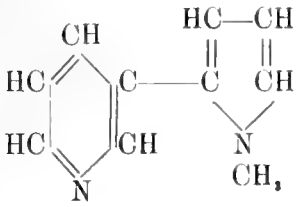
I. II.

PICTET and CRÉPIEUX [1]) give the above structure to this C (β-pyridyl)

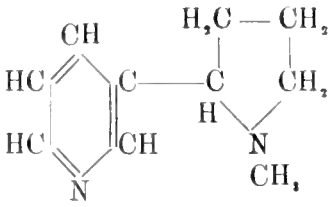
[1]) Ber. d. deutsch. chem. Ges. **28**, 1904 (1895).

pyrrole, in which it is, therefore, assumed that the pyridine nucleus is united at the α C atom of the pyrrole nucleus.

In how far this assumption is justified, will be discussed afterwards. The preparation of these substances did not offer any special difficulty; on the other hand, the conversion of C(pyridyl) pyrrole (II) into the methyl derivative, nicotyrine (III), was difficult to realize:



III.



IV.

When it is tried to methylate the pyrrol derivative at the nitrogen atom by treating the potassium-compound with methyl-iodide, there is also a molecule of methyl iodide combined with the nitrogen atom of the pyridine nucleus, so that the iodine methylate of nicotyrine is formed, from which afterwards methyl iodide must be split off.

PICTET and ROTSCHY[1]) have obtained but very little of the nicotyrine by this method. For the continuation of his experiments PICTET has, therefore, made use of a nicotyrine preparation which was prepared by oxidation from nicotine (IV) by BLAU's method.

A similar procedure is of course impossible in our case. In the end PICTET and ROTSCHY have succeeded in reducing nicotyrine to nicotine by an indirect way through making use of iodine and bromine substitution products.

Hence if this synthesis is repeated, starting from α-amino-pyridine, an isomer of the nicotine can be built up, in which the pyridine nucleus is substituted at the α-place.

As α-amino-pyridine is at present an easily accessible substance, it seemed not impossible to obtain sufficient quantities of all the intermediate products, so that it may also be expected that it will be possible to prepare so much of the final product that its properties can be properly studied.

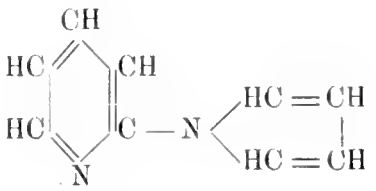
§ 2. *The preparation of N-(α-pyridyl)-pyrrole.*

For the preparation of N-(α-pyridyl)-pyrrole we have heated 25 gr. of α-amino-pyridine with 28 gr. of mucic acid. First the salt of

[1]) Ber. d. deutsch. chem. Ges. **37**, 1225 (1904).

mucic acid with 2 mol. α-amino-pyridine is formed. At a temperature of about 140° this salt begins to decompose: with separation of water and carbon dioxide the pyrrole derivative is formed while 1 mol. amino pyridine is split off. Hence a distillate is obtained which contains besides water, the required pyrrole derivative and amino pyridine. We have subjected the reaction product to fractionated distillation at 15 m.m. pressure. The first fraction of 104—130° is chiefly α-amino pyridine. At 140—145° distills a liquid, of a slight yellow colour, which solidifies to a white crystalline mass on being cooled in ice. The melting-point of these crystalls is 17° C.

This substance is N-α(pyridyl)-pyrrole, to which the following structure formula (V) applies.

V 

The freshly distilled N-(α-pyridyl)-pyrrole is a colourless liquid, which, however, assumes a dark colour after some time. The boiling-point at 760 mm. lies at 260—261°.

This substance is sparingly soluble in cold water, volatile with water vapour, and readily soluble in all organic solvents. A pine-chip moistened with hydrochloric acid is coloured red-violet by the vapour of N-(α-pyridyl)-pyrrole; with a hydrochloric acid solution of dimethylaminobenzaldehyde there arises a red-violet colour, which later on changes into a dirty green. These colour reactions are considered as characteristic of pyrrol derivatives. By potassium permanganate this compound is rapidly oxidized already at the ordinary temperature.

The values of 19.58% N and 19.34% N were found for the nitrogen percentage of this preparation, the calculated percentage for $C_9H_8N_2$ being 19.44%. We have prepared a picrate of this substance which melts at 143°. We obtained the iodine methylate of the N (α-pyridyl)-pyrrole by heating it in a sealed tube at 100° with the calculated quantity of methyl iodide. The reaction product was recrystallised from alcohol: yellowish white prisms, melting-point 141°—142°.

The isomeric N (β-pyridyl)-pyrrole prepared by Pictet and Crépieux has been described by these investigators as a liquid with

a boiling-point of 250.5—251° at 730 mm., which does not solidify at —10°.

The yield of N(α-pyridyl) pyrrole was in our experiments from 7 to 8 gr. out of 25 gr. of α-aminopyridine.

We found, however, that there is formed another substance besides this pyrrole derivative in the reaction between mucic acid and amino pyridine. During the distillation of the reaction product a liquid went over at 170°—190° and 15 mm., which crystallized at room temperature. After recrystallisation from alcohol this substance had a melting-point of 95°, and appeared to be α-α'-dipyridyl amine. The formation of this compound during the heating of the mucic acid salt of amino pyridine seems to be analogous to the formation of diphenylamine from aniline and hydrochloric aniline.

We have actually obtained α-α-dipyridyl amine by heating equivalent quantities of α-amino pyridine and the hydrochloric acid salt of this base in a sealed tube for two hours at 300°. We hope to return to this reaction on another occasion.

§ 3. *The conversion of N(α-pyridyl)-pyrrole into two isomeric C(α-pyridyl)-pyrroles.*

It was found long ago by Ciamician [1]) and his collaborators that the N-derivatives of pyrrole can be transformed into C-derivatives by the action of high temperatures.

Ciamician and Magnaghi [2]) heated N-acetyl pyrrole in a sealed tube at 250—280° and found that part of the starting material was changed into pyrryl methyl ketone:

That the acetyl rest actually occupies the α-position in the pyrrole nucleus, results from the observation that the bromation product of this pyrryl methyl ketone yields the imide of di-bromomaleic acid by oxidation with nitric acid [3]). Also some other pyrrole derivatives, in which an acylrest is combined with the nitrogen atom, were transformed into α-pyrrylketones on heating.

It was found later by Pictet and his collaborators that N-methyl pyrrole, N-phenyl-pyrrole, and similar substituted derivatives of

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[1]) Cf. Ciamician. Ber. d. deutsch. chem. Ges. **37**, 4200 (1904).

[2]) Ibid. **18**, 1828 (1885).

[3]) Ciamician and Silber. ibid. **20**, 2594 (1887).

pyrrole can be transformed into C-derivatives by distillation through a red-hot tube.

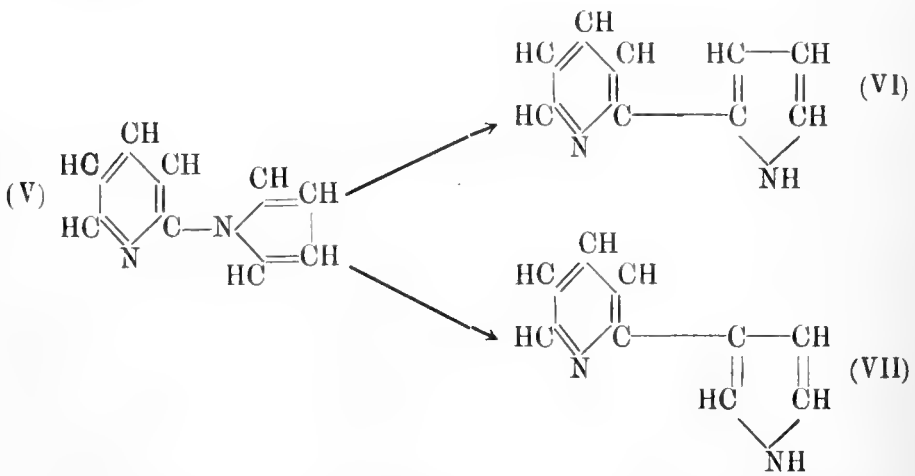
In all these intra-molecular arrangements only one C-derivative was found, whereas it would be theoretically possible that two isomeric pyrroles would be formed, since the hydrocarbon rest might be united at the α- or at the β-carbon atom of the pyrrole nucleus.

From N-methyl pyrrole the α-C-methyl pyrrole was obtained by PICTET. The structure of α-C-methyl pyrrole had already been determined by ZANETTI, by converting this substance into the dioxime of levulic aldehyde.

PICTET and CRÉPIEUX assume on grounds of analogy that in the C-phenyl pyrrole which they obtained from N-phenyl-pyrrole, the phenylgroup is united at the α-carbon atom of the pyrrole nucleus, and that the same thing holds for the C-pyridyl pyrrole (II), which they obtained from N-β(pyridyl)- pyrrole (I). A direct experimental evidence, for this view was not given.

As regards PICTET and CRÉPIEUX' β-pyridyl-α-pyrrole, the structure which these investigators assign to it, is undoubtedly supported by the fact that they have obtained nicotyrine (III) from this β-pyridyl pyrrole, as the structural formula (IV) of nicotine has been made very probable by PINNER's researches.

We found however that two isomeric C-pyridyl-pyroles are formed in the transformation of N(α-pyridyl) pyrrole, one of which melts at 93° and the other at 132—132.5°. This reaction must be represented by the following scheme:



To which of our isomers formula VI applies and to which formula VII should be assigned, has not yet been established. It

will not be very easy to decide this point. In former researches it has been tacitly understood that in the transformation of a N-derivative of pyrrole in a C-derivative, it is always the α-compound that is formed. In consequence of our observations the validity of these conclusion has become doubtful.

We shall now give a short description of our experiments on these reactions.

In the first place we have determined the most favourable temperature for the transformation of N(α-pyridyl) pyrrole into the C(α-pyridyl) pyrroles, as in PICTET's papers the reaction temperature is only vaguely indicated as "heated to redness" or "faintly red-hot". The best procedure appeared to be as follows:

25 gr. of N(α-pyridyl)pyrrole are distilled through a glass tube filled with pieces of pumice, which is heated at 670°—690° C. in an electrical oven. Part of the substance is decomposed, which shows itself in the formation of dense white vapours. The distillate consists of a black liquid, which soon solidifies at room temperature. This reaction product was distilled with steam, in which a white crystalline substance passed over, which was filtered off. This substance appeared to be very sparingly soluble in cold water. The crude product melted at 84°; after recrystallisation from a mixture of benzene and ligroine the melting-point is 90°. The yield of this substance was about 12 gr. The aqueous distillate contained only very little unchanged N(α-pyridyl) pyrrole. A second substance remained behind in the distillation flask, which is not volatile with water-vapour, and which after recrystallisation from hot water melts at 132—132.5°.

Properties of the pyrridyl-pyrrole melting at 90°.

This substance is obtained from benzene, to which some ligroine has been added, in hard, very shiny, colourless octohedrical crystals. We found 19,41 % for the nitrogen content; 19,44 % was calculated for $C_9H_8N_2$.

This substance is readily soluble in alcohol, ether, chloroform acetone and benzene; less easily in hot water and ligroine, very little in petroleum ether. These solutions exhibit a blue fluorescence, except the aqueous and alcoholic solution. A solution of β-pyridyl α-pyrrole also shows fluorescence according to PICTET and CRÉPIEUX.

Our pyrrole derivative does not give a colour reaction with a pine-chip moistened with hydrochloric acid; with a hydrochloric acid

solution of dimethyl-aminobenzaldehyde there appears, however, a red-violet colour.

Metallic potassium acts on this substance: a potassium compound is formed, as is to be expected. For this purpose we dissolved the substance in toluene, and let the potassium act at the boiling temperature of the solution. At first the action proceeds pretty rapidly, but it soon slows down, so that the heating must be prolonged. The potassium compound was deposited as an insoluble yellow-brown powder.

In order to ascertain the structure of the C-(*α*-pyridyl)-pyrrole, we have oxidized two grammes of this substance with potassium permanganate in sulphuric acid solution. The oxidation takes place very readily at the ordinary temperature. Out of the reaction product we have isolated the characteristic violet copper salt of picolinic acid, and from this salt we freed the picolinic acid itself by addition of sulphuretted hydrogen. The picolinic acid thus obtained was sublimated in order to purify it. The sublimated preparation melted at 134°.2, while we found 136°.8 for the melting-point of picolinic acid obtained by oxidation of picoline. The melting-point of the mixture of these two preparations was 132.5°—133°. The nitrogen percentage of our preparation that melted at 134.2, was 11.25 % (calculated for picolinic acid 11.38 %). In spite of the slightly too low melting-point there is no doubt of the identity of our preparation; the characteristically crystallizing platinum salt had exactly the same appearance as the platinum salt of the picolinic acid prepared from picoline. It appears from this that in the pyrrole derivative melting at 90° the pyrrole nucleus is united to the *α*-C-atom of the pyridine nucleus.

We have prepared a picrate from this pyridyl pyrrole, which was obtained after recrystallisation from alcohol as fine, yellow needles of the melting-point 227—228°.

We have prepared the iodine methylate of the pyridyl pyrrole melting at 90° by heating this pyrrole derivative in methyl alcoholic solution with an excess of methyl iodide at 100° for three hours. After evaporation of the solvent and of the superfluous methyl iodide the reaction product was recrystallized from methyl alcohol; in this way yellow-brown hard prism-shaped crystals were obtained, which melt at 148°. We found 9,6 % for the nitrogen content, and 44.7 % for the iodine content. The calculated values for $C_{10}H_{11}N_2J$ are N: 9,73 %; J: 44,37 %. This substance has, therefore, been formed by the addition of one molecule of methyl iodide; the group CH_3J is combined with the nitrogen atom of the pyridine nucleus.

Properties of the pyridyl-pyrrole melting at 132° 5.

This substance, which as we already remarked, is not volatile with water vapour, and is separated in this way from the isomer melting at 90°, crystallizes from alcohol or benzene in leaves joined to rosettes; from hot water long needles are obtained.

This base is readily soluble in alcohol, ether, acetone, chloroform, and benzene; not so easily in ligroine and hot water, very little soluble in low-boiling petroleum ether. As far as the solubility properties are concerned, there is, therefore, a close agreement with the isomer melting at 90°. The ethereal solution shows a blue fluorescence.

We found 19,34 % and 19,62 % for the nitrogen content (calculated for $C_9H_8N_2$: 19,44 % N). This base does not give a colour reaction with a pine-chip moistened with hydrochloric acid; with a hydrochloric acid solution of dimethylamino-benzaldehyde there appears, however, a cherry-red colour, which has changed into blueviolet after a day.

That this substance, too, possesses a pyrrole nucleus, appears again from the behaviour towards metallic potassium. The base was dissolved in toluene and the calculated quantity of potassium was added. The potassium dissolves with vigorous generation of hydrogen; the reaction is much more rapid than with the isomer of melting-point of 90°. The potassium compound is deposited as a white powder.

We have oxidized the pyridyl pyrrole of melting-point 132.5 in the same way with potassium permanganate in an acid solution, as we already described for the isomer of melting-point of 90°. From the pyridyl pyrrole melting at 132°.5 we likewise obtained picolinic acid, which melted at 136°.8 after sublimation, and was identical with the picolinic acid from picoline.

It results from these experiments that the two substances that are formed from N-(*α*-pyridyl)-pyrrole, are two isomeric C-(*α*-pyridyl)-pyrroles, which are distinguished in this that the pyrrole-nucleus in one substance is substituted at the *α*-place, and in the other substance at the *β*-place, as is expressed in formulae (VI) and (VII).

We may also mention that in this reaction chiefly the isomer melting at 90° is formed; the quantity of the isomer melting at 132°.5 is small.

§ 4. *The methylation of the C-(α-pyridyl)-pyrole of melting-point* 90°.

The next step in the synthesis of a substance isomeric with nicotine is that the hydrogen atom of the imide group of the pyrrole-nucleus is replaced by the methyl rest.

The difficulties experienced by PICTET and CRÉPIEUX when they endeavoured to realize the reaction, were already pointed out in the introduction. We met with the same difficulties in our case. The potassium compound of the pyridyl pyrrole melting at 90° was heated with an excess of methyl iodide in a sealed tube at 100° for three hours. The reaction product was freed from superfluous excess of methyl iodide and solved in water. On evaporation of the aqueous solution crystals were separated, while potassium iodide was present in the mother liquor. These crystals were purified by recrystallisation from a small quantity of water. Yellow-brown crystals were obtained, melting at 186°. Analysis gave 8.95 for the nitrogen percentage, and 42,55 for the iodine percentage. Calculated for $C_{11}H_{13}N_2I$: Nitrogen 9,34 %, iodine 42.30 %.

This substance is, therefore, the iodine methylate of C (α-pyridyl)-N-methyl-pyrrole: $(CH_3I)\,N{-}C_5H_4\,.\,C_4H_3N\,.\,CH_3$.

Just as in PICTET and CRÉPIEUX' experiments not only was the nitrogen atom of the pyrrole nucleus methylated, but also a molecule of methyl iodide had combined with the nitrogen atom of the pyridine-nucleus.

This iodine methylate is easily soluble in water, sparingly in alcohol, very little soluble in the other usual organic solvents.

In order to split off the group CH_3I out of this compound, we have followed the method which PICTET and ROTSCHY [1]) already applied, i.e. heating with quick lime.

The jodine-methylate was mixed with quick lime, and slowly heated in a retort. Soon a liquid distilled over, which was received in ether, in order to separate it from a little of the unchanged methyl iodide compound, which had also been distilled over in a small quantity. After evaporation of the ethereal solution there was left a light yellow liquid; we have converted this base into the picrate, which melted at 143° after a double recrystallisation from alcohol. We found 18.19 for the nitrogen percentage of this substance, while 18.09 % of nitrogen is calculated for the monopicrate of C (α-pyridyl)-N-methyl-pyrrole,

We have, accordingly, very probably obtained the required methyl derivative, which must, therefore, be an isomer of nicotyrine.

It seems, however, possible to carry out the methylation of the C (α-pyridyl) pyrrole in such a way that the C (α-pyridyl) N-methyl-pyrrole is obtained without the necessity of following the indirect way over the iodine methylate.

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[1]) L. c.

It had, indeed, already appeared that the addition of methyl iodide to the pyridyl pyrrole of melting point 90° only takes place at a higher temperature, whereas PICTET and CRÉPIEUX' pyridyl pyrrole combines with methyl iodide already at the ordinary temperature.

For this reason we have heated a mixture of pyridyl pyrrole potassium with methyl iodide in molecular quantities in a sealed tube at 50°. The reaction mixture was a solid mass, in which pyridyl pyrrole potassium and the above mentioned methyl iodide compound of C-(pyridyl)-N-methyl-pyrrole were present. It was, however, possible to extract by means of ether a little of a yellow oil from this reaction mixture. This liquid was received in alcohol, and picric acid was added; a picrate crystallized out, which melted at 142° when it had been recrystallized out of alcohol, and appeared to be identical with the picrate of the C (*α*-pyridyl)-N-methyl-pyrrole described above, as appeared from the melting point of the mixture of both preparations.

We shall first of all set ourselves the task of preparing a larger quantity of this C(*α*-pyridyl)-N-methyl-pyrrole, and examining its properties closely. We shall further try to determine the structure of the two isomeric pyridyl pyrroles more exactly.

A full communication of this investigation will appear in the Recueil des Travaux chimiques des Pays Bas.

Organic-chemical Laboratory of the University.

Amsterdam, March 1923.

Bacteriology. — "*The splitting of lipoids by Bacteria.*" (First communication.) By G. M. KRAAY and L. K. WOLFF. (Communicated by Prof. C. EYKMAN.)

(Communicated at the meeting of June 30, 1923).

The splitting of fats by bacteria has often been investigated and the behaviour of the lipases has properly been recorded. However no literature dealing with the splitting of lipoides by *bacteria* is known to us. Also in general physiological chemistry little information is given concerning the splitting of lipoids (lecithin) by enzymes, apart from the beautiful researches by DELEZENNE and FOURNEAU about the splitting of lecithin by serpent venom. In many respects we thought it of interest to investigate the action of bacteria on lipoids, the formation of strong blood poisons being possible, as DELEZENNE and FOURNEAU found as the result of the action of serpent venom on lecithin. We first tried to find out whether some fat-splitting bacteria are able to split lecithin and further if there exist among the non-fat-splitters some that will split lecithin.

Considering our working method this; we mostly used plates with lecithin agar obtained by shaking up a small quantity of lecithin and ordinary nutrient agar (about 0.5 gram per 100 gr.) at about 50° C. If the lecithin is affected an area is formed all around the streeks of inoculation.

It appears on microscopical examination that this area contains per surface unit more grains than are to by found anywhere else in the culture medium. Plates with yolk of egg cannot be used; the fat contents of yolk of egg cannot be used; the fat contents of yolk of egg makes one unable to distinguish lecithin-splitters from fat-splitters. Our results are summarized in the following table. Our conclusions based upon this table are: there exists fat-splitting bacteria unable to affect lecithin; lecithin-splitting bacteria unable to act upon fat, bacteria unable to act upon both fat and lecithin, and bacteria able to act upon both. (See table on p. 437).

The latter bacillus, a very strong lecithin splitter, but quite unable to split fat has been isolated by us from brackish water; this bacillus resembles much the bac. piscium pyogenes described by MATZUSCHITA.

	Splitting of	
	fat	lecithin
bact. typhi	—	—
„ coli	—	—
„ dysenteriae Shiga	—	—
„ prodigiosus	+	+
„ pyocyaneus	+	+
„ fluor. liquef.	+	+
„ proteus[1])	—	—
staphylocc. pyogenes	+	—
spir. El Tor.	—	+
„ Dunbar	—	+
„ Cholerae	—	—
Bac. piscium pyogenes?	—	+

We have not yet resolved the question, how the lecithin is broken down; we can only say that as a result of the splitting by the here above mentioned bacteria no hemolysines are formed. We could not find a link between hemolysis by bacteria and lipolysis or lipoidolysis; we found a staphylocc. which had lost its hemolytic property but not its lipolytic character and on the other hand one of our colistrains behaved hemolytic but was inactive on fat or lecithin, our bac. piscium pyogenes splitted lecithin but had no hemolytic action.

No fatty acids could be titrated in broth containing splitted lecithin (B. piscium prog.). This result is in agreement with observations on the non-hemolytic action of the splitted lecithin, because if lecithin is splitted in such a manner that (unsaturated) fatty acids are formed, a hemolytic action must take place.

We still want to mention that the power of splitting of the bacteria in the table, has been tried on cholesterol and lanoline, the latter was affected only by a staph. pyog., the former only by B. pyocyaneus.

June 1923. *Laboratory of hygiene of the University of Amsterdam.*

[1]) One of our proteus strains affected fat.

Physiology. — "*The Presence of Cardio-regulative Nerves in Petromyzon fluviatilis*". By J. B. ZWAARDEMAKER. (Communicated by Prof. H. ZWAARDEMAKER.)

(Communicated at the meeting of March 24, 1923).

In the 2nd edition of his "Physiologie des Kreislaufs" TIGERSTEDT [1]) remarks that inhibitory cardiac nerves are present in nearly all vertebrates. Only among the cyclostomata some exceptions are known. GREENE [2]) found that in Myxine electrical stimulation, starting from the brain, the spinal cord or the vagi did not affect the frequency of the heart-beat. CARLSON [3]) corroborated this finding and tried to extend the investigation to another group of cyclostomata, viz. the petromyzonta. At first he could work only on the larval form, in which cardio-regulative nerves appeared to be absent. Afterwards he examined adult animals [4]). When, in these experiments, he applied an electrical stimulus to the medulla oblongata on the level of the vagus nucleus, he noted a brief standstill, which was followed by an accelerated rhythm. From this he concludes that "the central nervous system is connected with the heart by ordinary augmentor and probably also by inhibitory nerves" (l. c. p. 231).

In the continuing volume of his "Vergleichende Anatomie der Myxinoiden" JOHANNES MÜLLER makes mention of a connection between. N. sympathicus and cardiac nerves [5]). He also adds some remarks about the N. vagus, for which I think it better to refer to the original work (l. c. p. 59 sqq.)

The first experiments which I made myself to ascertain whether in petromyzon fluviatilis any influence is exerted by the central nervous system upon the heart's action, yielded a negative result, which was in accordance with GREENE and with the first set of experiments performed by CARLSON [6]). However, I have been in a position to extend my research. In order to preclude

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[1]) R. TIGERSTEDT, Die Physiologie des Kreislaufs II p. 319.

[2]) CH. W. GREENE, Amer. Journ. of Physiol. VI p. 318 1901.

[3]) A. J. CARLSON, Zeitschr. f. allg. Physiol. IV p. 259 1904.

[4]) A. J. CARLSON, Amer. Journ. of Physiol. XVI p. 230 1906.

[5]) J. MÜLLER, Fortsetzung der vergleichenden Anatomie der Myxinoiden p. 57. Berlin 1838.

[6]) J. B. ZWAARDEMAKER, Physiologendag Amsterdam Dec. 1922.

movements of the animal I curarized it beforehand. Paralysis of the skeletal muscles can, in fishes, be effected only with very large doses [1]). For my animals I used 4 mgr. tubo curari of which, 2 mgr., injected intraperitoneally, was sufficient to paralyze a 220 gr.-rat. after 7 minutes. This also plays an influence upon the vagus-function [2]), but this inconvenience could readily be obviated by the technique followed, because the synapses of the vagus are restored sooner than the motor innervation.

After the injection the animal was let alone until no "Stellreflexe" were distinguishable any longer. Also the gills are completely motionless then. At that juncture the cerebrum is severed from the rest of the nervous system by an incision posteriorly along the eyes. After this the cerebrum and the spinal cord are laid bare down to the second gill-hole. Now a straight glass cannula is inserted into the Vena cava dextra, through which the animal, in ventral position, is perfused during some time with RINGER's fluid, containing $6^1/_2$ gr. NaCl, 200 mgr. $NaHCO_3$, 200 mgr. $CaCl_2$, 200 mgr. KCl [3]). The surplus of curari is hereby gradually washed out. Through a window in the cartilagenous pericardium [4]) the atrium is fixed to a lever beneath the animal. Now two thin platinum electrodes are fixed, so as to be well visible, at the level where stimulation produces the effect aimed at. With strongly curarized animals it sometimes takes rather a long time before any effect can be distinguished. At that moment, however, the animal is perfectly quiet, and the experimenter can be sure that only the movements of the heart are registered. In subsequent periods of the perfusion also the contraction of the gills can be distinguished. The electrodes are connected with the secondary coil of an inductorium of DUBOIS-REYMOND, provided with a NEEF-hammer. An accumulator is connected up in the

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[1]) *a.* J. SCHIFFER, Arch. f. Anat. u. Physiol. p. 453, 1868.
b. J. STEINER, ibid 1875.
c. BOLL, Mon. Ber. d. Kgl. Preuss. Akad. d. Wissensch. Nov. 1875.
d. J. STEINER, Das americanische Pfeilgift Curare p. 56.
c. and *d.* After R. BOEHM's article in Handbuch der experimentellen Pharmacologie II 1. Hälfte p. 183.

[2]) R. BOEHM, l. c. p. 202.

[3]) J. B. ZWAARDEMAKER, Diss. Utrecht 1922.

[4]) When the pericardium is being opened it all at once changes colour. Originally the heart is seen to loom vaguely through the transparent cartilaginous tissue with a bluish tint; after the opening the pericardium shows its own milkwhite colour, while the atrium now appears to lie at tbe bottom of the cavity. Apparently in the pericardium a negative pressure obtains, which of course is lost at the opening, so that the atrium partly collapses.

primary circuit. The Pfeilsignal, which was used sometimes (e.g. in the first figure), could not be placed in shunt, so it came in the primary circuit. The obtained coil-distances (C. d.) are smaller than when no signal is connected up. On stimulation we note a considerable acceleration shortly after the stimulus has been set up. If the stimulus continues a short time only (in fig. 1 5 seconds) the acceleration will be seen to disappear soon and to be substituted by a retardation; in case the latter increases, the heart is brought to a standstill. After cessation of the negative chronotropic effect,

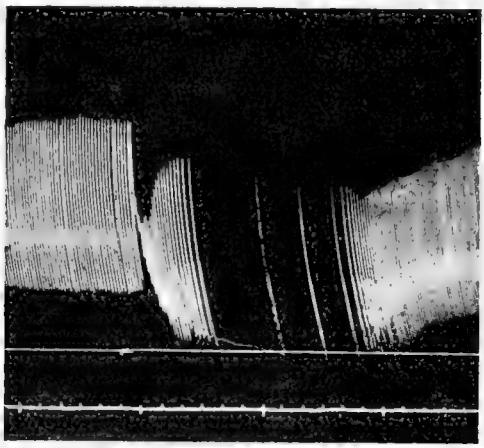

Fig. 1.
Accelerans-vagus effect.
Petromyzon fluviatilis. Perfused with RINGER's mixture. Stimulation for 5 seconds of medulla oblongata of the level of the exit of the N. vagus. C. d. 100.
The tracings from above downward: record of atrium movement
" " stimulus signal
" " time line 10 sec.

a new rhythm appears, more rapid than the original. A little later it gives way to the old rhythm. In fig. 1 the rhythm prior to the stimulation is ± 45 beats per minute, after the standstill the frequency amounts to 55. The action of side-currents upon the heartmuscle need not be taken into consideration in these experiments, because the effect appears only when a sharply defined area in the medulla oblongata is stimulated and the effect is destroyed again by a slight displacement of the electrodes. Besides this a great influence is exerted by summation. A stimulus, for instance, that produces no effect after 5 seconds, causes a distinct standstill after a longer period.

When instead of presenting a short stimulus, the current is sent through permanently, at first a marked quickening of the rhythm will be noted, attended with a marked positive inotropic effect. This is apparently an accelerans effect.

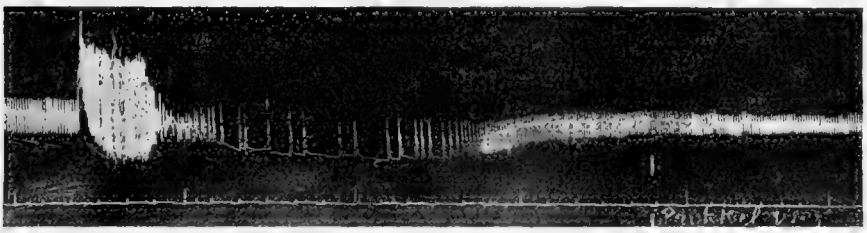

Fig. 2.

Petromyzon fluviatilis.

Fatigue of accelerans and vagus through permanent stimulation from medulla oblongata. The stimulus starts at the first elevations. C. d. 143. This continues as far as the stroke. Time 10 sec.

When breaking the current during this period a standstill will rapidly ensue, which will disappear again directly after fresh stimulation. When, however, the current passes continuously, a slower rhythm will appear after some time spontaneously (in fig. 2 ± 30 seconds), while at the same time the height of the contractions diminishes gradually. It is the transition to a distinct vagus-effect. When this rhythm has also continued for some time (in fig. 2 about 1 min.), it will change into a rhythm that is only slightly quicker than the normal, or does not differ from it at all, and will persist unaltered after the breaking of the current.

When perfusing the animal with a potassium-free uranium-containing, instead of a potassium-containing fluid we shall see that the phenomena are practically the same in the K-, and in the U-condition. First we see an acceleration, then a retardation, which in some cases is followed again by an acceleration. This, however, is never so pronounced as at the beginning of the stimulation.

What has been said above goes to show that:

1. in Petromyzon fluviatilis cardio-regulative nerves are present.

2. with the technique employed after the removal of curari the excitability of the cardiac nerves returns sooner than that of the motor nerves.

3. in the curarized animal the latent period of the accelerans is shorter than that of the vagus.

4. with long-continued stimulation the accelerans-effect is noticeable before the vagus-effect.

5. with brief stimulation the vagus effect appears only after cessation of the stimulation.

6. after cessation of the vagus-action an acceleration will sometimes follow, which is perhaps due to a longer after-effect of the accelerans-stimulation.

Chemistry. — "*The Light Oxidation of Alcohol* (III). *The Photo-Catalytic Influence of some Series of Ketones on the light Oxidation of Ethyl Alcohol*". By W. D. COHEN. (Communicated by Prof. J. BÖESEKEN).

(Communicated at the meeting of May 26, 1923).

Introduction. A first communication on this subject appeared in these proceedings [1]) already several years ago; a continuation of this was published by BÖESEKEN [2]). In this paper the theoretical grounds on which these researches are founded, are set forth in extenso [3]), and we may, therefore, refer to this treatise for a study of them.

It was now my purpose to examine what relation exists between the configuration of a ketone and its photo-catalytic influence on the oxidation of a definite alcohol, and for this reason I studied the influence of some series of ketones on the velocity of oxidation of ethyl alcohol, to be able, if possible to arrive at a conclusion with regard to the constitutive requirements which a ketone must satisfy to be able to act as a photo-catalyst under the circumstances specified later, which at the same time establishes its photo-chemical attackability. This question has, indeed, already been mentioned more than once before [4]), but the comparatively small regularity in the observed phenomena rendered an extension of the research in this direction very desirable.

The light-thermostat. In the reaction:

$$\text{Light} + \text{Ketone} + \text{Alcohol} + \text{Oxygen} = \text{Ketone} + \text{Aldehyde} + \text{Water}$$

a certain quantity of oxygen disappears, and the rapidity with which the oxygen is absorbed, is under for the rest fixed circumstances, a measure for the photo-catalytic activity of the examined ketone.

The light thermostat (fig. 1) consists of a copper trough, provided with two windows placed opposite each other in the longitudinal walls, which make a continual observation of the reaction vessel

[1]) BÖESEKEN and COHEN, These Proc. XVIII, p. 1640.

[2]) BÖESEKEN, Rec. 40, 433 (1921).

[3]) Ibid, 437.

[4]) COHEN, Rec. 39, 258 (1920). Chem. Weekblad 13, 902 (1916).

possible, and a window in the bottom for the illumination. The thermostat rests on an iron framework, which has become an entirely

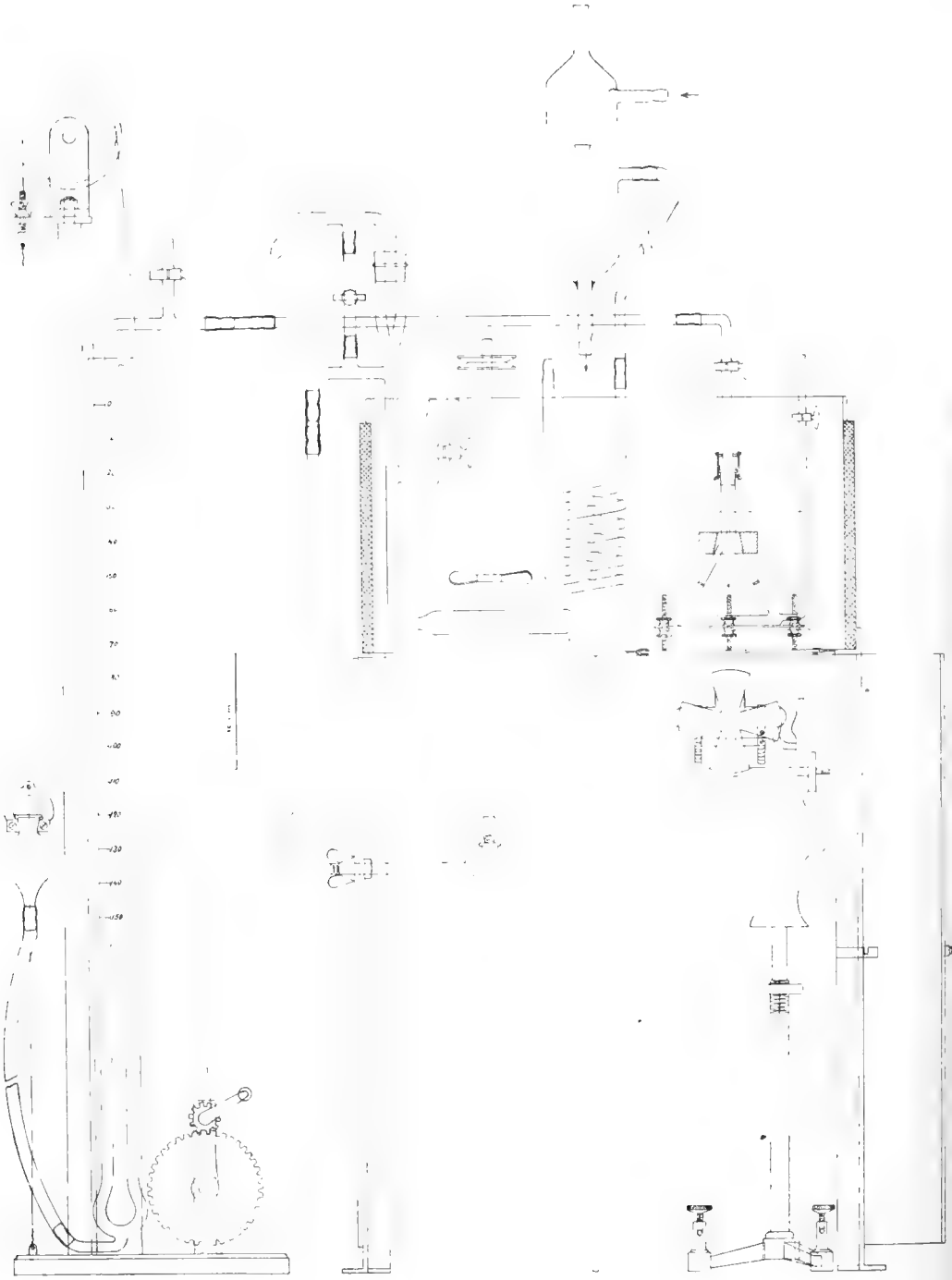

Fig. 1.

closed space by a cover of incombustible material. This space is divided in two by a vertical partition. On the left there are found two gas-burners connected with a thermo-regulator, and on the right there is adjusted a Heraeus quartz lamp. To work this the wall in the lefthand side of the framework is made like a door (drawn halfopen in the figure); in the front partition at the place of the incandescent body there is a ventilator which works by suction and serves to cool the lamp. The water in the thermostat can further be cooled by means of a cooling spiral, through which water flows under constant pressure, a screw stirrer ensuring thorough mixing in the trough; besides the windows, the vertical walls are insulated with felt. Ventilator and stirring apparatus are worked by separate regulatable motors. The temperature of the thermostat can be kept constant at $35 + {}^{1}/_{100}{}^{\circ}$, which temperature has been chosen, because at this temperature the thermostat can be regulated most accurately.

As reaction vessel I, at first, used the before described stirring-apparatus[1]) (fig. 2); it possesses the drawback, however, that the surface of illumination is small, the accuracy of the measurement being seriously impaired by the rapid contamination of the mercury in the mercury feal. Therefore I tried to modify the reaction vessel in such a way that also without intensive mixing of gas and liquid, an alcoholic liquid could be obtained, which remains saturated with oxygen, or contains at least such an excess of oxygen that there can be no question of measuring a velocity of diffusion instead of a velocity of reaction.

This is possible when the thickness of the liquid layer is taken very small (about 1 mm.). According to fig. 3 a reaction vessel is then obtained, which chiefly consists of a flask with a perfectly flat bottom; the dimensions being such that 5 cc. of liquid give a thickness of layer of 5 mm. The neck is narrow and possesses a ground piece to which a bent capillary tube with tap can be attached. Near the bottom there is further a side tube with tap, through which the whole apparatus can be filled with oxygen. Besides there is a filling body in the flask, to make the gas-volume as small as possible in proportion to the surface of illumination; this considerably enhances the accuracy of the measurement. For definite purposes this filling body has been made to a second reaction vessel within the former; then an apparatus is obtained as is shown in fig. 4.

By the aid of a narrow tube the reaction vessel is connected with the micromanometer. The lefthand leg of this has a capacity

[1]) These Proc. XVIII, p. 1642.

of 1,5 cc. and is divided into 150 parts. Each space between two dividing lines represents, therefore, a capacity of 0.01 cc. The adjustment is obtained by moving the flask up and down by means of a hoisting apparatus, the position of the meniscus in the two

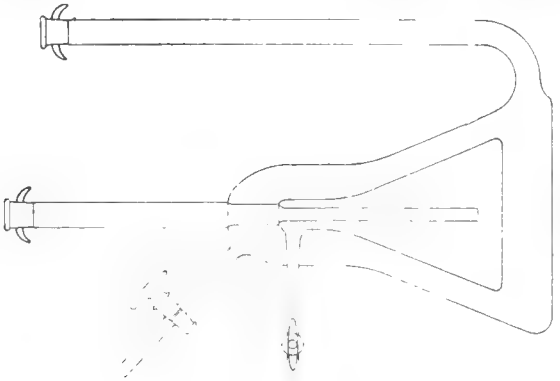

Fig. 5.

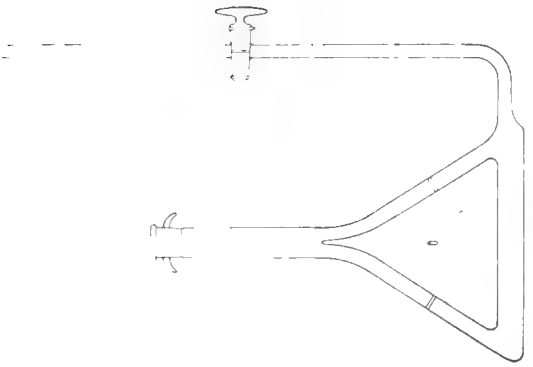

Fig. 3.

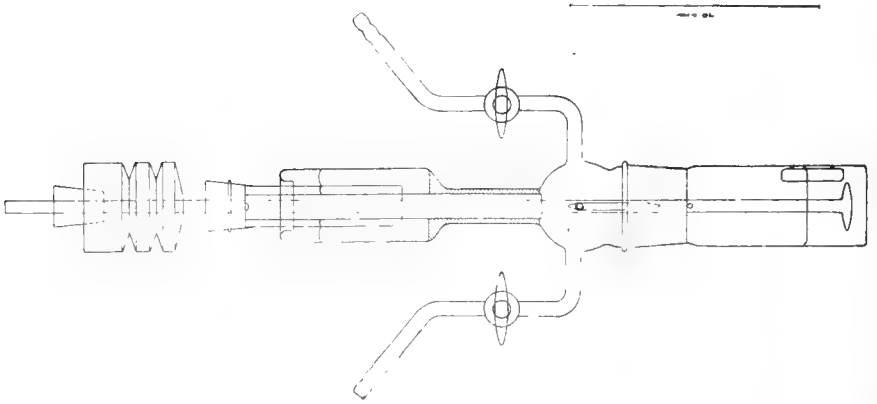

Fig. 2.

legs of the manometer being verified by a mirror behind it, on which horizontal lines are drawn at distances of 1 mm. When the apparatus has been properly cleaned and filled with distilled mercury, an accuracy of adjustment can be attained of 1 or 2 hundredths, which as well as the influence of the temperature lies within the limits of the error of observation. The calibration error of the apparatus was so small that it could be neglected.

After having been weighted with a copper ring, the reaction vessel is placed on a glass table, which itself rests on the bottom plate of the thermostat. The table can be put in a horizontal position by means of three adjusting screws, through which the thin liquid layer entirely covers the bottom surface of the reaction vessel. After being lit the incandescent body of the lamp is always placed in a horizontal position; the lamp burns at a terminal voltage of 110 Volts and a series resistance of about 20 Ω constant at 2,7 Amp./40 Volt. Lamp and reaction vessel are always at the same distance from each other; in my experiments the distance from the bottom side of the reaction vessel to the window was 20 mm., and from the upper side of the lamp to the window 25 mm.; taking into account the thickness of the glass, the mutual distance from lamp to object was about 50 mm.

The measurements.

a. The preparations. They were prepared for the greater part by myself or under my supervision, and purified as carefully as possible. As the way of preparing is known for all of them, we may refer for this to the records of the literature published on this subject. When it was possible, at least two preparations of different origin were examined, or the preparation was again recrystallized or distilled after the measurement; the values found were not considered as definitive until they were perfectly constant and reproducible; save for a single exception this was always the case.

b. As solvent, resp. liquid that is to be oxidized, was used absolute ethylalcohol, not because its being absolute was quite indispensable for the success of the reaction — for water is formed during the reaction — but in order to start always from a solution of constant proporties. In my earlier investigations I had come to the conclusion that water would be a strong anti-catalyst, at least for the photo-chemical reduction [1]). At the time I did not yet know the photo-catalytic alcohol oxidation by molecular oxygen, nor that

[1]) Cohen Rec. 39, 244 (1920).

this reaction and the keton reduction were primarily the same, and that it is, therefore, illogical to assume that water would be an anti-catalyst in the ketone reduction. It has really appeared in a new investigation, that there would be no question at least of a considerable anti-catalytic action of water, but that the error made before, which has, unfortunately, already been adopted in the handbooks [1]), must be attributed to a wrong interpretation of the experiments made at the time.

It seems to me of use to discuss this a little more at length, if it were only to point out how easily certain phenomena are overlooked in the study of a reaction. For at first I made my experiments on the photochemical ketone reduction in such a way that I illuminated the 96 % alcoholic solution in a thin layer in *open* flasks, but did not observe then anything of the crystallisation of the sparingly soluble pinacone already described by CIAMICIAN [2]). This succeeded however without any difficulty when I used absolute alcohol — as CIAMICIAN also did —, and besides worked in *closed* apparatus, hence with exclusion of oxygen. I then drew the very plausible conclusion, which proved erroneous afterwards, that water would be a strong anti-catalyst, and quite overlooked the interesting photo-catalytic alcohol oxidation in which — the results of this paper are a convincing proof of this — aldehyde *does* appear, but no pinacone [3]), and which was not discovered until a few years later.

c. In order to be able always to have a great excess of oxygen at our disposal, the reaction vessel after addition of 5 cc. of the solution to be examined, is filled with oxygen which is saturated with alcohol vapour in a washing bottle. Under these circumstances the solution always remains more than sufficiently saturated with oxygen; it is, however, without influence on the result of the measurements, if the gas in the reaction vessel is air or oxygen; for the sake of safety oxygen was, however, always taken.

The measurements, the results of which are combined in the following table, extend chiefly over the following series of ketones:

a. benzophenon and its hydration products in the nucleus,

b. acetophenon and some alkyl-, and also phenyl substitution products in the CH_3-group,

c. the phenyl substitution products of acetone,

d. the simplest aliphatic, aromatic, and fat aromatic α-β-diketones,

e. some α-β-γ-triketones.

The figures over the horizontal division line indicate the molar

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[1]) HOUBEN—WEYL. Die Methoden der organischen Chemie 2te Aufl. (1922), Band II pag. 983.

[2]) CIAMICIAN and SILBER, Ber. 33 2911 (1900); 34 1530 (1901); 44 1288 (1911).

[3]) BÖESEKEN and COHEN, l.c.

concentration of the ketone, the values under it representing the oxygen absorption, expressed in cc. per hour. They are the mean of a great many mutually concordant observations.

1. Benzophenon.

1 (saturated)	3/4	1/2	1/4	1/8	1/16	1/32	1/64
10.30	**12.00**	**12.00**	**11.90**	9.00	5.60	3.65	2.28

2. Phenylcyclohexylketone.

1 (saturated)	1/2	1/4	1/8
5.15	**5.00**	2.82	1.12

3. Dicyclohexylketone. Inactive in all concentrations.

4. Phenyl n. hexylketone.

2	1	1/2	1/4
1.00	**0.97**	0.68	0.22

5. Di-n hexylketone. Inactive in all concentrations.

6. Acetophenon.

2	1 1/2	1	1/2	1/4
1.30	**1.40**	**1.42**	1.03	0.22

7. Propiophenon.

2	1	1/2	1/4
1.10	**1.11**	0.92	0.20

8 Phenylbenzylketone.

1/2 (saturated)	1/4	1/8
5.05	**4.85**	2.35

9. Diphenylacetophenon.

1/32 (saturated)	1/64
3.13	0.78

10. Triphenylacetophenon (β-benzpinacoline). Inactive.

11. Acetone. Inactive in all concentrations.

12. Monophenylacetone.

3	2	1
0.50	**0.48**	0.35

13. Symm Diphenylacetone (dibenzylketone).

2	1	1/2
1.76	1.75	0.85

14. Asymm. Diphenylacetone.

1 (saturated)	1/2
0.03	0.01

15. Triphenylacetone 1.1.2.

1/4 (saturated)
0.05

16. Symm Tetraphenylacetone [1]).

1/50 (saturated)
0.17

17. Phenylfurylketone.

2	1	1/2
0.07	0.10	0.10

18. Diacetyl.

4	3	2	1 1/2	1	3/4	1/2	1/4	1/8	1/16	1/32
16.00	15.30	15.30	15.10	14.90	14.10	10.60	6.40	2.60	0.64	0.16

19. Benzil.

1/4 (saturated)	1/8	1/16
3.20	1.44	0.52

20. Acetylbenzoyl [2]).

4	3 1/2	3	2	1 1/2	1	3/4	1/2	1/4	1/8	1/16
8.60	11.70	12.90	12.60	12.80	13.10	10.80	8.50	6.05	4.15	2.08

[1]) Prof. STAUDINGER, Zürich, had the kindness to send me a small quantity of this preparation.

[2]) By illuminating an alcoholic solution of acetylbenzoyl in a sealed tube the corresponding photoreduction product can be very easily obtained. The substance consists of very fine colourless crystal needles, sparingly soluble in alcohol, and is perfectly stable at the air in dry condition. For the rest the compound is quite comparable with the corresponding reduction product of diacetyl (Comp. Chem.

21. Furil.

1/16 (saturated)
2.80

Remark. After some time's illumination the absorption, which was constant at first, descends to 0; in this the ketone itself is attacked with decoloration of the liquid

22. Benzfuril.

2	1	1/2	1/16
5.80	6.30	6.20	2.20

23. Terephtalophenon.

0.01 (saturated)	0.006	0.004
2.80	3.55	3.45

24. Isophtalophenon.

0.1 (saturated)	0.01
2.80	1.48

25. Phenanthrenequinone.

0.02 (saturated)	0.01
10.50	6.25

Remark. Behaves like furil.

26. Anthraquinone.

0.004 (saturated)
0.67

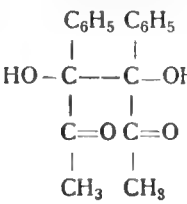
27. Camphorquinone. The activity varies with the origin of the preparations. Some five varied at 1/8 mol. from 0.19—0.43. With still lower concentrations, and also with very small ones the activity is practically not perceptible

28. Fluorenone. In all concentrations — also very small ones — inactive.

29. x-Hydrindon.

2	1	1/2
0.19	0.17	0.07

Weekbl. 13, 594 (1916); it melts amidst decomposition at 116°—124°. It is still uncertain whether the structure formula is:



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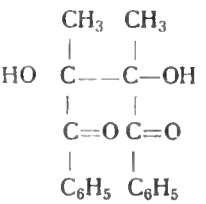


30. β-Hydrindon. Is useless as a photocatalyst, as this substance itself is very readily attacked by oxygen in alcoholic solution.

31. Indanedion 1.2.

$^1/_8$ (saturated)	$^1/_{16}$
0.92	0.39

32. Pentanetriketone. Inactive.

33. Diphenyltriketone. Inactive.

34. Alloxane. Exceedingly slight activity.

These data allow us to draw the following conclusions:

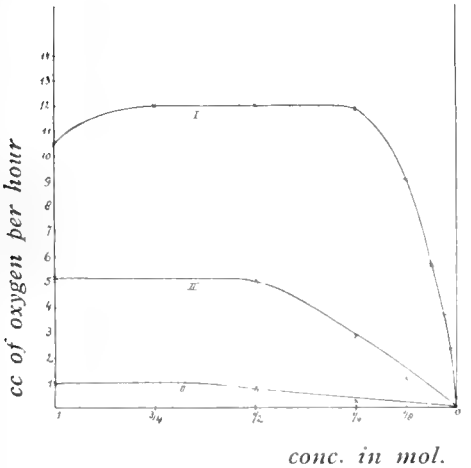
a. The velocities of activation are independent of the concentration of the ketone (printed in bold type in the tables) within comparatively wide limits, quite corresponding to the reduction velocities found before[1]). This phenomenon does not, indeed, manifest itself in all the examined cases, but it should not be forgotten that the circumstances of the experiment necessitate a certain degree of activity and solubility of the ketone to reach the maximum velocity of activation.

Clear examples in which the oxygen absorption remains constant within wide limits, are benzophenon, diacetyl, and benzoyl acetyl (compare the graphical representations in fig. 5 and 6). We sometimes see the activity diminish again in very high ketone concentrations (20) or in the neighbourhood of the point of saturation (1), which must then be attributed to mutual disturbances of the ketone molecules[2]). The diminution of activity in lower concentrations must simply be accounted for by the absence of a sufficient quantity of activable ketone molecules, in which part of the available light is left unused. That really in the concentration region of the maximum activation all the photo-active light is absorbed by a layer only 1 mm. thick, I have been able to prove very clearly by means of the reaction vessel according to fig. 4, which can, therefore, be perfectly compared with the "mantle tubes" described formerly for the photo-chemical reduction. When e.g. an alcoholic (or a benzolic) solution of benzophenon in a concentration necessary for the maximum activation is brought into the outer reaction vessel, a benzophenon solution in the inner reaction vessel appears to absorb no trace of oxygen; the absorption begins, however, to become immediately perceptible, as soon as the ketone concentration in the outer

[1]) Cohen, Rec. 39, 253 (1920).

[2]) Ibid. p. 273.

vessel descends below the critical. In the region of maximum activation all the photo active light is, therefore, arrested by a layer of 1 mm., and this takes place independent of the solvent used. These phenomena are in perfect harmony with what was found before in the ketone reduction. Corresponding experiments with diacetyl and benzoyl acetyl lead to perfectly the same results.



I. benzophenon
II. phenylcyclohexyl ketone } Fig. 5.
III. phenyl. n. hexyl ketone

III. " " "
IV. acetophenon
V. acetylbenzoyl } Fig. 6.
VI. diacetyl

Fig. 5.

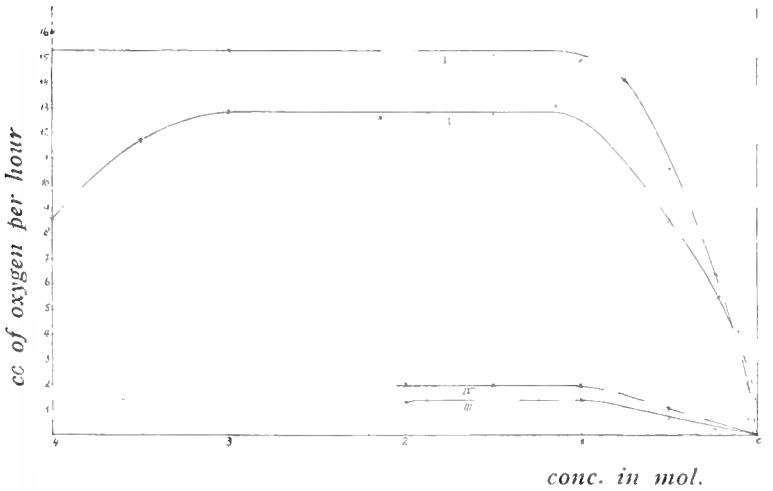


Fig. 6.

b. For the photo-activity of the mono-ketones the "aromatic", character is in general decisive [1]), constitutive factors being of influence by the side of it. Thus the photo-activity of benzophenon has been reduced to about half its value, when one of the nuclei

[1]) Cohen, Chem. Weekbl. 13, 902 (1916).

has been hydrated (2) (fig. 5), and it has quite disappeared in the dicyclohexyl ketone (3). That for the rest the cyclohexyl nucleus weakens the activity of the phenyl nucleus less than a purely aliphatic group, is proved by the much smaller activity of phenyl n. hexyl ketone (4) (fig. 5), which may be put on a line with the activity of acetophenon and propiophenon (6,7) (fig. 6). On introduction of C_6H_5-groups into the CH_2-group of acetophenon, the activity at first greatly increases (8,9), suddenly becoming 0 in triphenyl acetone. It is, indeed, known that β-benzpinacoline lacks all the ketone characteristics. In the phenyl substitution products of acetone (quite inactive in themselves, just as di-n-hexyl keton (5, 11)), the introduction of only one phenyl group appears to make the compound photo-active (12). Of the higher phenyl substitution products, the molecules built symmetrically show the greatest activity (compare 13 and 16 with 14 and 15).

c. The photo-activity of the α-β-di ketones is a much more general property, and bound neither to the specifically aliphatic or aromatic character, nor in particular to the more or less symmetrical structure of the molecule. The introduction of a second $C=O$ group has mostly a greatly strengthening influence on the photo-activity (compare 18 and 20 with 11 and 6), in which possible disturbing influences issuing from the rest of the molecule, are thrown into the background. In this connection it is e. g. interesting to point out that phenanthrene quinone (25), which is to be considered as a particularly ortho-substituted benzil, far exceeds all the examined ketones with regard to its relative activity, whereas fluorenon (28), which may be compared with it, is perfectly inactive. The opposite case presents itself in the comparison of benzil (19) with regard to benzophenon (1), where the di-ketone compared with the mono-ketone is less active. It may, however, be possible that in consequence of the slight solubility of benzil in alcohol the maximum activation concentration cannot be reached.

Of great importance is also the activity of the α-β-di-ketones, which carry one or two furane-nuclei (21 and 22), which furnishes a new proof of the great resemblance in properties of the furane and benzol-derivatives.

d. Thus we see that the phenyl and furyl groups do not exert a disturbing influence on each other in the α-β-di-ketone; this influence is, however, evidently particularly strong in the corresponding mono-ketone, phenyl furyl ketone (17), which presents a very small activity. Here the above-mentioned influence of the symmetry of the molecule on the photo-activity of the mono-ketone is very

pronounced. This influence of the symmetry was already observed more than once in the photo-chemical reduction of the substitution products of benzophenon, but it had not been recognized as such [1]). To give a further support to this view it has been tried to make di-furyl ketone, as this compound would have to possess an activity equivalent to benzophenon. Unfortunately all attempts to obtain this substance have failed so far [2]), but in this connection attention may already be drawn to the much greater activity of terephtalophenon compared with isophtalophenon (23 and 24).

e. A somewhat separate place among the α-β-diketones occupies camphorquinone, the activity of which is unexpectedly slight, and moreover not reproducible. The greater or smaller purity of the preparations seems to be of great influence.

f. α-Hydrindon (29) and indane dion 1 . 2 . (31), considered as internal condensation products of resp. propiophenon (7) and acetyl benzoyl (20), present a greatly diminished activity. β-Hydrindon cannot be used as object of comparison with mono-phenyl acetone on account of its great oxidisibility.

g. The photo-activity of the examined α-β-γ-tri-ketones is zero, or so small as to be negligible (32, 33, 34) [3]). This phenomenon must, without any doubt, be attributed to the paralysis of the middle C=O group caused by the solvent [4]), through which the compound has practically quite lost its favourable properties of double α-β-diketone. [5])

Laboratory of Organic Chemistry
of the Technical University.

Delft, April 1923.

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[1]) Compare. COHEN, Rec. 39, 258 (1920).

[2]) FREUNDLER, Bl. (3) 17, 612 (1897).

[3]) Compare for the photo-chemical reduction of alloxane Ciamician and Silber, Ber. 36, 1581 (1903).

[4]) At first pentane tri-ketone and di-phenyl tri-ketone dissolve in absolute alcohol with a dark yellow colour, after standing some time the colour of the solution changes into light yellow, in which very probably alcohol addition products $CH_3-CO-C\langle^{OH}_{OC_2H_5}\rangle-CO-CH_3$ and $C_6H_5-CO-C\langle^{OH}_{OC_2H_5}\rangle-CO-C_6H_5$ which are analogous to the hydrate, are formed.

[5]) Comp. SACHS, Ber. 34, 3052 (1901); 35, 3311 (1902); VON PECHMANN, Ber. 23, 3380 (1890); WIELAND, Ber. 37, 1531 (1904); BILTZ, Ber. 45, 3662 (1912).

Mathematics. — "*On Power Series of the Form:* $x^{p_0} - x^{p_1} + x^{p_2} - \ldots$" By M. J. BELINFANTE. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of April 28, 1923).

Introduction.

It is a well-known theorem of FROBENIUS that if Σa_n is summable of order 1, $\left(\text{i.e. if } \lim_{n=\infty} \frac{s_1 + s_2 + \ldots + s_n}{n} = s, \text{ where } s_n = a_1 + a_2 + \ldots + a_n\right)$ then $\lim. \sum_1^\infty a_n x^n = s$, provided x approaches 1 by real values from below (which we denote by $x \to 1$).[1])

Under the same conditions we have:[2])

$$\lim_{x \to 1} \sum_1^\infty a_n x^{p_n} = s$$

provided $p_1 < p_2 < \ldots$ are integers which satisfy the condition:

$$\nu\,(p_\nu - p_{\nu-1}) < p_\nu . k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Some condition of the form (1) is necessary as may be seen from the following example, where our condition is broken and $\sum_1^\infty a_n x^{p_n}$ has no limit as $x \to 1$.[3])

Put $p_\nu = 2^\nu$ and $a_n = (-1)^{n+1}$, then we have:

$$\lim_{n=\infty} \frac{s_1 + s_2 + \ldots s_n}{n} = \tfrac{1}{2}$$

while $\sum_1^\infty a_n x^{p_n} = x - x^2 + x^4 - x^8 + \ldots$ oscillates between limits at least as wide as 0,498 and 0,502, if $x \to 1$[3]).

Thus we are led to the question: what is the connexion between the exponent-series $p_0, p_1, p_2, p_3, \ldots$ and the existence or non-existence of

$$\lim_{x \to 1} (x^{p_0} - x^{p_1} + x^{p_2} - \ldots)$$

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[1]) BROMWICH, Theory of infinite series, p. 312.

[2]) BROMWICH, op. cit., p. 388.

[3]) BROMWICH, op. cit., p. 498 example 30.

HARDY [1]) has investigated several particular exponent-series with particular methods that cannot be applied to other exponent-series, for instance the series of FIBONACCI:

$$1, 2, 3, 5, 8, \ldots$$

The only general result HARDY could reach was the non-existence of a limit if:

$$p_{\nu+1} > k \,.\, \nu \,.\, p_\nu \,.\, log.\, p_\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

but HARDY's example quoted above (where $p_\nu = 2^\nu$), shows the non-existence of a limit notwithstanding the condition (2) is not satisfied.

In the present paper another condition is given (§ 2), with the aid of a theorem of LITTLEWOOD which is treated in § 1.

§ 1.

LITTLEWOOD has proved the following theorem: [2])

Theorem 1. *If* $|n a_n| < k$, *and* $\lim_{x \to 1} \sum_1^\infty a_n x^n = s$, *then* $\sum_1^\infty a_n$ *converges to* s.

For our purpose we want the following extension which has also been enunciated by LITTLEWOOD: [3])

Theorem 2. *If the mean-values* [4]) *of order* $k-1$ *of* Σa_n *are limited and* $\lim_{x \to 1} \sum_1^\infty a_n x^n = s$, *then* Σa_n *is summable of order* k.

LITTLEWOOD states that the proof of theorem 2 follows the lines of his proof of theorem 1. The latter being rather long and tedious, it seems not without interest to show that theorem 2 is an immediate consequence of theorem 1.

Adopting the notation of our article "*On a Generalisation of* TAUBER's *Theorem concerning Power Series*" [5]), we have the following relations between the mean-values $A_n^{(k)}$ and the functions q_k:

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[1]) Quarterly Journal, vol. 38, p. 269, 1907.

[2]) Proceedings of the London Mathematical Society Ser. 2, Vol. 9, p. 434—448, 1911.

[3]) Proc. of the Lond. Math. Soc., l.c., p. 448.

[4]) For definitions of the mean-values of order p we refer to BROMWICH, op. cit., § 122, 123 and LANDAU, Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie, § 5.

[5]) Proceedings Vol. XXVI (p. 216—225).

$$\varphi_k(x) = \sum_1^\infty [A_n^{(k)} - A_{n-1}^{(k)}] \cdot x^n \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\varphi_0(x) = \sum_1^\infty a_n x^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

$$\varphi_k(x) + (1-x) \cdot \varphi_k'(x) = \varphi_{k-1}(x) \quad . \quad . \quad . \quad . \quad (2)$$

$$n \cdot [A_n^{(k)} - A_{n-1}^{(k)}] = A_n^{(k-1)} - A_{n-1}^{(k)} \quad . \quad . \quad . \quad . \quad (3)$$

With the aid of (2) we have proved [1]) that $\lim_{x \to 1} \cdot \sum_1^\infty a_n x^n = s$ implies

$$\lim_{x \to 1} \varphi_k(x) = s.$$

Now, if moreover:

$$n \cdot [A_n^{(k)} - A_{n-1}^{(k)}] | < c,$$

we have by theorem 1 that $\sum_1^\infty [A_n^{(k)} - A_{n-1}^{(k)}]$ converges to s, i. e.:

$$\lim_{n=\infty} A_n^{(k)} = s$$

or: Σa_n is summable of order k.

Since

$$A_n^{(k)} = \frac{A_1^{(k-1)} + A_2^{(k-1)}}{n} + \ldots + \frac{A_n^{(k-1)}}{n}$$

we infer from $|A_i^{(k-1)}| < c$: $|A_i^{(k)}| < c$ and by (3):

$$| n [A_n^{(k)} - A_{n-1}^{(k)}] | < 2c.$$

Hence we see that:

$$| A_i^{(k-1)} | < c \text{ and } \lim_{x \to 1} \sum_1^\infty a_n x^n = s$$

imply that Σa_n is summable of order k.

We use in § 2 the particular case that $k = 1$. It then runs:

Theorem 3: *If* $\lim_{x \to 1} \sum_1^\infty a_n x^n = s$, *and* $|s_n| < c$, *then* $\lim_{n=\infty} \sigma_n = s$, *where*

$$\sigma_n = \frac{s_1 + s_2 + \ldots + s_n}{n}.$$

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[1]) Loc. cit. p. 222.

§ 2.

We now prove the following theorem:

Theorem 4. *If* $1 < k_1 \leq \frac{r_{n+1}}{r_n} \leq k_2$, and $k_1 > 1 + \frac{k_2^2 - 1}{2k_2}$, *then* $f(x) = x^{r_0} - x^{r_1} + x^{r_2} - \ldots$ *does not tend to a limit as* $x \to 1$ [1]).

Proof: We show that the series of coefficients of $f(x)$ (which consists of the terms 1, $(r_1 - r_0 - 1)$ zeros, -1, $(r_2 - r_1 - 1)$ zeros, 1, and so on...) is *not* summable of the first order, i.e. that σ_n does not tend to a limit as $n \to \infty$. Then it is impossible that $f(x)$ should tend to a limit as $x \to 1$, for this implies by theorem 3 the existence of $\lim_{n=\infty} \sigma_n$ [2]).

We show that σ_n does not tend to a limit if $n \to \infty$, by calculating two positive numbers γ and m so that:

$$\sigma_{r_{2p+1}} - \sigma_{r_{2p}} > \gamma \quad \text{if} \quad p > m.$$

We have:

$$\sigma_n = \frac{s_1 + s_2 + \ldots + s_n}{n} = \frac{n a_1 + (n-1) a_2 + \ldots + [n - (n-1)] a_n}{n}$$

$$\sigma_{r_{2p}} = \frac{r_{2p} - [r_{2p} - r_1 + 1] + [r_{2p} - r_2 + 1] - \ldots + [r_{2p} - r_{2p} + 1]}{r_{2p}}$$

$$= 1 + \frac{r_1 - r_2 + r_3 - \ldots - r_{2p}}{r_{2p}} = 1 - \frac{r_{2p} - r_{2p-1} + r_{2p-2} - \ldots - r_1}{r_{2p}}$$

Since $1 < k_1 \leq \frac{r_{\nu+1}}{r_\nu} \leq k_2$, it follows that $r_{\nu+1} \geq r_\nu k_1$ and $r_{\nu+1} - r_\nu \geq (k_1 - 1) r_\nu$. Substituting this in the expression for $\sigma_{r_{2p}}$ we have:

$$\sigma_{r_{2p}} \leq 1 - \frac{(k_1 - 1) r_{2p-1} + (k_1 - 1) r_{2p-3} + \ldots + (k_1 - 1) r_1}{r_{2p}}$$

$$< 1 - \frac{k_1 - 1}{k_2} \cdot \frac{r_{2p-1} + r_{2p-3} + \ldots + r_1}{r_{2p-1}}$$

$$< 1 - \frac{k_1 - 1}{k_2} \left[1 + \frac{1}{k_2^2} + \frac{1}{k_4^2} + \ldots + \frac{1}{k_2^{2p-2}} \right]$$

$$\leq 1 - \frac{k_1 - 1}{k_2} \cdot \frac{1 - \frac{1}{k_2^{2p}}}{1 - \frac{1}{k_2^2}}$$

[1]) We suppose $r_0 = 1$.

[2]) The condition $|s_n| < c$ is satisfied since s_n is 1 or 0.

$$\sigma_{r_{2p+1}} = \frac{r_{2p+1} - [r_{2p+1} - r_1 + 1] + [r_{2p+1} - r_2 + 1] - \ldots - [r_{2p+1} - (r_{2p+1} - 1)]}{r_{2p+1}}$$

$$= \frac{-1 + r_1 - r_2 + r_3 - \ldots + r_{2p+1}}{r_{2p+1}} = \frac{r_{2p+1} - r_{2p} + r_{2p-1} - \ldots + r_1 - 1}{r_{2p+1}}$$

$$\sigma_{r_{2p+1}} \geq \frac{(k_1 - 1)\, r_{2p} + (k_1 - 1)\, r_{2p-2} + \ldots + (k_1 - 1)\, r_2 + k_1 - 1}{r_{2p+1}}$$

$$\geq \frac{k_1 - 1}{k_2} \cdot \frac{r_{2p} + r_{2p-2} + \ldots + r_2 + 1}{r_{2p}}$$

$$\geq \frac{k_1 - 1}{k_2} \cdot \left[1 + \frac{1}{k_2^2} + \frac{1}{k_2^4} + \ldots + \frac{1}{k_2^{2p}} \right]$$

$$\geq \frac{k - 1}{k_2} \cdot \frac{1 - \frac{1}{k_2^{2p+2}}}{1 - \frac{1}{k_2^2}}$$

$$\sigma_{r_{2p+1}} - \sigma_{r_{2p}} \geq \frac{k_1 - 1}{k_2} \cdot \frac{1 - \frac{1}{k_2^{2p+2}}}{1 - \frac{1}{k_2^2}} - 1 + \frac{k - 1}{k_2} \cdot \frac{1 - \frac{1}{k_2^{2p}}}{1 - \frac{1}{k_2^2}}$$

$$\geq \frac{k_1 - 1}{k_2} \cdot \frac{2}{1 - \frac{1}{k_2^2}} - 1 - \frac{k_1 - 1}{k_2} \cdot \frac{1}{k_2^{2p}} \cdot \frac{1 + \frac{1}{k_2^2}}{1 - \frac{1}{k_2^2}}$$

Put $\frac{k_1 - 1}{k_2} \cdot \frac{2}{1 - \frac{1}{k_2^2}} - 1 = c$, then it follows from $k_1 > 1 + \frac{k_2^2 - 1}{2k_2}$ that $c > 0$. Hence we have:

$$\sigma_{r_{2p+1}} - \sigma_{r_{2p}} \geq c - \frac{\frac{c+1}{2} \cdot \left(1 + \frac{1}{k_2^2}\right)}{k_2^{2p}}.$$

Since $k_2 > 1$ the second term tends to zero as $p \to \infty$; hence it is possible to calculate whatever be γ between 0 and c an integer m so that:

$$\sigma_{r_{2p+1}} - \sigma_{r_{2p}} > \gamma > 0 \qquad \text{if} \qquad p > m.$$

Remark 1.

Of course it is sufficient that the relation $1 < k_1 \leq \frac{r_{n+1}}{r_n} \leq k_2$ is only satisfied provided $n >$ some finite number g, since the addition of a finite number of terms does not influence the existence or non-existence of a limit.

Thus the function $x - x^2 + x^3 - x^5 + x^8 - x^{13} + \ldots$ does not tend to a limit as $x \to 1$ since

$$1\frac{3}{5} \leq \frac{r_{n+1}}{r_n} \leq 1\frac{2}{3} \text{ if } n > 5 \text{ and } 1\frac{3}{5} > 1 + \frac{\left(1\frac{2}{3}\right)^2 - 1}{2 \cdot \left(1\frac{2}{3}\right)}.$$

Remark 2.

Strictly spoken we have proved theorem 2 only if the HÖLDER-mean-values are limited. Now the existence of a "Hölder-limit" of order k implies the existence of a "Cesaró-limit" of order k and vice-versa [1]; hence if we prove that the HÖLDER mean-values of order p are limited provided the CESARÓ mean-values of the same order are limited, then our theorem is proved for both classes of mean-values.

Now we have (see LANDAU l.c.):

$$H_n^{(k)} = T_2\, T_3 \ldots T_k\, (C_n^{(k)}) \quad \ldots \ldots \quad (1)$$

where $H_n^{(k)}$ is the n^{th} HÖLDER mean-value and $C_n^{(k)}$ is the n^{th} CESARÓ mean-value of order k, and:

$$T_p\,(x_n) = \frac{p-1}{p} \cdot \frac{x_1 + x_2 + \ldots + x_n}{n} + \frac{1}{p} \cdot x_n \quad \ldots \quad (2)$$

From (2) we deduce that $|x_i| < c$ implies $|T_k(x_i)| < c$; hence it follows from (1) that $|C_n^{(k)}| < c$ implies $|H_n^{(k)}| < c$.

[1] Theorem of KNOPP-SCHNEE. See LANDAU, l.c.

Psychology. — "*On Subjective Rhythmisation.*" By F. F. HAZELHOFF and Miss HELEEN WIERSMA. (Communicated by Prof. E. D. WIERSMA).

(Communicated at the meeting of May 26, 1923).

Rhythmical perceptions, corresponding to rhythmical phenomena in the outer world, are at all times produced in numberless varieties by the rhythmical play of all sorts of physical and physiological processes. (Succession of day and night, of summer and winter, the heart-beat, the respiration, music etc.).

However, our perceptions are not always true reproductions of the reality round about us, as is borne out by many sensory illusions. The present authors are disposed to class among these illusions of perception also the peculiar phenomenon that we can perceive rhythmically a series of absolutely equal and regularly successive stimuli, which phenomenon has been termed "subjective Rhythmisation".

We purpose to discuss this phenomenon from a psychological view-point.

It was especially BOLTON (1) and MEUMANN (2) who pointed out the fact that regular series of auditory stimuli, i.e. auditory stimuli, precisely uniform quantitatively as well as qualitatively, and succeeding one another after equal intervals, can be perceived rhythmically, in any event when the rate of succession lies within certain bounds. KOFFKA (3) demonstrated the same with regard to visual stimuli, satisfying the same conditions.

This rhythmical perception of regular stimuli, however, is not restricted to visual-, and auditory sensations. The same rhythmisation also occurs in the department of touch. This we were able to demonstrate by a simple experiment.

Experiment: A regularly rotating axis is provided with a pointer that slightly touches the motionless hand of the observer at every revolution. The hand receives at every time precisely the same tick after precisely equal pauses (the rate of rotation may be regulated at will). The whole apparatus is hidden from the eye of the observer by a screen; neither can he see the pointer touching his hand.

By means of antiphones every auditory sensation is precluded so that the observer is entirely thrown back upon tactual sensations.

When we select a favourable rate of rotation, say 1 or 2 revolutions per second, subjective rhythmisation will soon reveal itself; at every time the observer will perceive with greater stress the 2nd or the 3rd touch, just as occurs with auditory-, and visual stimuli. It is also evident that with slow rates the observer will more readily apprehend a 2-rhythm; with increased rates, however, there is a greater aptitude for a 3-, or a 4-rhythm.

After some practice the observer will be enabled to work up to a different rhythm at every moment, either rising or falling. Suggestion imparted by his surroundings is of very great influence, but with a special rate of successive stimuli he generally selects a definite rhythm, most often a falling 2-rhythm (— ⏑), or a falling 3-rhythm (— ⏑ ⏑). However, other more complicated rhythms may occur.

Simple though this experiment may be, it is a great help to explain the nature of "subjective Rhythmisation", as it shows that this phenomenon is not restricted to auditory-, or visual sensations. It prompts us to assume that all sensory stimuli, which fulfil certain conditions, may be perceived rhythmically.

Let us first of all find an answer to the question what subjective rhythmisation really is. A periodic recurrence of "Betonungsunterschiede", "Innerliche zusammenfassung" (Meumann (2), and other much used terms are merely periphrastic designations of our experiences, they do not, however, explain their genesis. Neither is any explanation afforded by the mental pictures of rhythmic movements (such as dancing, the gallop of a horse and others), which in subjective rhythmisation are often aroused through association (Koffka (3)).

Introspection and the exact record of our experience, will have to show us the way here. We, therefore, made the following experiments; the observers were besides ourselves, 3 medical students.

Experiment: The observer is subjected to a regular series of auditory stimuli (we choose sound stimuli because sounds, indeed, are the better material for the perception of rhythm). The stimuli are applied in succession at a certain rate, about 1 to 2 stimuli per second, that quick or slow rate being selected with which the observer apprehends a certain rhythm most distinctly He is instructed to attend to what he hears, and to record accurately what he experiences by soft taps coinciding with every tick he hears upon a copper layer with a copper rod. As soon as the rod touches the layer an electric current, attracting an electromagnet, passes through. When the rod is raised again the current is broken, and the electromagnet returns to its original position. The electromagnet is provided with an inked pointer, which records the up-and-down movements on a rotating kymograph. In the curve thus obtained, the moment can be read at which the circuit is made (descending

movement of the pointer), as well as the time during which the current remains closed. The moment when the circuit is made is the moment when the observer imitates the tap, i. e. when he announces the moment at which he apprehends the sound stimulus; the time of the electric current represents the duration of auditory sensation. It is obvious that we can hardly expect absolute precision in this report, but important conclusions may be drawn from it with certainty, as already appears from the subjoined short portions of some curves.

The middle curve shows the reproduction of the auditory sensation of the observer, the upper one is an illustration of the stimulus itself. This stimulus consisted of a series of ticks given by an electrical hammer. The moment the regularly moving hammer touches the layer, thereby eliciting the sound-stimulus can readily be registered electrically in the same way as the taps given by the observer. The bottom curve shows the time in $^1/_{25}$ seconds:

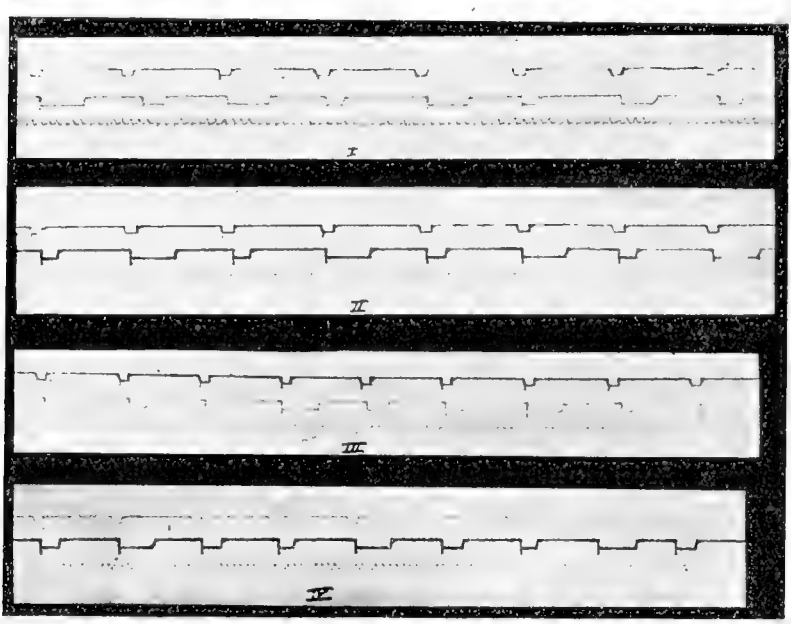

Fig. 1.

I. Falling 2-rhythm $\acute{-}\ \breve{}$

II. Rising 2-rhythm $\breve{}\ \acute{-}$

III. Falling 3-rhythm $\acute{-}\ \breve{}\ \breve{}$

IV. Amhybrachic 3-rhythm $\breve{}\ \acute{-}\ \breve{}$

We have measured the intervals of time between the records of the moments at which the observer announces the perception of the auditory stimulus.

The results of the measurements are given in the annexed tables:

I. Falling 2 rhythm (⏑ ⏑).

Observer.	Number of measured 2-rhythm.	Time in 1/25 sec. between: beginning of non-accented and accented stimulus.	Time in 1/25 sec. between: beginning of accented and non-accented stimulus.	On an average the accented stimulus is perceived sooner than the non-accented in sec.:
B.	20	221	241	**0,020**
H.	20	249	264	**0,015**
Ho.	20	272	287	**0,015**
R.	25	284	292	**0,007**
W.	10	167	174	**0,014**

II. Rising 2-rhythm (⏑ ⏑).

Observer.	Number of measured 2-rhythm.	Time in 1/25 sec. between: beginning of accented and non accented stimulus.	Time in 1/25 sec. between: beginning of non-accented and accented stimulus.	On an average the accented stimulus is perceived sooner than the non-accented in sec.:
B.	25	330	315	**0,012**
Ho.	40	577	537	**0,020**
R.	40	470	444	**0,013**
W.	40	554	516	**0,019**

III. Falling 3-rhythm (⏑ ⏑ ⏑).

Observer.	Number of measured 3-rhythm.	Time in 1/25 sec. between: beginning 2nd non-accented and accented stimulus.	Time in 1/25 sec. between: beginning accented and 1st non-accented stimulus.	Time in 1/25 sec. between: beginning 1st and 2nd non-accented stimulus.	On an average the accented stimulus is perceived sooner than the: 1st non-accented in sec.:	On an average the accented stimulus is perceived sooner than the: 2nd non-accented in sec.:
B.	30	355	416	413	**0,028**	0,052
H.	10	141	168	145	**0,068**	0,044
Ho.	10	99	114	113	**0,040**	0,036
W.	20	267	272	271	**0,004**	0,006

IV. 3-rhythm. Amphibrachic (⏑ – ⏑).

Observer.	Number of measured 3-rhythm.	Time in 1/25 sec. between:			On an average the accented stimulus is perceived sooner than:	
		beginning 2nd non accented and 1st non-accented stimulus.	beginning 1st non-accented and accented stimulus.	beginning accented and 2nd non accented stimulus.	1st non accented in sec.:	2nd non accented in sec.:
B.	40	488	469	505	**0,017**	**0,018**
H.	30	372	348	390	**0,029**	**0,027**
W.	40	460	450	494	**0,018**	**0,026**

In all these cases, in which "subjective rhythmisation" readily manifested itself, and in which the observer reported his experience as accurately as could be, it is evident that

1°. the observer perceives the accented stimulus sooner than the non-accented

2°. that the perception aroused by the accented stimulus lasts longer, as is shown directly by readings from the curves, so that special measurements in this respect we considered superfluous.

That the observer perceives the accented stimulus with greater intensity is not expressed in the above curves, the deflection of the electromagnet being the same at every time. That this is a fact, however, is made manifest by introspection. As shown by the following curve this is also easily demonstrable by another method of recording, viz. by using a tambour instead of a copper layer for the accompanying taps of the observer. Owing to the slowness of the recording pointer these curves do not indicate small time-differences accurately; for this reason the measurements of time were taken from electrically registered curves only.

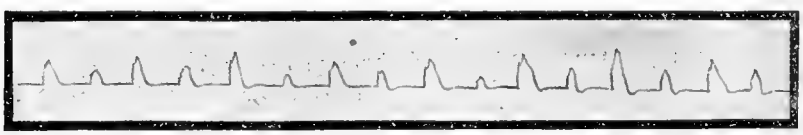

Fig. 2.
Falling 2-rhythm – ⏑

In summary, then, the results of our curves illustrating the perceptions, lead to the following conclusions:

1. The subjective accented stimulus is perceived sooner.
2. It is perceived with greater intensity.
3. It is of longer duration.

Psychologically these three characteristics are easy to understand and may be explained upon the basis that a keener attention is given to one stimulus than to another, for we know that our perceptions are aroused sooner, that they become more intense, and that their after-effect lasts longer, according as our attention is more closely concentrated upon the stimulus.

This, in our judgment, is the nature of "subjective rhythmisation": *we attend more keenly to one stimulus than the other.*

Some points still require further elucidation. First of all the so-called "*Innerliche zusammenfassung*", the running-together of the impressions to form groups, which generally start with the subjective accented stimulus. This is also a temporal grouping, in such a sense that the elements of the group seem to follow each other at a quicker rate, while at every time there is a longer pause between two groups (MEUMANN (2)). We believe this grouping to be of a secondary nature and to result from the fact that the after-effect of the subjective accented stimulus is prolonged. This causes the pause between the termination of the accented stimulus and the beginning of the non-accented stimulus to be shorter than in the opposite case. This also accounts for the fact that rhythmisation always occurs in the sense of a falling rhythm, the shorter pause then falling within the group, or rather in consequence of the short pause it seems to us as if the stimuli by which it is bounded, belong together; conversely the long pause causes a separation between two groups. Another question that arises concerns the *cause of subjective rhythmisation*: why do we attend to one stimulus more keenly than to the other, and why is this alternation regular?

Our capacity of receiving outside impressions is limited. Of a large number of simultaneous stimuli to which we are exposed, we perceive only a part. Some attain a high state of consciousness, others are driven into the background of consciousness. When the stimuli are weak and affect us only for a short time, the quantity need not be large for a selection.

Experiment: two, three or more lines or points, perfectly equal *inter se*, are presented to the observer for a very brief period of time. When the lines or points are not very clear, and the exposure is short enough, only a few will be perceived well, the others appear to us much less distinct, or we do not see them at all.

In so short a period of time we cannot divide our attention among several weak stimuli so as to perceive all of them with the same distinctness. Now when such weak stimuli are presented in rapid succession, we may expect the same; also when they follow each other at short intervals, we cannot perceive all of them and we must make a selection. In our experimentation we also observed that it is exactly series of weak and obscure stimuli that are best adapted to subjective rhythmisation.

Now, *why* is this *accentuation regular?*

A periodically recurring stimulus is easy to perceive; we are beforehand predisposed to the impression, as we know when it is coming. When for instance of a series of stimuli we clearly apprehend the first and the third, we are better prepared for the fifth and the seventh. We may change this accommodation at will every moment so that we apprehend a 3- instead of a 2-rhythm, or we may substitute a rising for a falling rhythm.

The foregoing offers an explanation for other familiar features of subjective rhythmisation.

The quicker the rate of succesion of stimuli the larger will be the groups in which they are included. We endeavour to apprehend well as many stimuli as is possible, the slower the movement the larger will be the number of stimuli we perceive distinctly. When the rate of succession increases a 2-rhythm is changed into a 3-rhythm, a 3-rhythm becomes a 4-rhythm, etc.

It is also evident that the rate of the movement must not exceed certain bounds. When the pauses between two stimuli are too long, all our attention may be directed to every separate stimulus, the perception of every stimulus attains its maximal intensity, so that no rhythmisation will occur. When the pauses are too short, we cannot single out any stimulus, they run together into a vague entity.

It is also clear now that a *sensation of relaxation* (a pleasurable relief) is aroused when, after being constrained to intently follow a series of stimuli, we perceive them rhythmically, because of the much smaller demand upon our attention.

In experimenting it will be noted that the observer's aptitude for the rhythmic perception of a series of stimuli is ever increasing. Ultimately a certain rhythm will stick to him, it has so to say become an obsession.

It is just the same with special series of regular stimuli, which continually affect us in every-day life, such as the ticking of a clock, in which every one recognizes a rhythm, without being able to break away from it.

SUMMARY.

Rhythmical perception of a series of perfectly equal stimuli, following each other in regular succession, is aroused by the different degree of attention given to the various stimuli.

This divided attention is a necessary result of the fact that of a large quantity of stimuli operating upon us, only a limited number can be attended to (constriction of consciousness).

At the outset we can divide our attention at will, in the long run we may be constrained to do so.

The primary characteristics of subjective rhythmisation are based upon the fact that one stimulus is perceived after a shorter interval, with greater intensity and for a longer period of time than the other. The formation of groups results from it.

Subjective rhythmical perception can be aroused not only by visual-, and auditory-, but also by tactual impressions of stimuli that satisfy certain conditions.

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Physiology. — "*Researches on the chemical causes of normal and pathological Haemolysis*". By R. BRINKMAN and A. v. SZENT-GYÖRGYI. (Communicated by Prof. H. J. HAMBURGER),

(Communicated at the meetings of February 23 and April 26, 1923).

I. *Isolation of the haemolytic substances of normal human blood.*

It has been known for a long time that it is possible to isolate from normal blood by means of fat-extraction methods groups of substances, which possess strongly haemolytic properties. The study of these substances must be important for the explanation of normal and pathological haemolysis, but a definite result revealing their structure and manner of action has not yet been obtained. NOGUCHI [1]), when extracting these substances supposed them to be soaps, but he only examined them in regard to immunological phenomena and his views were not supported by later investigators [2]). Others were thinking of substances with a phosphatid structure, but could not give sufficiently conclusive proofs [3]).

A more exact investigation of the chemical constitution and the physico-chemical form, in which they exist in the blood, is wanted to be able to determine the physio-pathological significance of these substances. We have started from the idea, that it must be desirable to isolate these substances in a form as pure as possible, to be able to determine their chemical and physico-chemical properties. The first condition to be fulfilled was complete extraction of the haemolytic substances. Afterwards the extracts were fractioned under the guidance of their more and more increasing activity. This activity was tested by dispersing the extracts in isotonic neutral phosphate mixture at 37° [4]).

The determination of the haemolytics was carried out in the following way:

The human blood obtained by venapunction was defibrinated and sharply centrifugalized at once The corpuscles were imbibed in fat-free filterpapers and

[1]) NOGUCHI, Biochem. Zeitschr. VI. 327, (1907)

[2]) See LANDSTEINER. Handbuch KOLLE-WASSERMANN II, 1291, (1913)

[3]) BRINKMAN, l. c.

[4]) See for the method BRINKMAN Arch. néerl. de Physiol. VI, 451, (1922).

dried at 37°. Afterwards the red corpuscles were extracted for one hour by petrol-ether at room temperature; in this way neutral fat and most of cholesterin are extracted without any loss of haemolytics. The following quantitative extraction of haemolytics was made in a specially constructed small apparatus for "boiling-point extraction", with always freshly destilled fluid, adaptable for small quantities. As extraction-liquid acetone was chosen in analogy to the use of this liquid in phosphatid chemistry. In order to extract the haemolytics completely with acetone a preceeding treatment with alcohol-vapour for half an hour was necessary; the than following acetone-extraction dissolves all haemolytics in two hours.

To complete purification the acetone-extract was concentrated to a small volume and allowed to stand for one night in ice. In this way most of the dissolved substances are precipitated but no haemolytics are found in the precipitate. The remaining strongly active fraction has the following properties: it can be dissolved in all typical lipoid-dissolving liquids, if the reaction is slightly acid, but in an alkaline medium the haemolytics are insoluble is petrol ether. The examined substances are not precipitated by cadmium, but they are precipitated quantitatively in aquous solution by barium and in acetone solution completely by an ammoniacal solution of acetate of lead, in the presence of not less than 30% of water

So we see, that the investigated haemolytics show the typical reactions of the higher fatty acids. The said precipetate contained no phosphorus, so that phosphatides can be excluded definitely, *and the haemolytics, which can be extracted from normal blood must be identified with higher fatty acids resp. their soaps.*

The solubilty of the Pb and Ba salts indicated, that a mixture of fatty acids must be present, containing no or one and also more than one double linkage. Further experiments must determine the constitution and procentual concentration of these substances.

Separation of the fraction of the fatty acids and of phosphatids can only be obtained by careful quantitative methods of working; it is probable, that complete separation was not got by former investigators.

In addition to these results we have examined once more the haemolytic action of pure lecithin. If was found that a praeparation of lecithin purified by the newest methods showed no haemolytic properties; the haemolytic action of common trade lecithin must be ascribed to impurities, and this also is the case if this substance is somewhat purified by the usual acetone-precipitation.

With the knowledge, that the haemolytics of lipoid blood extract are higher fatty acids it is possible to isolate them in a simpler way. This may be done by the following method: the dried blood, sucked in filterpaper (5 cc. of blood) is treated with absolute alcohol for one hour in the boiling-point extraction apparatus. The extract

is concentrated to 5 cc. and than 5 cc. of a solution is added, which contains 0,2 n. Na_2CO_3 and 0,2 n. NaOH (in water). After five minutes the mixture is thoroughly shaken with 5 cc. of petrol ether; in this way neutral fats and cholesterin are eliminated completely and phosphatids for the greater part. The remaining alcoholic extract is acidulated with 0.5 cc. of HCl conc. and shaken with 2×5 cc. of petrol ether; afterwards 1 cc. of benzol is added to the alcoholic extract, and this is once more shaken with 5 cc. of petrol ether. The three petrol ether fractions thus obtained contain practically all normal haemolytics.

If this extract is dried and the residue emulgated in neutral isotonic phosphate mixture, then the amount of fatty acids obtained from 1 cc. of blood and emulgated in 1 cc. of phosphate solution may be diluted to $^1/_{64}$ and is still capable to haemolyse 1 % of blood corpuscles completely in half an hour at 37°.

CONCLUSION.

It is possible by means of lipoid extraction to isolate from normal blood substances wich are strongly haemolytic. These substances solely consist of higher fatty acids. A simple method is indicated for their quantitative extraction.

II. *The form in which strongly haemolytic fatty acids are contained in normal blood.*

In the previous communication it was stated, that a rather large quantity of intensively haemolytic higher fatty acids can be isolated from normal blood. It will be obvious that in normal blood this action must be completely on or nearly completely prevented; the mechanism of this inactivation is not definitely known. In this relation the formation of a protein-fatty acid compound was generally supposed, but we did not know if these combinations could exist in the blood plasma and if their haemolytic character has disappeared in this way. The knowledge of this inactivation must be important for the analysis of normal and pathological haemolysis, because insufficiency of the inactivation-mechanism must be dangerous to the corpuscles.

In order to investigate in which way the fatty acids are bound in the blood, we have made use of the high degree of capillary activity of these compounds; and this in the first place because this

surface activity is a property to which combining power and haemolytic action are intimately related, and secondly because the surface tension of small a amounts of blood can be measured accurately and easily by the torsion balance method [1]).

The surface-tension of a neutral highly diluted solution of fatty acids is much lower than the static surface tension of blood or serum. This fact already indicates that the fatty acids of blood cannot occur in the free state but must be bound somewhere. The minutest trace of free fatty acid must reveal itself immediately by a marked decrease of static surface tension, but the capillary-activity of the protein-fatty acid compounds also is, as far as we know, so intensive, the a decrease of surface tension plasma-air should be observed if an added fatty acid was bound as protein-compound.

The possible combination of protein and fatty acids may supposed to be primarely chemical or adsorptive; the last form would be probable by the intensive surface activity of these substances.

In the following table it is shown how the surface tension serum-air changes when small quantities of a diluted neutral emulsion of fatty acids are added.

Surface tension of fresh human serum.	52	Dyne c.M.
+ 0.001 N oleic acid in neutral emulsion.	52	„
+ 0.002 N „ „ „ „	52	„
+ 0.003 N „ „ „ „	52	„
+ 0.004 N „ „ „ „	51	„
+ 0.005 N „ „ „ „	47	„
+ 0.006 N „ „ „ „	45	„
+ 0.007 N „ „ „ „	42	„
+ 0.008 N „ „ „ „	41	„
+ 0.009 N „ „ „ „	39	„
+ 0.010 N „ „ „ „	39	„

It is seen, that 0,004 N. oleic acid may be added to serum without change of surface tension; when more acid is given, a gradual decrease of surface tension takes place till the value of ± 40 dynes/cm. has been reached. A further lowering is only to be seen, if large amounts of fatty acid are added. We will not delay upon the explanation of the gradual decrease of tension, but try to investigate the mechanism by which the plasma can preserve its original tension. Any marked lowering of this tension must be considered abnormal.

The constancy of surface tension indicates that the researched

[1]) Brinkman and Van Dam. Münch. med Woch. 1550 (1921).

fatty acid compound can not lower the surface tension of water to less that 52 dynes/cm. By this observation the hypothesis of inactivation of fatty acids by protein solely is proved to be insufficient. Therefore we had to think of other possible compounds und found a sufficient explanation for constancy in the formation of calcium soaps. The existence of this proces of inactivation was found in the following way:

A. If the Calcium of serum or blood is precipitated by addition of oxalate of ammonia, the surface tension can not be held constant if small quantities of oleic acid are added. This is to be seen in the next table.

Surface tension of fresh human oxalate plasma	49 d. c.M.
+ 0.001 N oleicacid in neutral emulsion	47 „
+ 0.002 N „ „ „ „	45 „
+ 0.003 N „ „ „ „	42 „
+ 0.004 N „ „ „ „	40 „
+ 0.005 N „ „ „ „	38 „

The same results are obtained, when NaFl plasma is used.

B. A salt-solution containing the same amount of Ca as Plasma can maintain its tension above 50 dynes on addition of a neutral emulsion of oleic acid at 37°, to the same extent as plasma can. This holds for a solution of $CaCl_2.6$ Aq. 0,05 % as well as for a solution composed of NaCl 0,7 %, $NaHCO_3$ 0,2 % KCl 0,02 %, $CaCl_2.6$ aq. 0.05 % and H_2CO_3 till $[H^{\cdot}] = 0{,}4.10^{-7}$ is reached.

The following table gives the surface tension of the said salt solution if small amounts of oleic acid were added very gradually.

Surface tension of the balanced salt-solution	74 d. c.M.
+ 0.0001 N oleicacid in neutral emulsion.	54 „
+ 10 × 0.0001 N „ „ „ „	53 „
+ 10 × 0.0001 N „ „ „ „	52 „
+ 10 × 0.0001 N „ „ „ „	52 „
+ 5 × 0.0001 N „ „ „ „	50 „
+ 5 × 0.0001 N „ „ „ „	39 „

If the surface tension of the saline shall not be lowered unter the plasma tension, it is necessary to add the oleic acid very gradually, and to leave the mixture for a half hour at 37° after each addition; only in this way it is possible to obtain a form of Oleate of Ca, whose capillary activity is low enough. But this condition is fulfilled in vivo.

The question now arose, wether this mechanism of inactivation would be equally important for the normal fatty acids of the blood as it proved to be for oleic acid. It is certain, that about one third part of the blood-calcium is present in the colloidal state; when we

consider the insolubility of Ca soaps it is possible, that the indiffusible part of the plasma Ca will consist wholly or partially of soaps. It is easily to be shown, that complete precipitation of the blood Ca is followed by a marked decrease of surface tension.

Surface tension of one cc. of freshly taken human serum is 53 dynes/cm; when 0,3 ccm. of a saturated solution of oxalate of ammonium is added, it decreases to 50, 48, 46 dynes/cm. The action of NaFl is similar.

Further if at little acid is added to the plasma, the fatty acids must be liberated from the eventually existing Ca soap compound. This proved to be the case; the amount of HCl necessary to lower the surface tension of serum from 52 to 45 dynes is exactly equivalent to the potential alkalinity of that serum. So it is probable that in normal blood also a great deal of the fatty acids are circulating in the form of Ca compounds. Direct chemical analysis will have to bring further evidence.

Till now we only examined the inactivation of oleic acid; the saturated fatty acids appear to be bound in the same way, but the physiologically important highly unsaturated linolenic acid give Ca salts, which lower the surface tension of the balanced salt solution to 38 Dynes. In accordance with this the blood or plasma it is not capable to maintain its surface tension if a small amount of isotonic neutral emulsion of linolenic acid is added, contrary to what occurs when oleic acid is given. This is seen from the following experiment.

Surface tension of fresh human serum	53	dynes	p. cm.
+ 0.001 N linolenic acid emulsion	47	„	„
+ 0.002 N „ „ „	46	„	„
+ 0.003 N „ „ „	46	„	„
+ 0.004 N „ „ „	44	„	„
+ 0.005 N „ „ „	43	„	„

Although linolenic acid also is in plasma subject to considerable capillary inactivation, this process is not so complete, that the surface tension can be maintained absolutely constant. This fact must be explained by the capillary activity and solubility of the linolenate of Ca.

By these circumstances the higher unsaturated fatty acids circulating in the blood must have a great biological importance, because their Ca inactivation is failing. Therefore these acids must be bound by plasma colloids or corpuscles with decrease of interfacial tension. If now the inactivation of fatty acids extracted from corpuscles is compared in serum and in salt solution with the process described, it appears that these substances have the same properties as saturated

acids, and oleic acid have, but that a very small fraction is present which acts in the plasma as would do linolenic acid. Addition of fatty acids extracted from blood lowers the surface tension of serum from 53 dynes to 49,5 dynes; when more extract is added, the tension remains as constant as if oleic acid were added. The extracted fatty acids lower the surface tension of the serum to that of total blood, for corpuscles also can decrease the surface tension of serum to 50 dynes. So the tension of *blood* is not decreased by extracted fatty acids. If it may be concluded, that a small fraction of highly unsaturated fatty accids is absorbed normally to the corpuscles, this must be verified by further investigation.

In a following communication we will describe the influence which the investigated mechanisms of inactivation have on normal and pathological haemolysis.

SUMMARY.

By means of determination of surface tension of blood and serum it was shown, that the normal fatty acids of the blood or also those added on purpose are bound in the form of Calcium soaps, by which mechanism their capillary activity is decreased considerably. It is very probable, that this formation of a Calcium compound is the cause of disappearance of haemolytic properties of stearic acid, palmitinic acid and oleic acid in serum. The inactivation by means of Ca is not present in the case of linolenic acid; by this circumstance the haemolytic character of this substance of serum will be much greater.

III. *Experimental anaemia caused by injections of linolenic acid.*

In a previous communication it was pointed out, that higher fatty acids in the blood generally are circulating as Ca compounds, and thus have lost their marked capillary activity. It was stated however, that the Ca soaps of the higher unsaturated fatty acids i.e. of linolenic acid, do not loose their capillary activity, and that by this reason we have to expect much greater haemolytic action in vivo of this substance.

It was shown, that linolenic acid is an intravital haemolytic substance, of great activity and that there is no direct inhibition of the action of linolenic acid in the plasma. It was known for a long time, that injection of the saturated fatty acid or of oleic acid can not cause a distinct intravital haemolysis, probably by the mechanism of Ca inactivation, described formerly. In the case of linolenic acid

the injection is followed by marked haemolytic symptoms, as appeared from the following experiments.

Intravenous injection. A rabbit of 3620 gr. is injected in the auricular vein 0.250 gr. of linolenic acid dispersed in 10 ccm. of isotonic phosphate mixture. After 10 minutes the surface tension of the blood, which otherwise is 54,5—55,5 dynes/cm is decreased to 50 dynes and the serum is coloured lightly reddish. If now the surface tension is measured with regular intervals, it is seen, that the surface tension can not rise to the normal value but always has a value of 50 dynes approximately. The haemoglobinaemia is increasing more and more. After twenty minutes a strong haemoglobinuria is observed, and the rabbit makes a very sick impression. One hour after injection the animal dies with symptoms of utmost anaemia and dyspnoe.

In this way we could prove by several experiments, that a rabbit is killed by intravenous injection of ± 100 mg. of linolenic acid per Kg. under symptoms of very strong haemolysis. If smaller quantities of linolenic acid are given intravenously, the rabbit is not killed at once, but a chronic haemolysis with severe anaemia, urobilinuria, etc. sets on. When the linolenic acid is given intravenously, there is always a certain chance, that a little too large dosis of linolenic acid will lead to a direct mortal haemolysis.

A severe chronic haemolytic anaemia is produced by the subcutaneous, or better intramuscular injection of the acid. In this case the greater part of the injected substance seems to be inactivated and the following discase develops itself.

A rabbit of 3450 gr. in good state of health. Number of red cells 5,400,000. Haemoglobin 60 (Sahli). The form of the red cells in plasma is purely biconcave; a very small degree of anisocytosis, no polychromatophilia, absence of normoblasts. The serum is colourless and does not contain bilirubin. No uroblin or urobilinogen in the urin. The surface tension of the blood is 55,4 dynes at 19°.

The rabbit is injected every day with 200 mgr. of linolenic acid intramuscularly. After the first injection the surface tension of the blood decreases to 51—52 dynes, and remains so during all the experiment. 2—3 days after beginning of the treatment an intensive urobilinuria sets on and does not disappear during the course of injections. The blood picture shows from the third day a more and more increasing anisocytosis and polychromatophilia, while the number of irregularly shaped cells and sphaeric cells is rising. After five days the number of red corpuscles was lowered to 2.500.000; after that the first regeneration showed itself with numerous normo-

blasts, and strong anisocytosis and polychromatophilia. The number of red cells at this time was 3.700.000, the haemoglobincontent 55. So the index had increased distinctly and this increase remains very marked during the course of injections.

Twelve days after the first injection the number of red cells had decreased again till 2.900.000 (Haemoglobin (45), and the second period of regeneration began. Now the blood picture demonstrated the typical symptoms as they are found in distinct pernicious anaemia. Especially macrocytosis, poikilocytosis, strongly disshaped corpuscles, polychromatophilic megalocyts and normoblasts were striking. Bilirubinaemia could only be traced in the rabbit in cases of strong acute haemolyses. In the more chronic forms this phenomenon is not observed, urobilinuria being very marked however.

The rabbit is emaciated and makes a sick impression. If the injections are stopped in the beginning, the anaemia may be cured; if the treatment is continued, the typical pernicious symptoms will last.

So there is no doubt, that intramuscular injection of linolenic acid causes a chronic haemolytic anaemia in a short time, the red picture of which is showing all typical marks of pernicious anaemia. The picture of white cells has not yet been researched till now. We shall have to make a more exact analysis of this anaemia by linolenic acid, but it may be stated already, that linolenic acid is a very severe haemolytic substance. As it was found in the previous communications, we must ascribe this intravital action to the fact, that the Ca soaps of higher unsaturated fatty acids are capillary active and haemolytic, contrary to the Ca soaps of palmitinic acid and oleic acid.

Now this acid, forming an important percentage of the phosphatid fatty acids, it is practically certain, that the formerly used praeparation of trade-lecithin could effect the described haemolytic action by the rather large content of linolenic acid. Linolenic acid is a substance, which is found in the biochemistry of fat and phosphatid metabolism and it is probable, that this acid is circulating in normal blood.

In fact we were able to demonstrate by means of specific extraction, that in the 0,6—0,7 mgr. of fatty acids, which are found in one ccm. of normal human blood there is always present a small fraction consisting of higher unsatured fatty acids.

It appeared further, that all other fatty acids of the blood are inactivated by serum, in regard to capillary active and haemolytic action, but this small fraction of higher unsaturated fatty acids can

not be inactivated completely, so that we must ascribe a great importance as a physiologically haemolytic substance to the normal circulating linolenic acid. We will try, to determine the concentration of linolenic acids in the severe human anaemias.

SUMMARY.

Intramuscular injection of 200 mg. of linolenic acid per day in the rabbit is followed in a short time by chronic haemolytic anaemia. The blood picture shows a striking resemblance to that in pernicious anaemia.

Intravenous injection of ± 100 mgr. of linolinic acid pro Kg. causes a letal haemolysis.

Chemistry. — "*Nitrogen fixation by means of the cyanide-process and atomic structure.*" By Dr. L. HAMBURGER. (Communicated by Prof. P. EHRENFEST).

(Communicated at the meeting of June 30, 1923).

a. *Introduction.* It is known that the reaction

$$MCO_3 + 4\,C + N_2 = M(CN)_2 + 3\,CO$$

forms the foundation for the nitrogen fixation by the so called cyanide process. For this conversion the temperature at which the capture of the nitrogen takes place with practically appreciable velocity appears to be very divergent, according as another M is chosen for the metal. H. LUNDÉN [1]) has also included rubidium and ceasium in his researches, and is of opinion that there is a relation between the boiling-points and atomic weights of the metals in question and the "cyanizing-temperature". It is, however, not possible to derive a quantitative relation on this foundation.

b. *Stages of the cyanizingreaction.* In order to arrive at a clear insight the fact should be considered that according to J. E. BUCHER [2]), two stages before all should be distinguished in the course of the reaction:

$$\text{I.}\quad MCO_3 + 2\,C = M + 3\,CO$$
$$\text{II.}\quad M + 2\,C + N_2 = M(CN)_2.$$

Of these reactions I bears an exceedingly endothermal character, II on the other hand, is strongly exothermal, II takes place practically momentarily (either with addition of a catalyst or without). Whether a practically appreciable reaction-velocity will appear, will therefore depend on I. The strongly endothermal character of I [3]), however, causes the temperature to remain pretty well constant, when the reaction sets in, till the reaction of MCO_3 has been completed. The quantities of energy required for this are so great, that, especially at comparable conditions, factors like energy-quantities,

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[1]) Cf. TH. THORSELL. Zeitschr. f. angew. Chem. **33**, 251 (1920).

[2]) J. E. BUCHER. Jl. of Ind. and Eng. Chem. **9**, 233 (1917).

[3]) In consequence of which the total reaction I + II is also still endothermal in a high degree.

required for division, dilution, solution, melting and evaporation remain of subordinate importance [1]). Hence notwithstanding the complicated character, we have to do with:

1. comparatively characteristic reaction temperatures;
2. with the decisive influence of metal-formations.

We can now divide I again into the following more elementary processes:

$$a.\ MCO_3 = MO + CO_2 \qquad c.\ O + C = CO$$
$$b.\ MO = M + O \qquad d.\ C + CO_2 = 2\,CO$$

in which c and d follow the same reaction equation for all metal cyanide-syntheses, whereas b is first of all the reaction which claims the lion's share of the energy-supply.

c. *The primary reaction.* From W. KOSSEL's [2]) point of view the course of b means only this, that under the influence of the supplied energy in the metal oxide the oxygen cedes again the negative electron taken from the metal. In our case this view entails the difficulty that this restitution would have to take place more easily with metal atoms with relatively great affinity to the returning electron than to more electro-positive elements. In reality, however, the reaction appears to set in more easily with increasing electro-positivity of the elements.

It is, however, not necessary to assume that in every metal-metalloid compound one or more of the metal-electrons has quite gone over to the metalloid. In many cases it may be more a question of partial transition, i. e. conditions will be found in which only partial separation ("Lockerung", dislocation) of the metal- (valency-) electron with regard to the metal atom must be assumed. In connection with the spectral interpretations by BOHR this may also be expressed thus, that the electron in question will on an average be at the disposal both of the oxygen atom and of the metal atom in a characteristic "abnormal" path.

The decreasing reaction temperature with the increasing electro-positivity of the metal leads us further to the assumption that pri-

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[1]) In the same way the dependence of the reaction-energy on the temperature may be neglected within a wide margin. The possibility of all these approximations stamps the cyanizing-reaction as one of the few chemical conversions, which like the rare ideal photo-chemical reactions are able to give experimentally demonstrable indications in favour of the theories which at the same time bear a fundamental and idealizing character [cf. e. g. F. WEIGERT, Zeitschr. f. Phys. **14**, 383 (1923)].

[2]) W, KOSSEL, Ann, d. Phys. **49**, 229 (1919).

marily the electron does not entirely return to the metal atom, but that inversely a complete separation of the valency-electron from the metal rest should be taken into account. It is this process that we shall subject to a fuller examination in what follows, and which we shall briefly denote by the expression of "primary reaction". [1])

Hence we come to realize the possibility of the conclusion that the dissociation of the metal oxide does not take place by the direct process of splitting up $MO = M + O$, but that with increase of energy first an *activation* sets in, manifesting itself as formation of ions. From this activated condition the further course of the reaction takes place.

Thus we are led to place ourselves on the basis of these theories of reaction-velocities which have appeared to possess remarkable validity, at least in definite cases; particularly we come to the "activation" basis given as of general validity first by Sv. ARRHENIUS [2]) and later among others by J. PERRIN [3]) in his "Lumière et Matière".

d. Ionisation Potential.

It is known that the ionisation potential of the vapour is a decisive quantity for the detaching of an electron from a free metal atom. Where there is reason to assume that the primary reaction takes place in the gasphase [4]) we will first of all try, in connection with the view about the primary reaction given under *c*, in how far the ionisation potential can also govern the cyanizing-reaction. For this purpose we will calculate the values of the quantity $\frac{V}{T}$ according to the table below, in which V represents the ionisation potential

[1]) A possible addition of the separated electron to the oxygen rest should be considered as a second stage. From J. FRANCK's researches on the collisions of slow electrons in electro-positive and noble gases we know that negative electrons are seized by oxygen, but are on the contrary under certain conditions relinquished by the electro-positive atom. In the same way a partially bound electron, which gets free through "critical" energy supply with small velocity [thus the lower limit of energy supply required to bring about the primary reaction may be indicated for brevity], may be bound to the oxygen atom. We leave this out of account in the next close examination of the primary reaction.

[2]) Sv. ARRHENIUS. Zeitschr. f. Phys. Chem. **4**, 226 (1889).

[3]) J. PERRIN. Ann. d. Phys. **11**. 5 (1919).

[4]) All the metal carbonates or oxides of the alkalies and earth-alkalies are greatly or appreciably volatile at the indicated reaction temperatures. In the dissociable carbonates the evaporation is promoted by the formation of oxydes in molecular distribution (formation "in statu nascendi" and transportation by the gas current). The parallelism between the cyanizing temperature and volatility of the carbonates resp. oxydes is striking.

of the free alkali-resp. earth alkali atoms, and T the cyanizing-reaction-temperature.[1]) It is seen from the fourth row of the table that the same order of magnitude is found everywhere for $\frac{V}{T}$.

Row 1	Metal	Li.	Na.	K.	Rb.	Ce.	Mg.	Ba.	Sr.	Ba.
2	Reaction-temp. in °K(T)	1370	1200	1100	970	870	2100?	1900	1670	1320
3	Ionisation-potent. in volts (V)	5.4	5.1	4.3	4.2	3.9	7.6	6.1	5.7	5.2
4	$\frac{V}{T}$. 10^3	4.2	4.2	4.0	4.2	4.3	3.2?	3.2	3.4	3.9
5	Excitation potential V'' in Volts	1.84	2.09	1.60	1.55	1.38	2.70	1.88	1.79	1.56
6	$\frac{V-V''}{T}$. 10^3	2.5+	2.5	2.5—	2.7	2.86	2.3	2.2	2.3	2.7

Considering the widely divergent circumstances the agreement may even be called remarkable, the more so as a perfectly sharply defined reaction temperature cannot be expected on theoretical ground either.

e. Dislocation potential.

The ionisation potential determines the energy required to detach an "outer" electron of a metal atom entirely from a normal path. As under *c* we arrived at the view that in the compound the electron in question is present in an abnormal path, the conclusion is obvious that not V, but a smaller quantity $V - V'$ can give a measure for the critical supply of energy, in which V' is a quantity which determines the difference of energy between the electron in the abnormal path of the compound and the electron in the free metal atom, that is in the normal path. We shall call this quantity briefly dislocation potential, the electron in the abnormal path will be called dislocated electron.

The separation of the dislocated electron from the metal rest must, in our opinion, require a quantity of energy that is proportional to

[1]) The value of the reaction temperature of CaO is taken from a communication by P. Schläpfer (Schweiz. Chem. 1919, Heft 29 (30), the values of the ionisation-potential are derived from a summary given bij J. Franck (Phys. Zeitschr. 22, 413 ('21). The values of T taken for *Mg* and *Ca* will be discussed elsewhere, among others because reduction- and cyanizing-temperature (resp. the temp. of the formation of metal cyanamid) differ considerably for (the compounds of) these elements.

$e\,(V-V')$ (in which e represents the charge of the electron), and which is derived from the available thermal energy of the medium. Putting the latter in approximation proportional to T, we find:

$$e\,(V-V') = kT,$$

so that the following relation is found:

$\frac{V-V'}{T}$ must be constant for all cyanizing reactions.

f. Excitation potential.

With the analogous structure of the "outer electron shells" of the homologous elements it is probable that in the metal oxide the dislocated electron as a rule and on an average will be in a corresponding abnormal path. With the available date we can, however, not say which. When we, however, compare the values of the excitation tension V'' of the different elements, which quantity is decisive for the energy-supply required to transfer an "outer" electron in the metal atom from the normal into the first abnormal path (Row 5 of the table) with the ionisation potential (row 3), the analogous course of these values with increasing electro-positivity of the elements, is striking. When we, therefore, introduce the quantity V—V'' instead of V—V' we should, reasoning in the same line, obtain practicable results not only for these cases in which the abnormal path of the valency electron in the compound would be identical with the first abnormal path, but also when the position of the dislocated electron would be identical with another abnormal path. A considerable difference between V'' and V' is, however, unlikely, because then the value of the quantity V—V'' would no longer be in accordance with the considerable energy-supply required if the primary reaction is to take place [1]).

In row 6 the values are recorded of the quantity $\frac{V-V''}{T}.10^3$. It is seen that the difference between the alkalies and the earth alkalies is smaller than in row 4. Considerations for a further correction wil be given elsewhere.

g. In conclusion we will remark that with the aid of RUTHERFORD-BOHR's atomic model we have endeavoured in the above to

[1]) This is the more cogent as moreover at the complete addition of the "outer" electron to the oxygen rest energy is liberated. We have not considered this more closely, because this increase of energy with regard to the oxygen rest may be put equal for all the metal oxydes considerated, and can therefore not give occasion to characteristic differences. [See also "note at the correction"].

give an example of the view that at least in definite cases, ionisation- and dislocation-potentials are not only decisive with regard to the possibility of reaction, but also with regard to reaction velocity and reaction temperature.

PERRIN sees photo-chemical action in every chemical action. Our insight into the structure of the atom makes us realize that the fundamental feature is the displacement of the electrons in it. This displacement may be brought about by radiation, but also in various other ways. Accordingly it is not justifiable to assign merely a part to light in the explanation of chemical action; other forms of energy also make their influence felt. In connection with the conception of an interaction between the different forms of energy, the possibility might, however be considered of a derivation, even though it be a formal one, of the relation $\frac{\text{critical energy supply}}{\text{reaction temperare}} =$ constant from the laws of radiation, in which case the directing lines given by R. C. TOLMAN [1]) and E. K. RIDEAL [2]) should be taken into account. This will be treated elsewhere.

Dordrecht, June 26th 1923.

Note at the correction. From recent determinations of the electron-affinity of some electronegative elements as well as from known electro-chemical date may be deduced — as will be shown elsewhere — that the addition potential of an electron to an atom of oxygen can at most be about 2 volt. This value confirms the assumptions given sub *f* and justifies the neglect of the addition potential of the valency-electron to the oxygen, the value of which in our cases (as a rule) can only be little.

July, 4th, '23.

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[1]) R. C. TOLMAN. Journ. Amer. Chem. Soc. 42, 2506 (1921).

[2]) E. K. RIDEAL. Phil. Mag. **42**, 156 (1921).

Bactériologie. — *„Culture du bactériophage sans intervention de bactéries vivantes"*, par F. D'HERELLE.

(Présenté par Mr. le Prof. W. EINTHOVEN dans la séance du 26 mai 1923).

On sait que l'activité des différentes souches du Bactériophage est essentiellement variable, cette activité se mesure par l'intensité de l'action destructrive vis-à-vis des bactéries sensibles. Un Bactériophage au maximum d'activité, introduit a l'unité dans une émulsion bactérienne, provoque la dissolution totale des bactéries présentes; le milieu est, l'action terminée, aussi limpide que du bouillon filtré et reste tel indéfiniment. Ces souches au maximum d'activité sont rares dans la Nature, c'est pourtant exclusivement a de telles souches qu'il faut s'addresser pour étudier et comprendre le mécanisme intime du phénomène de bactèriophagie qui, avec des souches moins actives, est déformé ou masqué par des phénomènes secondaires liés à la résistance des bactéries.

En possession d'une souche du Bactériophage possédant une activité maxima vis-à-vis d'un Staphylocoque blanc, j'ai tenté la culture de ce Bactériophage aux dèpens d'une émulsion de la bactérie sensible, non plus vivante comme dans toutes les expériences jusqu'ici realisées, mais morte.

Le Staphylocoque sensible est cultivé sur bouillon gélosé; après 24 heures d'incubation la surface de la gélose est lavée avec une petite quantité d'eau salée à 9 p. 1000: on obtient une émulsion épaisse qui, répartie dans des tubes qui sont scellés, est maintenue pendant une heure dans un bain-marie règlé à 58—60° C. La sterilité est vérifiée par ensemencements.

Cette émulsion épaisse est répartie dans des tubes renfermant 10 c.c. d'eau distillée sterilisée, salée à 9p. 1000, de manière a obtenir une opacité correspondant environ à 200 millions de Staphylocoques tués par c. c. Tel est le milieu employé pour la culture du Bactériophage.

Un tube de ce milieu est ensemencé avec 10^{-2} c.c. d'une émulsion en bouillon du Staphylocoque vivant, dissoute sous l'action du Bactériophage en expérience puis filtrée. Le tube est placé à l'étuve à 37° pendant 24 heures. 10^{-2} c.c. de la première èmulsion est alors introduit dans un second tube, renfermant 10 c. c. de l'émulsion

du Staphylocoque tué par la chaleur. Les passages successifs étan ensuite continués de même manière.

Au dixième passage, le taux, facilement calculable, de la dilution du centième de c. c. du liquide renfermant le Bactériophage, introduit dans le premier tube de la série, est de 10^{-30}. Si le Bactériophage se trouve encore dans ce dixième tube, on peut être certain qu'il y a eu culture car, du liquide introduit dans le premier tube, il ne pourrait rester dans ce dixième tube qu'une fraction d'un électron, par suite des dilutions successives. D'autre part on ne peut admettre que le Bacteriophage soit constitué par des corpuscules ne représentant qu'une fraction d'un électron.

Or, après incubation, 13^{-7} c. c. de l'émulsion du dizième passage, introduit dans uns émulsion en bouillon du Staphylocoque vivant provoque une bactériolyse totale: preuve que le Bactériophage s'est maintenue à travers les passages. Il n'a donc pu se multiplier qu'aux dépens des bactéries mortes.

Je suis arrivé actuellement au vingt-troisième passage (dilution du liquide primitif 10^{-69}): après 24 heures d'incubation l'activité de l'émulsion est la même qu'au dizième passage. c'est à dire que 10^{-7} c. c. introduit dans une émulsion en bouillon de la bacterie vivants provoque la bactériolyse.

Je me suis naturellement assuré que l'émulsion seule du Staphylocoque tué par la chaleur, de même d'ailleurs que vivant, ne possède aucune propriété bactériolytique.

Le Bactériophage se cultive donc indubitablement aux dépens de bactéries mortes. Contrairement à ce qui se produit lorsqu'il se cultive aux dépens de bactéries vivantes, les bactéries mortes ne sont pas dissoutes.

Les caractères du Bactériophage qui s'est développé dans une émulsion de bacteries mortes sont fort différents de ceux qu'il présente quand il se développe aux dépens de bactéries vivantes: dans ce dernier cas le Bactériophage présente une résistance à l'action de la chaleur et des antiseptiques, analogue à celle des spores bactériennes; cultivé aux dépens de bactéries mortes il est au contraire très sensible: il est tué par une exposition de quinze minutes à une température de 56° C. de même que par un séjour de dix heures dans l'eau renfermant 20 p. 100 d'alcool ou d'acétone; il ne résiste pas plus longtemps dans l'eau saturée d'éther ou renferment des traces d'iode. De plus il ne traverse plus les bougies de porcelaine, non pas a cause de ses dimentions (l'examen de préparation colorée ne montre aucun corpuscule visible), mais, vraisemblablement, porcequ'il est adsorbé par le filtre, ce qui se produit d'ailleurs pour

d'autres ultravirus. Par contre, la virulence ne semble pas atteinte car, mis en présence de bactéries vivantes, il provoque leur dissolution d'une manière très active: il récupère alors ses propriétés de résistance à la chaleur et aux antiseptiques et il traverse les bougies de porcelaine, même serrées.

Ces différences de résistance confirment ce que des expériences antérieures semblaient déja indiquer; le Bactériophage présente deux formes: une forme végétative et une forme de résistance. Toutes deux coexistent dans les cultures aux dépens de la bactérie vivante: les formes de résistance ne pouvant vraisemblablement se produire que *dans* les bactéries vivantes. Dans les cultures en présence de bactéries mortes, seules existent les formes végétatives qui, des lors, se reproduisent uniquement par scissiparité.

J'ai effectué des expériences complementaires qui montrent que la culture du Bactériophage peut également s'effectuer dans des macérations du Staphylocoque tué *centrifugées*, c'est à dire dans un liquide de macération débarassé des corps microbiens: la forme végétative se cultive donc, non pas dans la bactérie tuée, mais dans le milieu, aux dépens des produits bactériens solubles.

Par des expériences antérieures, j'ai montré que le Bactériophage est un être autonôme, possédant des caractéres propres, indépendants de la bactérie qui subit son action, se qui donnait la preuve, qu'il se multiplie en milieu hétérogène. Ce fait implique le pouvoir d'assimilation chimique, caractère fondamental qui suffit pour caractériser la nature vivante de l'être qui le possède. Les présentes expériences ne font que confirmer la nature vivante du Bactériophage.

Nombre d'auteurs ont voulu expliquer le phénomène de bactériophagie et la reproduction du principe actif en cours d'action, comme un phénomène lié au métabolisme microbien. Cette explication, déjà contredite par le fait de l'autonomie du Bacteriophage, tombe définitivement devant le fait de la culture dans de l'eau salée ne renfermant que des bactéries mortes, ou même leurs seuls produits solubles, car une bactérie morte, ou ses produits solubles, ne sont plus qu'un assemblage de substances chimiques inertes, incapables d'aucun acte de métabolisme [1]).

(*Institut d'Hygiène tropicale de l'Université de Leyde*).

[1]) Je tiens naturellement à la disposition des expérimentateurs la souche du Bactériophage avec laquelle j'ai realisé ces expériences.

Géologie. — *„Description de Crustacés décapodes nouveaux des terrains tertiaires de Borneo"*, par V. VAN STRAELEN.

(Présenté par Mr. le Prof. H. A. BROUWER dans la séance du 26 mai 1923).

Les Crustacés décapodes décrits dans cette note, ont été recueillis par M. le Dr. G. L. L. KEMMERLING, au cours d'un voyage d'exploration effectué en 1912, dans le bassin du fleuve Barito, au S. E. de l'île Borneo [1]). Ces fossiles, conservés au Musée géologique de la Technische Hoogeschool à Delft, m'ont été obligeamment communiqués par M. le Professeur G. A. F. MOLENGRAAFF, directeur de ce Musée.

Famille : RANINIDAE Dana 1852.
Genre : RANINA Lamarck 1818.
Sous-genre : LOPHORANINA Fabiani 1910.

Ranina (*Lophoranina*) *Kemmerlingi* nov. sp. (Fig. 1, 2a, b.).
= *Ranina* sp., in G. L. L. Kemmerling (l.c. p. 740, pas fig.).

Fig. 1.
Ranina sp. (schéma). — $^1/_2$ grandeur naturelle.
Céphalothorax indiquant en grisé, la région à laquelle appartient le fragment décrit sous le nom. de R. KEMMERLINGI.

Les restes assez fragmentaires de cette espèce se réduisent à la partie droite de la région postérieure du céphalothorax et d'un article d'un péréiopode droit, probablement le troisième.

Les crêtes du céphalothorax, caractéristiques du sous-genre, sont disposées transversalement. Elles sont onduleuses, irrégulières, ne présentant aucun parallélisme et concaves vers l'avant, tout au moins dans la moitié postérieure du céphalothorax. Ces crêtes sont garnies en avant par un grand nombre de tubercules subépineux. Le bord du céphalothorax, souligné par un sillon, est granuleux ainsi que l'article, probablement un carpopodite, du péréiopode encore conservé.

[1]) G. L. L. KEMMERLING. Topografische en Geologische Beschrijving van het Stroomgebied van de Barito, in Hoofdzaak wat de Doesoenlanden betreft (Tijdschrift van het Koninklijk Nederlandsch Aardrijkskundig Genootschap, 2de ser., Deel XXXII, p. p. 575–641 & p. p. 717—774, carte et nombreuses figures dans le texte, Leiden 1915).

C'est une espèce de grande taille, dont les dimensions devaient atteindre celles que présentent souvent des formes actuelles, telles que *Ranina serrata* Lamarck des mers du Japon. Les autres repré-

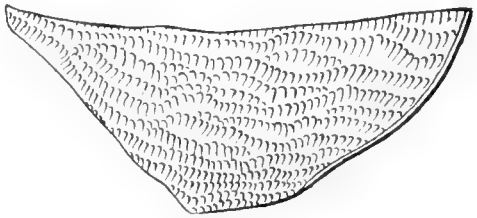

Fig. 2a.

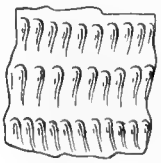

Fig. 2b.

Ranina (Lophoranina) Kemmerlingi nov. sp. — Face dorsale.
2a. Fragment du céphalothorax. — Grandeur naturelle.
2b. Crêtes du céphalothorax. — × 3.

sentants fossiles du genre *Ranina*, dont l'espèce de Borneo se rapproche le plus par les caractères de son ornementation, sont :

Ranina laevifrons Bittner, du Lutétien du Vicentin,

R. Bittneri Loerenthey, du Bartonien du Vicentin et de la Hongrie,

R. Reussi H. Woodward, du Bartonien de la Hongrie,

R. Marestiana Koenig, du Priabonien du Vicentin,

R. porifera H. Woodward, de l'Oligocène inférieur de l'île Trinidad.

R. Kemmerlingi se distingue :

de *R. laevifrons* par ses tubercules arrondis à peine spiniformes,

de *R. Bittneri* par ses tubercules moins serrés et dépourvus de ponctuation,

de *R. Reussi* par ses crêtes plus nombreuses, plus serrées et ses tubercules moins distants,

de *R. Marestiana* par ses crêtes plus serrées et garnies de tubercules plus espacés mais plus volumineux,

de *R. porifera* par l'absence de ponctuation en avant des tubercules des crêtes.

Type. Musée géologique de la Technische Hoogeschool à Delft, échantillon No 6561 et 6562, empreinte et contre empreinte.

Gisement. Etage γ de R. D. M. Verbeek, probablement Oligocène.

Localité. Vallée du fleuve Barito (Borneo).

Famille : Calappidae Dana 1852.

Genre : Calappilia A. Milne Edwards 1873.

Calappilia borneoensis nov. sp. (Fig. 3).

Les marnes calcarifères avec débris de végétaux de l'étage β de R. D. M. Verbeek, renferment parfois de nombreux débris appartenant à un Brachyoure de petite taille. Ces restes toujours incomplets, sont à rapporter au genre *Calappilia* A. Milne Edwards.

La région frontale est étroite et se prolonge en avant par un faible rostre, les orbites semblent avoir été larges et peu profondes. Les sillons limitant une région gastro-cardiaque étroite sont peu profonds, les régions branchiales sont relativement étendues. La surface du céphalothorax est ornée de petits tubercules arrondis, d'autant plus saillants qu'ils sont plus rapprochés des bords, leur nombre augmentant dans les régions postérieures du céphalothorax. L'espace compris entre les tubercules est occupé par de fines granulations. Les bords latéraux rencontrent le bord postérieur sous un angle à peu près droit et se prolongent postérieurement, par une épine. La face sternale n'est pas connue.

Fig. 3. *Calappilia borneoensis* nov. sp. Face dorsale. × 2. Reconstitution à l'aide de fragments provenant de 5 individus.

Les fragments de céphalothorax sont accompagnés de débris de péréiopodes, trop morcelés pour qu'on puisse les décrire. Tout ce qu'il est possible de voir est que comparativement au corps de l'animal, ces péréiopodes étaient extrêmement développés. Jusqu' à présent, on ne possède pas d'autres renseignements sur les appendices du genre *Calappilia*.

Ces caractères sont suffisants pour distinguer cette forme, de toutes les espèces de *Calappilia* décrites jusqu'à ce jour. Le genre a été rencontré depuis l'Eocène moyen jusqu' à l'Oligocène moyen. On en connait les espèces suivantes:

Calappilia incisa Bittner, du Lutétien du Vicentin,
C. dacica Bittner, du Bartonien de la Hongrie,
C. verrucosa J. Boehm, de l'Eocène supérieur de Java,
C. perlata Noetling, du Tongrien du Samland,
C. vicetina Fabiani, du Tongrien du Vicentin,
C. sexdentata A. Milne Edwards et
C. varucosa A. Milne Edwards, du Rupélien de Biarritz.

L'espèce de Borneo se distingue de toutes les *Calappilia* sauf de *C. varucosa* A. Milne Edwards, par l'absence de nombreux tubercules spiniformes sur les bords latéraux du céphalothorax. Son ornementation la rapproche également de cette espèce de Biarritz,

elle s'en écarte cependant par ses tubercules plus saillants et moins également répartis sur toute la surface du céphalothorax. D'autre part, elle se distingue de *C. verrucosa* J. Boehm par son céphalothorax plus circulaire et quasiment hémisphérique.

Type. Musée géologique de la Technische Hoogeschool à Delft, échantillon No. 6563.

Cotypes. Echantillons No. 6564, 6565, 6566, 6567.

Gisement. Etage β de R. D. M. Verbeek = étage marneux (Mergeletage) = Eocène, probablement Lutétien.

Localité. 2 Kilomètres à l'Ouest de Kampong Lemoe (Borneo).

Neurology. — "*A partial foetus removed from a child.*" By Prof. C. WINKLER.

(Communicated at the meeting of June 30, 1923).

A few months ago a child of nearly three months, was brought in my clinic, having a fluctuating tumour in the neck and a not very intensive internal hydrocephalus.

Apparently it suffered from spina bifida, as the transverse processus of the 2d and 3d cervical vertebrae stood far apart and the processus spinosi were missing. The examination of the tumour made it probable that a myelo-cysto-cele might be found in it.

For the rest this healthy child had normal breathing, responded to pin-pricks with mimic facial expressions and spontaneously moved its four extremities.

The tumour, filled up with liquor, was opened by Dr. WALLER. He found in the middle of the fluid a strongly vasculated stalk, nearly 1 c.M. in diameter, connecting the deep tissues in the midline with the external wall of the tumour-cyst. After underbinding the stalk in the depth, he removed stalk and cystic tumour. In a week the child recovered. As I saw it again, six weeks after the operation, it appeared to be a rather normal child of circa five months of age.

The removed specimen was given to me.

A section made through the middle of the stalk proved, that it was a spinal cord surrounded by an intensely vasculated membrane (fig. 1*a*). In this spinal cord the columna posterior had disappeared and the dorsal wall of the central canal was open. The form of the central canal was as this figure shows.

The lateral, the anterior column and the grey matter were easily recognisable. In the lateral column the area of LISSAUER, the spino-cerebellar tracts and the surroundings of the grey substance are myelinisated. In the anterior funiculi was seen a strongly marrowed commissura anterior, and the tecto-spinal path has also gained marrow. Both, not medullated, pyramidal tracts are recognisable.

The substantia Rolando is strongly developed. The anterior horns contain atrophic large cells.

From this cord, ventral and dorsal medullated rootlets take origin.

Examining sections through the central end of the stalk (fig. 1*b*), the central canal widens. The lateral part of the medulla disappears and only a ventral rest of nervous tissue remains lined by the

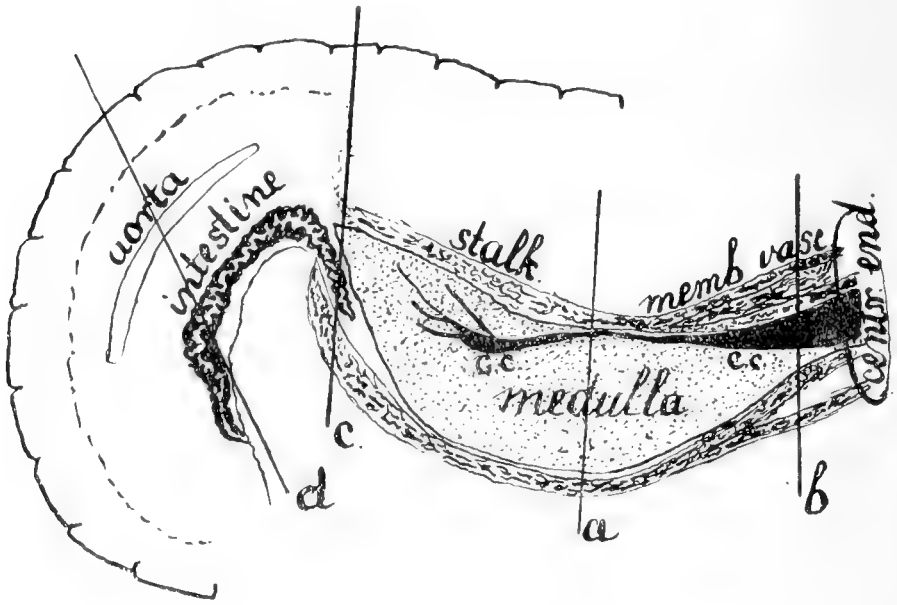

Fig. 1. Wall of the tumor and stalk.

ependyma of the central canal, now irregularly shaped and wound in an irregular way. The membrana vascularis also divides in two membranes, leaving a hole between them.

Examining sections through the stalk, towards its entrance in the skin (fig. 1*c*), the central canal soon closes dorsally. Its shape changes into another form, then it ends into many branches, one of which may be followed, lying excentrically, to the end.

At that moment the nervous tissue is represented by *a*. strongly medullated fibres of the medullar columns *b*. medullated posterior roots with well developed spinal ganglia (fig. 1*c*).

At the moment that the stalk reaches the skin, there is found, ventrally from what seems to me to be the caudal end of the spinal cord, a tube, which soon appears to be the intestinal tube.

Sectioning the wall of the tumour, caudally from the entrance of the stalk, it appears to contain the caudal end of an imperfectly developed, partly atrophied, foetus.

In foto 2 is seen, that cutis and subcutaneous tissue with hair-follicles and sudorific glands is separated from the new tissue by a system of lacunae, filled up with blood and bordered by endo-

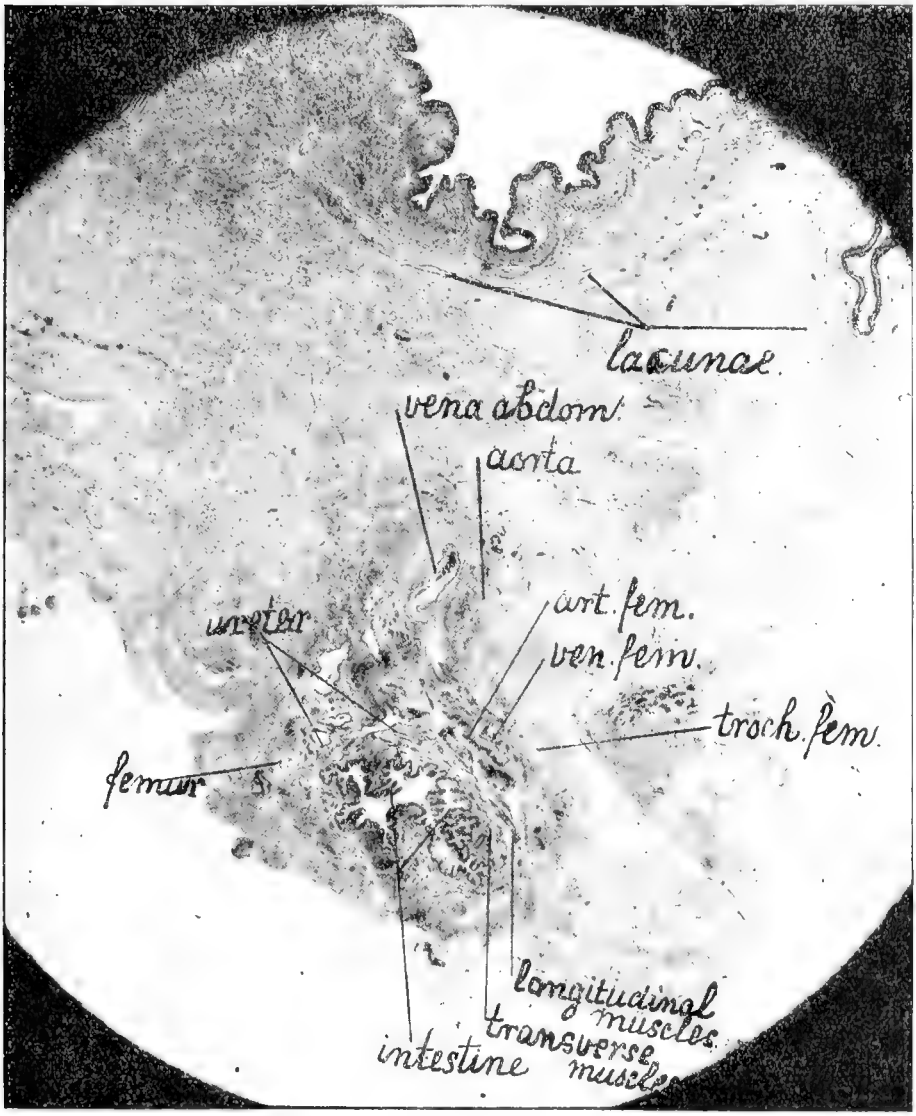

Fig. 2. Foto from a section through the wall of the tumour (see fig. 1*d*.)

thelium-cells. Most striking however is the deeper part. A transverse section of a tube is found there, whose internal surface is irregularly wound.

It is formed by a single layer of cylindric epithelium, placed upon a membrana basilaris and bordered towards the lumen of the

tube by a transparent band with small transverse lines — a hem of cell-cilia. The loose connecting tissue, building the basilar membrane upon which the epithelium-cells repose, is surrounded by a transversal and by a longitudinal muscle-layer. I consider this tube to be the intestinal tube.

Dorsally from this tube are found the large vessels, aorta and vena abdominalis. In the foto (fig. I *d*) the section touches the left femur; at the right side the trochanter femoris is found. Also both ureters and more caudalward the bladder is seen.

In this way it appears that the wall of the tumour contains the caudal end of an insufficiently developed foetus, connected to the well developed child by a stalk, containing the caudal end of a medulla spinalis.

I presume a double monstrum, a duplicitas posterior is here present. The single head of this monstrum was followed by a double caudal part of the body. The one half of the body developed normally.

The other half atrophied. A relatively well developed medulla remained in the stalk, the caudal end of the foetus was found in the wall of the fluctuating tumour.

Hence this female child carried its atrophied twin-sister at her back. The superfluous atrophic foetus was removed and it is not impossible, that the remaining child may grow up normally.

ERRATUM.

On p. 310 of this volume line 13 from the top to omit the words *and Wolffian Ducts* and to read: by the kidney-tubules (TITSCHACK 1922, a. s. o.).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXVI

Nos. 7 and 8.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vol. XXXII).

CONTENTS.

Physics. — "*A Relation between the Spectra of Ionized Potassium and Argon.*" (Second Communication). By H. W. J. DIK and Prof. P. ZEEMAN.

(Communicated at the meeting of June 30, 1923).

The observations of the spectrum of potassium vapour under the influence of the discharge without electrodes have now been completed. These measurements go up to λ 2342,3 Å. They, too, have been made with a quartz spectrograph. We begin Table IV with 3514.0, so that Table I of our first communication [1]) and Table IV for a small part overlap. The values of Table IV are more accurate, and have been obtained by direct comparison with the standard iron lines.

TABLE IV. Potassium lines on discharge without electrodes.

Intensity.				λ	ν	Remarks.
EV	S	McL	D			
—	1		9	3514.0	28458	
—	—	1	9	3490.8	28647	
1	1		10	3480.9	28728	
1	1		10	3476.4	28765	
—	1		9	3468.3	28833	
—	—	—	3	3457.4	28923	
—	—	—	2	3447.8	29003	
10	2		—	3447.5		arc-line
	3		—	3446.5		arc-line
6	3		20	3439.9	29070	
1	2		15	3433.2	29128	
—	—	—	2	3427.0	29180	

[1]) These Proceedings, Vol. XXV, p. 67.

TABLE IV (*Continued*).

Intensity.				λ	ν	Remarks.
EV	S	McL	D			
1	1		15	3421.9	29223	
—	—	—		3421.0	29231	
—	—	—	1	3417.0	29266	
2	2		15	3404.2	29376	
—	1		9	3392.6	29476	
6	4		15	3384.6	29545	
6	4		15	3380.3	29583	
1	3		15	3373.5	29643	
—	6		20	3363.9	29727	
1	8			3362.5	29739	
	2		2	3358.6	29774	
—	2		2	3356.2	29796	
8	5	6	15	3345.0	29895	
—	—	—	2	3338.0	29958	
—	—	3	2	3336.1	29975	
1	—	—	—	3326.4		
—	—	3		3324.7		
1	4	—	15	3322.2	30101	
3	4	5	15	3311.9	30194	
—	—	3	15	3301.2	30292	
3	3	5	15	3289.8	30397	
—	—	—	1	3285.5	30437	
—	3	3	10	3278.6	30501	
—	2	3	10	3261.9	30657	
—	—	3	3	3258.6	30688	
—	—	—	3	3253.9	30732	
—	—	3	—	3244.5		
—	2	—	10	3241.2	30853	
—	—	—	4	3237.8	30885	

TABLE IV (*Continued*).

Intensity.				λ	ν	Remarks.
EV	S	McL	D			
—	—	—	1	3226.9	30989	
1	1	1	8	3224.2	31016	
2	2	0	10	3220.2	31054	
—	1	—	3	3218.5	31071	
2	1	—	—	3217.5		
—	—	0	—	3213.0		
1	4	—	10	3209 0	31162	
—	—	4	—	3205.6		
1	3	—	10	3201.8	31232	
2	2	5	10	3190.0	31348	
—	2	.	10	3187.7	31371	
—	—	2	3	3171.8	31528	
1	1	4	9	3169.6	31550	
1	1	3	6	3157.0	31676	
—	—	2	—	3148.6		
—	—	—	1	3145.1	31795	
3	—	—	4	3142.7	31820	
4	2	4	15	3128.8	31961	
—	—	—	5	3109.7	32157	
5	4	—	15	3104.9	32208	
1	1	6	4	3102.9	32228	
1	1	2	8	3074.7	32524	
1	—	—	—	3067.3		
6	5	5	20	3061.7	32661	
1	2	0	10	3056.5	32717	
1	3	2	15	3051.9	32767	
—	—	2	3	3047.0	32819	
1	1	3	9	3030.4	32999	
1	2	3	10	3023.5	33074	

TABLE IV (*Continued*).

Intensity.				λ	ν	Remarks.
EV	S	McL	D			
4	3	3	15	2992.4	33418	
1	1	2	9	2986.3	33487	
—	—	2	4	2965.1	33726	
—	1	—	—	2963.4		
—	—	1	4	2954.3	33849	
—	—	—	1	2950.9	33888	
—	—	—	1	2942.9	33980	
1	1	2	10	2938.6	34030	
—	—	—	3	2927.9	34154	
—	—	1	4	2925.8	34179	
—	—	2	8	2903.4	34442	
—	—	—	1	2898.9	34495	
—	—	—	1	2893.9	34555	
—	—	—	3	2880.2	34720	
—	—	—	1	2877.5	34753	
—	—	—	2	2872.5	34813	
1	1	2	10	2854.4	35034	
—	—	—	4	2851.7	35066	
—	—	—	1	2847.7	35117	
—	—	3	2	2834.7	35277	
1	1	—	—	2833.3		
—	—	2	3	2824.3	35408	
	—	—	3	2821.9	35437	
1	1	3	10	2819.3	35469	
—	—	—	5	2809.0	35600	
—	—	—	9	2806.3	35634	
—	—	—	5	2804 6	35656	
—	—	5	—	2803.5		
—	—	—	3	2800.3	35711	

TABLE IV (*Continued*).

Intensity.				λ	ν	Remarks.
EV	S	McL	D			
1	1	—	—	2780.2		
—	—	1	5	2777.9	35998	
—	—	—	1	2776.1	36022	
—	—	1	—	2768.1		
—	—	—	1	2763.4	36187	
—	1	2	9	2743.5	36450	
1	1	4	5	2736.2	36547	
—	—	1	—	2732.0		
1	1	3	9	2690.0	37175	
1	—	3	3	2662.8	37555	
—	—	1	1	2657.1	37636	
1	1	4	10	2635.1	37950	
—	—	0	—	2630.0		
1	1	2	1	2613.8	38259	
—	—	—	3	2572.4	38874	
—	—	1	—	2569.8		
—	—	—	3	2561.3	39043	
—	—	2	—	2559.2		
2	1	4	10	2550.0	39215	
—	—	—	1	2542.0	39339	
—	1	—	—	2538.7		
—	—	3	5	2536.4	39425	
—	—	—	5	2504.6	39927	
—	—	2	—	2502.4		
—	—	—	1	2485.5	40234	
—	1	3	8	2473.2	40434	
—	—	3	—	2470.4		
—	—	2	1	2452.7	40771	
—	—	—	1	2448.9	40834	

TABLE IV (*Continued*).

Intensity.				λ	ν	Remarks
EV	S	McL.	D			
—	—	4	1	2447.2	40864	
1	1	2	7	2440.0	40984	
—	—	4	1	2436.7	41039	
—	—	—	2	2431.1	41134	
—	—	—	1	2415.4	41401	
—	—	—	1	2414.4	41417	
—	—	3	—	2410.4		
—	—	—	1	2404.5	41588	
—	—	3	—	2402.0		
—	—	3	—	2393.4		
—	—	—	1	2389.1	41857	
1	1	—	2	2379.2	42031	
	—	4	—	2376.3		
	—	5	2	2369.6	42202	
	—	2	—	2365.8		
	—	7	—	2362.6		
1	1	—	—	2358.9		
1	1	3	—	2350.3		
—	—	4	—	2348.3		
1	—	—	—	2344.7		
1	1	3	1	2342.3	42693	

The constant differences seem soon to stop below λ 3000. This may be in connection with the appearance of the second spark spectrum of potassium.

We have, however, also started an investigation of the lines that satisfy formulae with fourfold and ninefold RIJDBERG constans. By this way the proof might be furnished that the observed spectrum belongs to once ionized potassium; besides, a quantitative comparison with the red argon spectrum may perhaps be possible.

Physics. — "*Further experiments with liquid helium. S. On the electric resistance of pure metals, etc.* XII. *Measurements concerning the electric resistance of indium in the temperature field of liquid helium*". (Comm. N°. 167*a* from the Physical Laboratory at Leiden). By W. TUYN and Prof. H. KAMERLINGH ONNES.

(Communicated at the meeting of June 30, 1923).

§ 1. *Purpose of the investigation. Method of construction of the resistances.* For the further detection of supra-conducting metals it seems desirable to investigate the behaviour of those elements which take a place near already known supra-conductors in the periodic system. Indium — above thallium and by the side of tin — seemed a suitable metal.

The chemically pure indium (4 grammes) was supplied by E. DE HAËN, G. m. b. H. [1]). From wire extruded from this to a thickness of 0.1$_7$ m.m. we constructed the resistances *In*—1922—*I*, W_0=4,704$_7$ Ω, *In*—1922—*II*, W_0=3,708$_7$ Ω and *In*—1922—*III*, W_0=3,799$_7$ Ω; the resistances were, however, not enclosed in helium gas. A fourth resistance, *In*—1922—*A*, W_0 = 4,609$_6$ Ω was obtained by winding another piece of the same wire also bifilarly on a glass tube; silk thread served here for insulation [2]). The values W_0 were determined on December 22nd 1922 in the way as described in Comm. N°. 160*a*.

§ 2. *Measurements in liquid helium.* The four resistances were placed in the cryostat provided with a stirring apparatus, represented in Comm. N°. 124*c*, fig. 4. The measurements took place by compensation of the potential at the extremities of a known and an unknown resistance connected in series, by the aid of a compensation

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[1]) The high amount of $\left(\frac{W}{W_0}\right)_{T=4°,2\,\mathrm{K.}}$ of *all* the resistances constructed from this made us doubt the purity of the indium supplied. On inquiry the firm told us in a letter dated March 22nd 1923 "that they had sent chemically pure indium metal, free from impurities".

[2]) Old indium wires are difficult to fuse together to obtain the four required extremities; treatment with HCl removes this difficulty.

apparatus free of thermo-electromotive forces by DIESSELHORST's method, supplied by O. WOLFF; the strength of the current through the resistances was 4 m.a. For the determination of the temperatures the vapour pressures of the helium-bath were measured, below 400 m.m. Hg. with the cathetometer; the corresponding temperatures were then derived by means of the formula of Comm. N°. 147*b*, p. 33 [1]).

The results of the measurements follow in the tables I, II, III and IV. Near the vanishing-point, where the successive temperature

TABLE I. Indium—1922—I.

Date.	p_{helium} in m.m. Hg.	T.	$w = \left(\frac{W}{W_0}\right)_{In-1922-I.}$ [2])
December 8, 1922	775.4	4.°23 K.	0.1373_2
December 20, 1922	394.4_9	3.60	0.1372_3
	339.3_9	3.48	0.1371_4
	310.1_1	3.41	0.1370_8
	308.8_3		0.1370_7
	307.4_0		0.1367_8
	306.8_9		0.1367_2
	305.4_5		0.1364_0
	304.0_0		0.1363_6
	301.5_9		0.1359_7
	299.5_9		0.132
December 8, 1922	299.4_6		0.120
December 20, 1922	298.1_8		0.016
	295.4_8	3.38	0.0000_3
December 8, 1922	12.4_0	1.87	0.00000

[1]) This formula has been calculated out of measurements performed 1913. If by the side of these measurements one takes those of 1911 into account, and interpolates graphically, temperatures are obtained which often considerably deviate from those calculated with the formula. The vanishing-point temperature of thallium e.g., graphically derived in this way in Comm. No. 160*a*, is 2,°32 K; the formula gives 2,°47 K. Until the vapour pressure curve of helium is more accurately known, we give the read vapour tensions, and state also how we have calculated the temperatures from them.

[2]) Below the vanishing-point the measured potential differences have been recalculated to resistances, as if OHM's law were valid.

TABLE II Indium—1922—II.

Date.	p_{helium} in m.m. Hg.	T.	$w=\left(\frac{W}{W_0}\right)_{In-1922-II.}$
December 20, 1922	394.3_7	3.60° K.	0.03392
	339.5_5	3.48	0.03387
	309.8_4	3.41	0.03387
	308.7_4		0.03385
	307.4_0		0.0202
	306.8_9		0.0067
	305.9_9		0.00000
	304.0_0		0.00000

TABLE III. Indium—1922—III.

Date.	p_{helium} in m.m. Hg.	T.	$w=\left(\frac{W}{W_0}\right)_{In-1922-III.}$
December 8, 1922	775.4	4.23° K.	0.03390
	333.7	3.46	0.03380_5
	310.5_4	3.42	0.03380_5
	309.0_5		0.03377
	307.6_9		0.0207
	305.9_0		0.0001_8
	304.7_8		0.00000
	12.4_0—12.5_8	1.87	0.00000

TABLE IV. Indium—1922—A.

Date.	p_{helium} in m.m. Hg.	T.	$w=\left(\frac{W}{W_0}\right)_{In-1922-A.}$
December 20, 1922	759.7	4.21° K.	0.03420
	394.3_7	3.60	0.03418
	339.5_5	3.48	0.03415
	309.6_6	3.41	0.03392
	308.9_3		0.0297
	307.1_4		0.0013_4
	307.0_2		0.0001_4
	306.2_3		0.0000_3
	304.0_0		0.00000

differences are small, we give only the vapour tensions. Sometimes the resistance is given here in fewer decimals than elsewhere; the slightest change in indication of the oil-regulator described in Comm. N°. 119 is the cause that the galvanometer in the region of the great decrease of resistance does not settle down.

The tables show (cf. also fig. 1) that the rest-resistance of *In*—1922—*I* above its vanishing-point temperature is much greater than that of the other wires, that for *In*—1922—*I* the temperature at which the resistance diminishes most, has been shifted about 0,02 degree with regard to the corresponding one for *In*—1922—*II* and —*III*, and that the fall extends over a larger temperature region. Calculations with the available data by the aid of

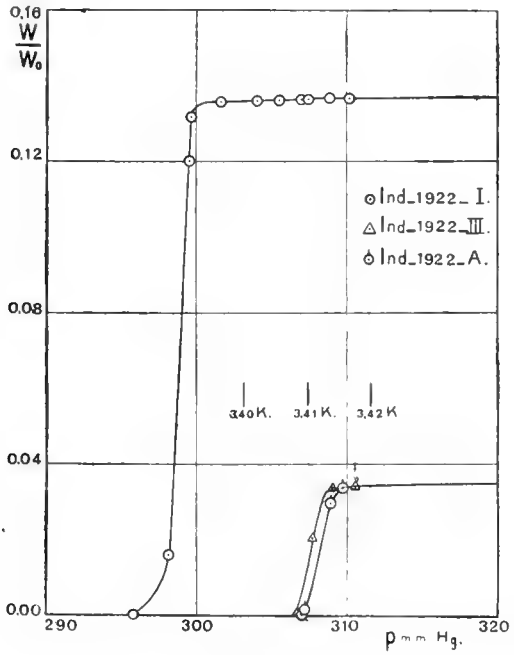

Fig. 1.

SILSBEE's hypothesis[1]) or by the aid of current-densities render improbable that the said displacement is caused by oxidation of *In*—1922—*I* throughout its length to such a degree that only a small nucleus of indium remained[2]); the ratio of the W^{0}'s in *In*—1922—*I*,

[1]) F. B. SILSBEE. Scient. Pap. Bur. of Stand. No. 307 (1917).

[2]) In contrast with the other wires *In*—1922—*I* presents a dull oxide-like surface. After the construction in July 1922 the resistance was preserved in benzine; though this was supposed to have been distilled, it seems to have contained impurities, which have attacked the wire.

$-II$ and $-III$ is incompatible with this[1]). It is improbable that the wire is strongly attacked over a small part, because then the question rises why the resistance of the better part of $In-1922-I$ does not disappear at the vanishing-point temperature of the two other indium resistances. This leads to the conception that the great rest-resistance of $In-1922-I$ is uniformly distributed throughout the whole wire. The equality inter se of this quantity over the three other wires makes this almost certain for them[2]). For the

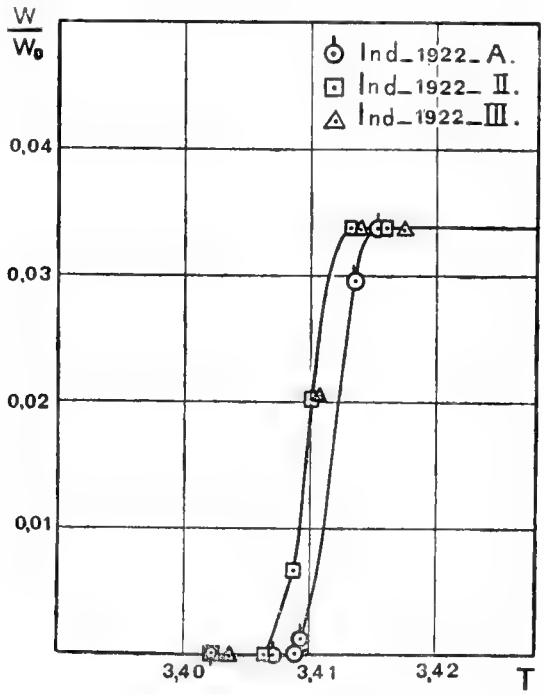

Fig. 2.

[1]) Measurements with $In-1922-A$ on the dependence of the magnetic threshold value of the temperature yield for indium roughly a required field of 1,4 gauss for a vanishing-point displacement of 0,02 degree. i is always 4 m.a. If in agreement with SILSBEE's hypothesis the inner magnetic field of $In-1922-I$ is to be 1,4 gauss larger than that of $In-1922-II$, the radius of $In-1922-I$ must have been reduced to about 0,005 m.m. by oxidation, which is incompatible with the ratios of the W_0's, when it is taken into consideration that the two resistances do not differ much in length.

The variation of the resistance with different intensities of the current has not been calculated in the experiments with indium wires. A current density 10-times greater in a tin-wire gave a vanishing-point displacement of about 0,02 degree according to Comm. 133*d*, table IX. With these values for indium the cross-section of $In-1922-II$ would have to be 10-times that of $In-1922-I$, which also gives a wrong ratio of the W_0's.

[2]) The great value and equality of this rest-resistance for all three wires made us doubt the purity of the supplied indium.

present a uniform distribution of the great rest-resistance of *In*—1922—*I* seems strange.

It also appears from the tables (cf. also fig. 2) that there exists a difference of 0,002 degree in vanishing-point temperature between *In*—1922—*A* on one side, and *In*—1922—*II* and —*III* on the other side. An explanation by the assumption of differences of temperature in the helium bath seems improbable. As far as the influence of the inner magnetic field is concerned, the windings lie at a distance of 0,4 for *In*—1922—*A*, at a distance of 2,2 m.m. for *In*—1922—*II* and —*III*; in definite parts of cross-section and area of a winding the inner magnetic field is weakened by that of adjacent windings, and the more so as they lie more closely together. On calculation [1]) this weakening appears too small to be able to account for the difference found in vanishing-point temperature between *In*—1922—*A* and *In*—1922—*II* and —*III*.

§ 3. *The supra-conducting metals in the periodic system of the elements.* The question rises whether the vanishing-point temperature has a periodic character. In the periodic system *In* lies above *Tl*, *Sn* above *Pb*; it is remarkable that the said temperature rises both going from *In* to *Sn*, and from *Tl* to *Pb*. Towards the left, from *Tl* to *Hg*, it also ascends; if this rise continues, the vanishing-point temperature of *Au* would lie higher than of *Hg*. Since *Au* did not become supra-conducting on cooling to 1°,5 K.[2]), the conclusion might be drawn that *Au* — perhaps with other metals — can never become so [3]).

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[1]) Cf. footnote 1, p. 508.

[2]) Cf. Comm. No. 120*b*, § 2.

[3]) In Comm. Suppl. No. 44, p. 35 the possibility is, on the other hand, given that the vanishing-point temperature of *Au* has not yet been reached on cooling to 1,°5 K.

Geology. — "*On the occurrence of diamond as an accessory mineral in olivine and anorthite bearing bombs, occurring in basaltic lava, ejected by the volcano Gunung Ruang (Sangir-Archipelago north of Celebes).*" By Dr. W. F. GISOLF. (Communicated by Prof. EUG. DUBOIS).

(Communicated at the meeting of June 30, 1923).

Dr. G. L. L. KEMMERLING, chief of the volcanological survey of the Dutch East Indian Archipelago, having collected some bombs out of the basaltic lavas from the volcano Gunung Ruang, composed of a mixture of a dark green to black mineral and glassy plagioclase, the latter in crystals of a size up to 1 cm., kindly intrusted those to the author for microscopical examination.

Two kinds of rock were collected; the first of these is dense and black and shows strong magnetic properties; examination with a magnifying glass reveals the presence of magnetite with a tinge of blue; under the microscope it proved to be composed of densely crowded grains of magnetite and between those one can indistinctly recognize strong pleochroitic hypersthene and green coloured monoclinic pyroxene.

The second kind, which contains much less magnetite, is composed of corroded olivine, fringed by a border of strong pleochroitic hypersthene; it also contains bottle-green monoclinic pyroxene. The plagioclase in both kinds of rock could be determined as belonging to members of the group which are very rich in anorthite.

In some of these rocks an accessory mineral occurs in a considerable number of minute grains. They may be seen most clearly in specimen 285 A in which the olivine can be recognized macroscopically; around the bomb a crust of the lava in which it is imbedded is to be seen; this lava has the composition of a basalt with bottle-green augite and very basic plagioclase.

The plagioclase is twinned according to the albite- and Carlsbad laws; one of the thin sections shows a plagioclase with three lamellae; the first shows in the conoscope, between crossed nicols, the emerging of the *Z*-axis within the field of view, slightly inclined to the surface of the section; the extinction amounts to 65 degrees. The second lamel shows in the conoscope the emerging of the *X*-axis, also

slightly inclined; the extinction is 32 degrees. The extinction of the third lamel is 85 degrees. These observations unmistakable point to a plagioclase very rich in anorthite.

The hypersthene is strongly pleochroitic from pale green to brown pink and has a double refraction rather strong for an orthorombic pyroxene. The hypersthene often contains freakishly formed grains of magnetite.

The olivine, as seen under the microscope, is colourless; it has a high double refraction and is always corroded.

The above mentioned accessory mineral occurs in well-shaped colourless crystals, which are most like to octahedrons, occasionally with pyramids on the planes, forming triakisoctahedrons; the size of these crystals is minute, in most cases they are thinner than the section is, about $^1/_{70}$ mm. The slide was difficult to cut. The crystals are isotropic, they diminish the colour of polarisation of the host-crystal, but are dark with the host-crystal between crossed nicols. They show a large black border in ordinary light, also when they occur in olivine, owing to their very high refraction.

In many cases the border is so large that only a cone of light emerges at or near the centre of the grains; this cone can be followed by moving the tube up and down. The mineral in question occurs both within olivine and anorthite; in the latter it is principally deposited on the planes of zonal structure. Besides the crystals also some irregular grains occur of the same substance, showing the same properties.

The hypersthene is, remarkable enough, devoid of this mineral or contains only some occasional grains.

A fragment of the rock with a flat side was chosen; under applications of pressure striae could be obtained on topaz and on corundum. Pressure had to be applied, because in preparing the slides it became evident that the grains of the mineral were easily jerked out; consequently many cavities are to be seen in the slides.

To resume: the mineral is isotropic, has an octahedral habitus, a very high index of refraction and a hardness exceeding that of corundum, if at least we may assume that the striae on these minerals are due to the minute grains; about this however little doubt is possible. From these observations we must conclude that the mineral is diamond; no further experiment being required, which would indeed be very difficult owing to the extreme minuteness of the grains.

Assuming this to be true, it seems to me, that it throws a wonderfull happy light upon the genesis of this mineral. As everywhere

else, the mother-rock has a peridotitic nature; but in this case there can be no question of layers of coal or shales, broken through by lava, fragments of which possibly could have been taken up in the lava and could be the source of the carbon in the rock.

The diamond is here a primary mineral and even older than the olivine.

The question left to be answered is this: why is the hypersthene free from grains of diamond, the olivine and anorthite containing them both? notwithstanding the fact that the hypersthene crystallized after the olivine and before the plagioclase.

The solution of this question is presumably, that the rock was originally wholly composed of olivine, and that in the cavities, formed by resorption, the anorthite crystallized; the olivine being resorbed the crystals of diamond were freed and suspended in the mother-liquor; the hypersthene has, by surface-tension, repelled these grains, which were collected in the anorthite, in which they occur as above stated, on the planes of growth or zonal structure. This being true, the reaction olivine → hypersthene + magnetite cannot have occurred in the solid phase because in that case there could have been no reason for the diamond to be driven out.

Mathematics. — *"The Complex of the Conics which cut Five Given Straight Lines."* By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of June 30, 1923).

§ 1. We can represent the conics k^2 cutting five given straight lines a_1, a_2, a_3, a_4, a_5 on the points of space by associating to each of these conics the pole K of its plane $\varkappa$ relative to a given quadratic surface O. To any point K there corresponds the conic k^2 in the polar plane $\varkappa$ of K passing through the points of intersection $A_1, A_2, \ldots, A_5$ of $a_1, a_2, \ldots, a_5$ with $\varkappa$.

For this representation the points of the straight lines $a'_1, a'_2, \ldots, a'_5$ which are associated to $a_1, a_2, \ldots, a_5$ relative to O, are singular. If we take for instance K on a'_1, $\varkappa$ passes through a_1 and A_1 becomes accordingly indefinite. To K there corresponds the pencil of conics k^2 passing through the points of intersection $A_2, \ldots, A_5$ of $\varkappa$ with $a_2, \ldots, a_5$. These are double conics of the system S_3 under consideration of ∞^3 individuals.

There are accordingly five straight lines a'_k of singular points of the second order. To a point of any of these straight lines there corresponds a pencil of double conics of S_3. Each of these straight lines is the representation of a system of ∞^2 conics the planes of which pass through one of the straight lines a_k and which cut the other four of these lines.

If we choose K on one of the two straight lines t'_{11} and t'_{12} cutting the lines a'_2, a'_3, a'_4, a'_5, e.g. on t'_{11}, $\varkappa$ passes through the associated straight line t_{11} intersecting $a_2, \ldots, a_5$, and this plane contains ∞^1 degenerate conics of S_3 associated to K consisting of t_{11} and a straight line through the point of intersection A_1 of $\varkappa$ and a_1.

There are accordingly ten straight lines $t'_{11}, t'_{12} \ldots, t'_{51}, t'_{52}$ of singular points of the first order. To a point of any of these lines there corresponds a pencil of degenerate conics and each of these straight lines is the representation of a system of ∞^2 degenerate conics of which one straight line is fixed and the other straight lines form a bilinear congruence.

§ 2. If K describes a straight line l', $\varkappa$ revolves round the asso-

ciated line l and k^2 has therefore always two points in common with l.

To a straight line l' of points K there corresponds accordingly a system S_1 of ∞^1 conics each of which cuts a line l twice and the straight lines $a_1, \ldots, a_5$ once.

Also the reverse is apparent.

If K describes a plane π, $\varkappa$ continues to pass through the pole P of π.

A plane π is the image of a system S_3 of ∞^3 conics the planes of which pass through a point P and which cut $a_1, \ldots, a_5$. Inversely such a system is represented on a plane.

To the conics of S_3 passing through a definite point P, the points of a plane curve k_P lying in the polar plane π of P are associated. In order to find the order of k_P, we try to find the number of conics of S_3 through P and through a definite point Q of a_1. The conics through P and Q intersecting a_3, a_4, a_5, form a surface of the fourth order. For a plane through PQ contains the non-degenerate conic of this surface passing through P,Q and the points of intersection of this plane with a_3, a_4, a_5, but also the straight line PQ which is a double line of the surface, because together with the two transversals of PQ, a_3, a_4 and a_5, it forms two degenerate conics of the surface. As a_2 intersects this surface in four points, there pass through a point Q of a_1 four conics of the system S_1 of the conics cutting $a_1, \ldots, a_5$ and passing through P. It follows from this that a plane through a'_1 cuts the curve k_P in four points outside a'_1. Further k_P has a double point on a'_1, which is associated to the double conic of S_1 lying in the plane through P and a_1 and passing through P and the points of intersection of this plane and $a_2, \ldots, a_5$. The two tangents to k_P at this double point are associated to the straight lines joining P to the two points in which the corresponding double conic cuts a_1. The curve k_P is accordingly of the sixth order. This curve intersects also the ten straight lines t'_{ik}, e. g. the line t'_{11} in the point corresponding to the degenerate conic consisting of t_{11} and the transversal of a_1 and t_{11} through P.

The system of the conics of S_3 passing through a point P, is accordingly represented on a plane curve of the sixth order which has double points on $a'_1, \ldots, a'_5$ and which cuts the ten straight lines $t'_{11}, \ldots, t'_{52}$.

As k_P has six points in common with an arbitrary plane, *six* of the planes of the conics of S_1 pass through an arbitrary point.

The system S_2' of the conics cutting a given straight line l, is represented on a surface O_l. The order of this surface, i. e, the

number of points of intersection with an arbitrary straight line m', is equal to the number of conics of S_3' the planes of which pass through a straight line m. From the order just found for k_P there follows that through a point P of m there pass six conics of S_3 the planes of which contain m. All conics of S_3 in planes through m, consequently form a surface which has m as a sextuple straight line and which is of the eighth order, as a plane through m contains one more conic of this surface. Consequently among the conics of S_3 intersecting l, there are *eight* the planes of which pass through m and the surface O_l associated to S_3' is accordingly of the eighth order. Evidently the pencil associated to a point of any of the straight lines $a_1', \ldots, a_5'$ contains a double conic of S_3' and in S_3' there always lies one individual of the pencil of degenerate conics associated to a point of one of the straight lines $t'_{11}, \ldots, t'_{52}$.

The system S_3' formed by the conics cutting a straight line l and $a_1, \ldots, a_5$, is therefore represented on a surface O_l of the eighth order, of which $a'_1, \ldots, a'_5$ are double straight lines and $t'_{11}, \ldots, t_{52}$ single straight lines. The two tangent planes at a point of one of the straight lines a' to O_l are the polar planes of the points where the double conic of S_3' corresponding to this point, intersects the straight line a associated to a'.

Finally we investigate the surface O_φ which is the image of the system S_3'' of the conics of S_3 touching a plane φ. The order of O_φ is again equal to the number of conics of S_3'' the planes of which pass through an arbitrary straight line m. The surface of the eighth order of the conics of which the planes pass through m and which cut $a_1, \ldots, a_5$, has in common with φ a curve k^8 of the eighth order which has a sextuple point in the point of intersection (m, φ) of m with φ. As each of the conics of this surface has in common with k^8 a pair of points lying on a straight line through (m, φ), the number of individuals touching φ is equal to the number of tangents which can be drawn out of (m, φ) to k^8, i.e. $8 \times 7 - 6 \times 7 = 14$. The system S_3'' contains consequently fourteen conics the planes of which pass through m and the order of O_φ is accordingly fourteen. Now S_3'' has two double conics in the pencil corresponding to a point of one of the five straight lines a' and this system has one individual in common with the pencil of degenerate conics corresponding to a point of one of the lines t'. This individual is a double conic of S_3''. For if we take a straight line m of its plane, it counts twice among the conics of S_3'' the planes of which pass through m. The above mentioned pencil of degenerate conics splits off from the system of the conics of S_3

cutting m twice, so that there remains a surface of the seventh order which intersects φ along a curve k^7 with a fivefold point in (m, φ). Instead of 14 we can now draw $7 \times 6 - 5 \times 6 = 12$ tangents out of (m, φ) to this curve. Hence a straight line m' through a point of a straight line t' has in this point two coinciding points of intersection with O_φ.

The system S''_2 formed by the conics of S_3 touching a plane φ, is represented on a surface O_φ of the fourteenth order of which $a'_1, \ldots, a'_5$ are quadruple straight lines and $t'_{11}, \ldots, t'_{52}$ double lines.

§ 3. From the investigated representation we can now in the first place derive the number of conics which cut five straight lines and which fulfil a threefold condition [1]).

$2 \times 2 = 4$ of the 48 points which a curve k_P has in common with a surface O_l fall in each of the double points of k_P and one in each of the ten points of intersection of k_P with the straight lines t'. Accordingly the curve k_P cuts a surface O_l in eighteen points which are not singular for the representation.

There are therefore eighteen conics passing through a given point and intersecting six given straight lines.

$2 \times 4 = 8$ of the 84 points in which a curve k_P intersects a surface O_φ, lie in each of the five double points of k_P and two in each of the ten points of intersection of k_P and the lines t'. Here we have therefore 24 points of intersection that are not singular for our representation.

There are accordingly 24 *conics passing through a given point, touching a given plane and intersecting five given straight lines.*

Of the curve of the order 64, which two surfaces O_l have in common, each of the straight lines a' splits off four times and each of the lines t' once. There remains, accordingly, a curve of the order 34, k^{34}, which is the representation of the system of the conics in S_3 cutting two given straight lines. The conics of this system of which the planes pass through an arbitrary point, are represented on the points of intersection of k^{34} with the polar plane of this point.

There are therefore 34 *conics which cut seven given straight lines and of which the planes pass through a given point.*

We have found in § 2 that there are eight conics which cut six given straight lines and of which the planes pass through a likewise given straight line. Hence the system associated to k^{34} contains eight

[1]) Cf. SCHUBERT: „*Kalkül der Abzählenden Geometrie*", p. 95.
JAN DE VRIES, These Proceedings, Vol. IV, p. 181.

double conics the planes of which pass through one of the lines a, and accordingly k^{34} has eight double points on each line a'.

Likewise the system corresponding to k^{34} contains pairs of degenerate conics of which the image points lie on one of the lines t'. For instance to points of t'_{11} there are associated the two conics consisting of t_{11} and the transversals of t_{11}, a_1 and the two directrices outside the lines a ours of the system of conics under consideration. Hence k^{34} cuts each of the lines t' in two points.

The curve k^{34} cuts a third surface O_l in 272 points. Four of these lie in each of the 40 double points of k^{34}, and 20 belong to the straight lines t. There are consequently 92 points of intersection that are not singular for the representation.

There are 92 *conics intersecting eight given straight lines.*

From the number of points of intersection of k^{34} with a surface O_φ that are not singular for the representation, there follows:

There are 116 *conics intersecting seven given straight lines and touching a given plane.*

A surface O_l and a surface O_φ have an intersection of the order 112. From this each of the straight lines a' splits off eight times and each of the lines t' twice. There remains a curve of the order 52.

There are 52 *conics which cut six given straight lines, touch a given plane, and of which the planes pass through a given point.*

Let us investigate the intersection of two surfaces O_φ more closely. It is of the order 196; each of the straight lines a' splits off sixteen times, each line t' four times. There remains, accordingly, a curve of the order 76, k^{76}.

There are 76 *conics which cut five straight lines, touch two given planes, and the planes of which pass through a given point.*

The curve k^{76} has as many double points on a'_1 as there are conics of which the planes pass through a_1, which cut $a_2, \ldots, a_5$, and which touch the planes φ_1 and φ_2. In order to find this number we remark in the first place that the conics through two points A and B of a_1 intersecting a_2 and touching φ_1 and φ_2, form a surface of the eighth order. For in each plane through a_1 there lie four conics satisfying these conditions, and a_1 is not a component part of any such a degenerate conic. Hence eight conics intersecting a_3 and a_4 and touching φ_1 and φ_2 pass through A and B and the line a_1 is an eightfold straight line of the surface formed by the conics through A intersecting a_1 outside A, cutting a_3 and a_4, and touching φ_1 and φ_2. This surface is of the sixteenth order as appears from its intersection with a plane through a_1. a_1 is therefore a sixteenfold straight line of the surface consisting of the conics the

planes of which pass through a_1, which cut a_2, a_3 and a_4, and which touch φ_1 and φ_2, and this surface is of the 24th order. The number of conics in question is therefore 24, and k^{76} has 24 double points on each of the lines a'. As for instance the line t_{11} is not a component part of any degenerate conic cutting $a_1, \ldots, a_5$ and touching φ_1 and φ_2, k^{76} has no point in common with any of the lines t'.

If we now determine the numbers of points of intersection of k^{76} with surfaces O_l and O_φ that are not singular for the representation, we find resp.:

There are 128 *conics intersecting six given straight lines and touching two given planes.*

There are 104 *conics intersecting five given straight lines and touching three given planes.*

§ 4. The genus of the system of conics through a given point P intersecting $a_1, \ldots, a_5$, is equal to that of the associated curve k_P, which is of the sixth order and has five double points; consequently it is *five*. According to the first theorem of § 3 these conics form a surface of the eighteenth order, Ω^{18}. To a conic of Ω^{18} we associate the two points in which it intersects a plane φ which therefore always belong to the curve k^{18} along which Ω^{18} is cut by φ. To the (1,2)-correspondence between the conics of Ω^{18} and the points of k^{18} arising in this way, we apply the formula of ZEUTHEN:

$$\eta_2 - \eta_1 = 2\alpha_1(p_2 - 1) - 2\alpha_2(p_1 - 1). \quad . \quad . \quad . \quad . \quad (1)$$

In this case $\alpha_1 = 1$, $\alpha_2 = 2$, $p_1 = 5$, $\eta_1 = 0$ and $\eta_2 =$ the number of conics of Ω^{18} touching φ, that is, according to § 3, 24. By substituting these values in (1) we find that p_2, i. e. the genus of k^{18}, is equal to 21. Hence the curve k^{18} has 115 double points. Among these each of the points of intersection of φ with a line a in which k^{18} has quadruple points, must be counted six times. Further there belong to them the ten points of intersection of φ with the five double conics of Ω^{18} of which the planes pass through one of the lines a, and the ten points where φ is cut by the double straight lines of Ω^{18} i. e. the transversals t_P of two of the lines a which pass through P and form a conic of Ω^{18} together with the two transversals of t_P and the three remaining lines a. There remain accordingly 65 double points.

The surface of the conics through a given point which cut five straight lines, has a double curve of the 65th *order.*

A plane φ through a_1 has in common with Ω^{18} besides a_1 a

curve of the order 14, k^{14}, the points of which may be associated univalently to the conics of Ω^{18} passing through them, so that k^{14} has the genus *five* and accordingly $\frac{13 \times 12}{2} - 5 = 73$ double points. Six of them lie in each of the four points of intersection of φ with one of the lines $a_2, \ldots, a_5$ and also there belong to them the four points of intersection outside a_1 of φ with the double conics of Ω^{18} the planes of which do not pass through a_1, and the points of intersection of φ with the six transversals through P of two of the lines $a_2, \ldots, a_5$. Besides these there are 39 more double points. Hence the double curve of Ω^{18} cuts the line a_1 in 26 points. These are points of a_1 through which there pass two conics of our system that have there a common tangent plane through a_1.

The surface Ω^{18} has a twelvefold point in P, as according to § 2 our system contains six conics that cut a straight line through P outside P. A plane through P intersects Ω^{18} in a curve of the order eighteen and the genus five as again the points of this curve may be associated univalently to the conics through them. This curve has consequently 131 double points. 66 of them lie in P, six in each of the points of intersection with the lines a, and also the points of intersection outside P of the five double conics with the plane must be counted. There remain accordingly 30 double points.

The double curve of Ω^{18} cuts each line a in 26 *points and has in P a* 35*-fold point.*

To the 35 branches of the double curve through P there correspond as many pairs of conics of Ω^{18} touching each other at this point. Outside P and a_1 it must have four more points in common with the plane (P, a_1). These lie in the points of intersection outside P of the double conic in the plane (P, a_1) and the two straight lines joining P to the points where the transversals of $a_2, \ldots, a_5$ cut the plane. For these two points of intersection are double points of the curve under consideration.

Analogously we can examine the double curves of the surface Ω^{34} consisting of the conics that cut six given straight lines and the planes of which pass through a given point, and of the surface Ω^{52} formed by the conics that cut five given straight lines, touch a given plane, and the planes of which pass through a given point.

§ 5. We shall first determine the genus of the curve k^{34} belonging to the intersection of two surfaces O_l and O_l'. The cone K^{34} projecting k^{34} out of an arbitrary point K, has in common with O_l

besides k^{34} a curve of the order 238, k^{238}. This curve has double points in each of the double points of k^{34}, because the entire intersection of K^{34} and O_l has a quadruple point in such a point. Further K^{34} cuts each of the lines a' in 18 more points and here k^{238} has double points. But this curve has 32 single points on each of the lines t'.

The surface O_l' is cut by k^{238} in

$$238 \times 8 - 5 \times 26 \times 4 - 10 \times 32 = 1064$$

points that are not singular for the representation. These are points of intersection of k^{34} and k^{238}; a part of them lie in the points where a generatrix of K^{34} touches the surface O_l, hence in the points of intersection of k^{34} with the polar plane of K relative to O_l that are not singular for the representation. As this polar surface is of the order seven and passes singly through the lines a', it cuts k^{34} in $7 \times 34 - 5 \times 2 \times 8 = 158$ non-singular points. The remaining 906 points of intersection of k^{34} and k^{238} are the points where the bisecants of k^{34} through K cut this curve. Hence there pass 453 bisecants of k^{34} through K, and in a plane there lie $\frac{34 \times 33}{2} = 561$ bisecants of this curve.

Accordingly:

The bitangents of the developable surface that is enveloped by the planes of the conics intersecting seven given straight lines, form a congruence (561, 453).

As K^{34} has $453 + 5 \times 8 = 493$ double generatrices, the genus of the curve k^{34}, hence also the genus of the system of the conics cutting the lines $a_1, \ldots, a_5$, l and l', is equal to: $16 \times 33 - 493 = 35$.

To each conic of the surface Ω^{92} corresponding to the curve k^{34}, we associate again the pair of points in which such a conic cuts an arbitrary plane φ; it belongs to the curve k^{92} along which Ω^{92} intersects the plane φ. We apply the formula:

$$\eta_2 - \eta_1 = 2\alpha_1 (p_2 - 1) - 2\alpha_2 (p_1 - 1) \quad . \quad . \quad . \quad . \quad (1)$$

to the correspondence (1,2) arising in this way between the conics of Ω^{92} and the points of k^{92}. Here $\eta_2 =$ the number of conics cutting seven straight lines and touching a plane; according to § 3 it is 116. Further $\eta_1 = 0$, $\alpha_1 = 1$, $\alpha_2 = 2$ and $p_1 = 35$. By the aid of these values there follows from (1) that the genus of k^{92} is equal to 127.

The number of double points of k^{92} is consequently $91 \times 45 - 127 = 3968$. As there pass eighteen conics of Ω^{92} through a point of one of the directrices of this surface, whence these directrices are

eighteenfold straight lines of Ω^{92}, k^{92} has eighteenfold points in the points of intersection of φ with these directrices and each of these points contains $\frac{18 \times 17}{1 \times 2} = 153$ out of the number of double points. The points of intersection of φ with the 70 double straight lines of Ω^{92}, i.e. the transversals d of four of the directrices, each of which forms a pair of two degenerate conics of Ω^{92} together with the transversals of d and the three remaining directrices, are double points of Ω^{92}, just as the 112 points of intersection of φ with the $7 \times 8 = 56$ double conics of k^{92} the planes of which pass through one of the directrices. There remain accordingly 2715 double points.

The surface formed by the conics intersecting seven given straight lines, has therefore also a double curve of the order 2715.

The intersection of Ω^{92} with a plane φ through a_1, consists besides of a_1 of a curve of the order 74, k^{74}. If we associate to a point of k^{74} the conic of Ω^{92} passing through it, there arises a (1,1)-correspondence between the conics of Ω^{92} and the points of k^{74}. The genus of k^{74} is accordingly 35 and the number of double points $73 \times 36 - 35 = 2593$. The points of intersection of φ and the six directrices of Ω^{92} outside a_1 are eighteenfold points of k^{74}, and each of them is therefore contained 153 times in the said number of double points. Also each intersection of φ with one of the thirty double straight lines of Ω^{92} that do not cut a_1, and each point of intersection outside a_1 of φ and one of the 48 double conics of Ω^{92} that cut a_1 only once, is a double point of k^{74}. There remain therefore 1597 double points. Hence:

The double curve of Ω^{92} *cuts each of the directrices of this surface in* 1118 *points.* These are points through which there pass two conics of our system that have there a common tangent plane through the directrix.

Analogously it is possible to examine the double curves of the surface Ω^{116} formed by the conics intersecting six given straight of the touching a given plane, and of the surface Ω^{138} consisting lines and conics intersecting five given straight lines and touching two given planes.

Mathematics. — "*On the Plane Pencils Containing Three Straight Lines of a given Algebraical Congruence of Rays*". By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of June 30, 1923).

§ 1. In his „*Kalkül der Abzählenden Geometrie*", p. 331, SCHUBERT finds that the vertices of the plane pencils containing three straight lines of the congruence which two complexes of rays of the orders m and m' have in common, form a surface of the order:

$$\tfrac{1}{3}\, mm'\,(mm'-2)\,(2mm'-3m-3m'+4),$$

and the planes of these pencils envelop a surface of the same class. In this paper we shall examine what these results become for an arbitrary algebraic congruence of rays. With a view to this we make use of the representation of a special linear complex C on a linear three-dimensional space R_3 which is described in STURM: „*Liniengeometrie*", I, on p. 269. First, however, we shall give a derivation of this representation which differs from the one l. c.

§ 2. If we associate to a straight line l with coordinates $p_1, \ldots p_6$ the point P in a linear five-dimensional space R of which the six above mentioned quantities are the homogeneous coordinates, a special linear complex C is represented on the intersection of a variety V with the equation

$$p_1 p_4 + p_2 p_5 + p_3 p_6 = 0$$

and one of its four-dimensional tangent spaces R_4.

This intersection is a quadratic hypercone K that has its vertex T in the point where R touches the variety V. As the generatrices of K intersect an arbitrary three-dimensional space in the points of a quadratic surface, K contains two systems of planes each of which projects one of the scrolls of the surface in question out of T. Two planes of the same system have only the vertex T in common, two planes of different systems a generatrix of K. The planes V_p of one system are the representation of the stars of rays of the complex C, which have therefore their vertices on the axis a of C, and the fields of C the planes of which pass through a, are associated to the planes V_v of the other system. The axis a of C and the

plane pencils of this complex containing a, correspond resp. to the vertex T of K and the generatrices of this hypercone. A straight line of K in a plane V_p represents a plane pencil of C the vertex of which lies on a, and a plane pencil of C of which the plane passes through a, is associated to a straight line of a plane V_v.

Now we assume on K a point S and in the four-dimensional space R_4 a three-dimensional space R_3. The representation mentioned in § 1 arises, when we associate to each straight line l the projection L of P out of S on R_3, if P is the point on K corresponding to l.[1])

§ 3. The straight line s of C of which S is the image point on K, is a *singular straight line* of the second order for the correspondence $(l—L)$. For all the points of the plane ϱ that the three-dimensional tangent space R of K at S, lying in R_4, has in common with R_3, are associated in R_3 to this straight line.

In R there lie the two planes V_p^1 and V_v^1 of K of which the intersection is the generatrix b_1 of K through S. To these planes there correspond resp. the star of C, that has its vertex in the point of intersection A of s and a, and the field of C consisting of the rays of the plane α that passes through s and a. The star A and the field α have in common the plane pencil (A, α) to which the straight line b_1 on K is associated.

The planes V_p^1 and V_v^1 cut ϱ resp. along the straight lines p_1 and v_1 each consisting of points that are singular for the correspondence $(l—L)$. For to each point L of p_1 there corresponds on K a straight line of V_p^1 through S, hence in C a plane pencil containing s, with vertex in A. Likewise a plane pencil in α containing s, is associated to each point L of v_1. The point of intersection B_1 of p_1 and v_1 is the image point L for all rays l of the plane pencil (A, α). In this way the ∞^2 straight lines of the star A correspond to the ∞^1 points of p_1, the ∞^2 rays of the field α to the ∞^1 points of v_1.

To a plane pencil with vertex on a a straight line on K in a plane V_p, which accordingly intersects V_{v_1}, is associated; consequently to such a plane pencil in R_3 corresponds a straight line cutting v_1. Inversely the plane through S and a straight line of R_3 cutting v_1, intersects the hypercone K along a straight line in V_{v_1} through S, to which there corresponds the plane pencil of C that is associated

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[1]) The method applied here, has been indicated for the rays of space by FELIX KLEIN. Cf. Mathem. Annalen, Bd. 5, p. 257.

to the singular point of intersection of the chosen straight line with v_1, and along a straight line cutting V_{v_1}, which lies therefore in a plane V_p and corresponds to a plane pencil of C the vertex of which lies on a. In the same way it is evident that the pencils of C in planes through a, are represented on the straight lines of R_3 which cut p_1, and that the plane pencils containing a are associated to the straight lines through the point of intersection B_1 of p_1 and v_1 (for a plane through SB_1 cuts the hypercone K outside SB_1 along a generatrix).

To a star of C, the vertex of which lies consequently on a, there corresponds on K a plane V_p that cuts V_{v_1} along a straight line and the projection of which on R_3 passes accordingly through v_1. Hence a plane through v_1 is associated to a star of C in R_3. It is easily seen that also the reverse holds good and that the fields of C, the planes of which pass through a, are represented on the planes of R_3 through p_1.

§ 4. A congruence $\Gamma(\alpha, \beta)$ of the order α and the class β has in common with C a scroll Ω of the order $\alpha + \beta$ that has a as an α-fold directrix. If further Γ has the rank r, there are r plane pencils through a containing two straight lines of Ω.

The curve γ in R_3 on which Ω is represented, cuts p_1 in the α points that are associated to the α generatrices of Ω which pass through A, and v_1 in the β points that correspond to the β generatrices of Ω in the plane (a, s). A plane through p_1 cuts γ outside p_1 in the β image points of the straight lines which the corresponding field of C has in common with Ω, and it appears in the same way that a plane through v_1 intersects the curve γ outside v_1 in α points. Hence the order of γ is $\alpha + \beta$.

To the r plane pencils through a that contain two straight lines of Ω, there correspond in R_3 as many bisecants of the curve γ through B_1. Besides the lines p_1 and v_1 which cut γ resp. α and β times pass through B_1. The number of apparent double points of γ is accordingly:

$$r + \tfrac{1}{2}\alpha(\alpha-1) + \tfrac{1}{2}\beta(\beta-1).$$

We shall just mention an application that STURM gives on p. 271 of his book quoted in § 1. The order of the focal surface of the congruence Γ is equal to the number of sheaves with vertices on a containing two straight lines of Γ, hence also of Ω, that are infinitely near to each other. These are represented on the planes through v_1 touching γ outside v_1. Hence the order of the focal surface of Γ is equal to the number of points of intersection outside γ of v_1 with

the surface of the tangents of γ. The order of the latter surface, that has γ as a double curve (cuspidal curve), is equal to

$$2(\alpha\beta - r).$$

We find this by substituting in the formula $n(n-1) - 2h$ for n the order $\alpha+\beta$ of γ and for h the above mentioned number of apparent double points of this curve. As v_1 cuts the surface under consideration on the double curve γ in β points, we find for the number of points of intersection outside γ, i. e. the order of the focal surface of the congruence Γ:

$$2\beta(\alpha-1) - 2r.$$

The class of the focal surface of Γ is equal to the number of planes through a containing two straight lines of Γ, hence also of Ω, that are infinitely near to each other, or equal to the number of planes through p_1 touching γ outside p_1. As p_1 cuts the curve γ in α points, we find for the class in question:

$$2\alpha(\beta-1) - 2r.$$

§ 5. In order to find the order of the surface formed by the vertices of the plane pencils containing three generatrices of Γ, we try to find the number of these plane pencils that have their vertices on a. These belong to C and are represented on the trisecants of γ that cut v_1 outside this curve.

The order of the surface Δ of the trisecants of γ is found by substituting in the formula:

$$(n-2)\{h - \tfrac{1}{6}n(n-1)\},$$

given by CAYLEY, for n the order $\alpha+\beta$ of γ and for h the number of apparent double points of this curve found in § 3. We find in this case:

$$(\alpha+\beta-2)\{r + \tfrac{1}{2}\alpha(\alpha-1) + \tfrac{1}{2}\beta(\beta-1) - \tfrac{1}{6}(\alpha+\beta)(\alpha+\beta-1)\}$$

or, after a simple reduction:

$$(\alpha+\beta-2)r + \tfrac{1}{3}\alpha(\alpha-1)(\alpha-2) + \tfrac{1}{3}\beta(\beta-1)(\beta-2).$$

In order to find the number of generatrices of Δ that cut v_1, we remark that these are the common straight lines of Δ and the special linear complex that has v_1 as axis. Now the axis of a special linear complex C may be considered as a double line of C. This follows in the first place from the representation of C on a hypercone K that has been described in § 2 and through which the axis of C is transformed into the vertex of K, but also from the well known property that $n-2$ generatrices of a scroll of the order n cut a straight line of this scroll. As further v_1 has β points in common

with γ, it is apparently a $\frac{\beta(\beta-1)(\beta-2)}{6}$-fold generatrix of Δ. The number of generatrices of Δ cutting v_1, is therefore found by diminishing the order-number found above, by:

$$\tfrac{1}{3}\beta(\beta-1)(\beta-2).$$

Hence there are

$$(\alpha+\beta-2)r+\tfrac{1}{3}\alpha(\alpha-1)(\alpha-2)$$

straight lines of Δ which cut v_1.

In the first place the straight line p_1 must be counted $\frac{\alpha(\alpha-1)(\alpha-2)}{6}$ times, for as this line has β points in common with γ it is an $\frac{\alpha(\alpha-1)(\alpha-2)}{6}$-fold generatrix of Δ. Further the number found above has to be diminished by the number of trisecants of γ that cut v_1 on γ. This is the case in each of the β points that γ has in common with v_1. We find the number of trisecants of γ passing through such a point, by the aid of the property that through a point of a twisted curve of the order n with h apparent double points, there pass $h-n+2$ straight lines that contain two more points of the curve, if we take into account that in our case for each of the said β points v_1 counts $\frac{(\beta-1)(\beta-2)}{2}$ times among the trisecants of γ passing through them, as v_1 contains $\beta-1$ more points of γ outside the point under consideration. Consequently

$$\beta\{r+\tfrac{1}{2}\alpha(\alpha-1)+\tfrac{1}{2}\beta(\beta-1)-\alpha-\beta+2-\tfrac{1}{2}(\beta-1)(\beta-2)\}$$

or

$$\beta\{r+\tfrac{1}{2}\alpha(\alpha-1)(\alpha-2)\}$$

trisecants of γ that cut v_1 on γ, must be taken apart.

If we subtract these two numbers of straight lines from the aforesaid number of straight lines of Δ that cut v_1, we find that

$$\tfrac{1}{6}(\alpha-2)\{6r-(\alpha-1)(3\beta-1)\}$$

trisecants of γ intersect v_1 outside this curve.

According to the beginning of this § we arrive at the following theorem:

The locus of the vertices of the plane pencils that have three straight lines in common with a congruence $\{\alpha,\beta\}$ of the rank r, is a surface of the order:

$$\tfrac{1}{6}(\alpha-2)\{6r-(\alpha-1)(3\beta-\alpha)\}.$$

§ 6. In order to show that the result found in § 5, is in accordance with the result of SCHUBERT, mentioned in § 1, we have to know the rank of the congruence $\Gamma(mm', mm')$ that two complexes C_1 and C_2 of the orders m and m' have in common. It might suffice to refer to SCHUBERT, *Kalkül der Abzählenden Geometrie*, where there is found on p. 330 a derivation of this number. We shall however show that the order of Γ may also be found by the aid of the representation used in this paper.

The surface Ω consisting of the straight lines of Γ which cut the axis a of C, is of the order $2mm'$ and has a as an mm'-fold straight line. It is the intersection of the two congruences $\Sigma_1(m, m)$ and $\Sigma_2(m', m')$ consisting of the straight lines out of C_1 and C_2 that cut a.

Σ_1 and Σ_2 are represented resp. on two surfaces S_1 and S_2 in R_3. As C_1, hence also Σ_1, contains m generatrices of an arbitrary plane pencil of C, all points of p_1 and v_1 are m-fold points of S_1 and all straight lines cutting p_1 and v_1 have m more points in common with S_1. S_1 has accordingly the order $2m$ and p_1 and v_1 are m-fold straight lines of S_1. In the same way S_2 has the order $2m'$ and p_1 and v_1 are m'-fold straight lines of this surface. The intersection of S_1 and S_2 consists of the straight lines p_1 and v_1, each counted mm' times, and the curve γ on which Ω is represented. This curve has the order $2mm'$ and has mm' points in common with each of the straight lines p_1 and v_1. We first determine the number of apparent double points of γ.

The cone A projecting γ out of an arbitrary point L of R_3, is of the order $2mm'$ and has in common with S_1 besides γ a curve ϱ of the order $4m^2m' - 2mm' = 2mm'(2m-1)$. The curve ϱ has $(m-1)$-fold points in the $2mm'$ points where γ cuts the lines p_1 or v_1, because the entire intersection of A and S_1 must have there m-fold points. Further A cuts each of the lines p_1 and v_1 in mm_1 more points, that are m-fold points for ϱ. As all these points are m'-fold for S_2, ϱ has $4mm'^2(2m-1) - 2mm'^2(m-1) - 2m^2m'^2 = 2mm'^2(2m-1)$ points of intersection with S_2 outside p_1 and v_1. These belong to γ and lie partly in the points where a generatrix of A touches the surfaces S_1 on γ, hence in the points of intersection with γ outside p_1 and v_1 of the first polar surface of L relative to S_1. As this polar surface is of the order $2m-1$ and has $(m-1)$-fold straight lines in p_1 and v_1, it cuts γ outside p_1 and v_1 in $2mm'(2m-1) - 2mm'(m-1) = 2m^2m'$ points. The remaining $2mm'^2(2m-1) - 2m^2m' = 2mm'(2mm' - m - m')$ points where ϱ and γ cut each other outside p_1 and v_1, are points that the bisecants of γ through L have

in common with this curve. The number of apparent double points of γ is therefore equal to $mm'(2mm'-m-m')$.

If we choose L in the point of intersection B_1 of p_1 and v_1, $\frac{mm'(mm'-1)}{2}$ of the chords of γ through this point coincide with each of the lines p_1 and v_1. Through B_1 there pass accordingly $mm'(m-1)(m'-1)$ bisecants of γ different from p_1 and v_1. According to § 3 these are the representation of as many plane pencils through a containing two straight lines of Ω, hence also of Γ. *The rank of the congruence Γ that two complexes of the orders m and m' have in common, is therefore equal to* $mm'(m-1)(m'-1)$.

If we substitute this number for r in the expression found in § 5, and if we make α and β equal to mm', we find indeed that the order of the surface formed by the vertices of the plane pencils containing three straight lines of the intersection of two complexes of rays of the orders m and m', is equal to:

$$\tfrac{1}{3}\,mm'\,(mm'-2)\,(2mm'-3m-3m'+4).$$

We get another check through the application of our formula to the congruence consisting of the straight lines passing through one of n given points. For this congruence $\alpha = n$ and $\beta = r = 0$. The locus of the vertices of the plane pencils which three straight lines have in common with this congruence, consists of the planes that may be passed through each triple of the given points. By the said substitutions in the formula of § 5, we find indeed the number of these planes, namely:

$$\tfrac{1}{6}\,n\,(n-1)\,(n-2).$$

To the theorem derived in § 5 there corresponds dually:

The planes of the plane pencils that have three straight lines in common with a congruence $\{\alpha, \beta\}$ of the rank r, envelop a surface of the class:

$$\tfrac{1}{6}\,(\beta-2)\,\{6r-(\beta-1)\,(3\alpha-\beta)\}.$$

Physics. — *"Transients of Magnetic Field in Supra-conductors"*. By G. BREIT. National Research Fellow U. S. A. (Communicated by Prof. H. A. LORENTZ).

(Communicated at the meeting of June 30, 1923).

It is known that supra-conductivity is determined not only by temperature but also by the magnetic field and the current density [1]).

In view of the considerations of SILSBEE and LANGEVIN it is probable that the only essential factors are the magnetic field and the temperature [2]).

This hypothesis will be adhered to below. The problems to be discussed are the calculations of the manner in which a strong magnetic field impressed from the outside on a supra-conductor destroys its supra-conductivity and the way in which the supra-conductivity is reestablished when the magnetic field is withdrawn.

If the view proposed by BRIDGMAN [3]) is correct there is an evolution or an absorption of heat whenever a change in the conductive state takes place. These phenomena being of unknown magnitude, they will be disregarded below. If experiments should fail to confirm the calculations here developed, the source of disagreement may be then looked for in the neglect of BRIDGMAN's latent heat.

The mathematical difficulty of the problem consists in the existence of two distinct states determined by the magnetic field. The purpose of this paper is to point out some special solutions (particular integrals) of the problem.

We shall employ the electromagnetic system of units. By H (a vector) and by σ we shall denote the magnetic field and the resistivity. The symbol H_c will be used for the threshold value of the field. The resistivity σ may have either of two values σ_1, σ_2 according as to whether $|H| > H_c$ or $|H| < H_c$. The value σ_2 is the microresidual resistivity and in a special case may be taken to be zero. The electric intensity at any point we shall denote by the

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[1]) H. KAMERLINGH ONNES, Proc. Amst. Acad. Sc. 16, (2) 1914. Leiden Comm. N°. 133, 139.

[2]) F. B. SILSBEE, Journal Washington Academy 6, 597—602, 1916. Bureau of Standards Scientific Paper N°. 307 (July 23, 1917).

[3]) Journal Washington Acad. Vol. 11, p. 455, 1921.

vector E. The current density is then $\frac{E}{\sigma}$. If t be the time, the fundamental equations of the problem are:

$$div\ H = 0 \qquad curl\ H = \frac{4\pi}{\sigma} E \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$div\ E = 0 \qquad curl\ E = -\frac{\partial H}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence

$$\left(\nabla^2 - \beta \frac{\partial}{\partial t}\right) H = 0 \quad , \quad \beta = \frac{4\pi}{\sigma} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and in the case of cylindrical symmetry, H being parallel to the axis, the distance from which is r

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \beta \frac{\partial}{\partial t}\right) H = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3^{\mathrm{I}})$$

If only small penetrations from the surface are investigated the approximate form

$$\frac{\partial^2 H}{\partial r^2} = B \frac{\partial H}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3^{\mathrm{II}})$$

may be used. The equations (3), (3^{I}), (3^{II}) are analogous to equations in heat conductions and it is therefore of interest to follow out this analogy somewhat closer. In the case of cylindrical symmetry and H parallel to the axis the electric intensity is by symmetry directed along a system of coaxial circles having the axis of symmetry for their common axis as shown on the figure (Fig. 1). Dropping now the meaning of E and H as vectors and denoting forthwith by E and H the absolute magnitudes of the electric and magnetic intensities, we have from (1) and (2)

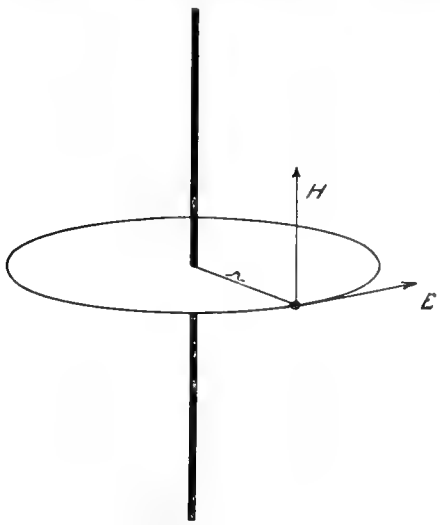

FIG. 1.

$$-\frac{\partial H}{\partial r} = \beta E \quad . \quad . \quad . \quad (5)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE) = -\frac{\partial H}{\partial t} \quad . \quad . \quad (6)$$

These equations are analogous to the equations in heat conduction:

$$-\frac{\partial\theta}{\partial r}=\frac{1}{K}F \quad \ldots\ldots\ldots \quad (5^{I})$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rF)=-\frac{\partial}{\partial t}(C\theta) \quad \ldots\ldots\ldots \quad (6^{I})$$

where θ is the temperature, F the flow of heat, K is the conductivity for heat, and C is the specific heat. The electrical problem is the analogue of the heat problem for a substance having unit specific heat and a conductivity for heat $=\frac{1}{\beta}=\frac{\sigma}{4\pi}$. Thus a perfect supra-conductor corresponds to $K=0$ i. e. to a perfect insulator for heat. This is another expression for the fact that the shielding properties of the supra-conductor are perfect.

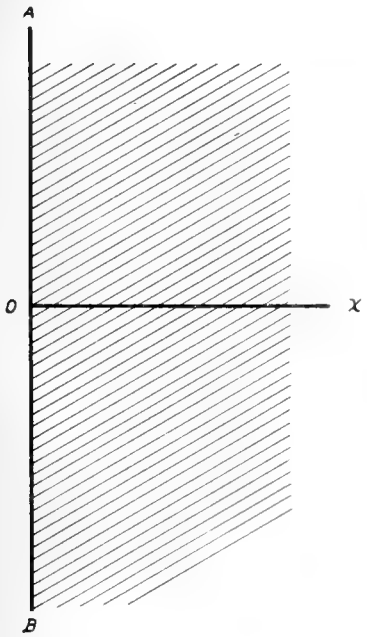

FIG. 2.

In view of the difficulty of treating the cylindrical case accurately we shall specialize the problem by investigating it within the approximation (3^{II}) i. e. neglecting the curvature of the surface within the depth of penetration, this makes the problem an essentially unidimensional one.

The shaded region on the right of the plane AB (see Fig. 2) is occupied by the metal. The axis OX is perpendicular to AB. The changes in the field are produced from the left side of AB. H is positive when vertical and upward. E is positive when into the plane of the paper. The relations between E and H are:

$$-\frac{\partial H}{\partial x}=\beta E. \quad \ldots \quad (7)$$

$$\frac{\partial E}{\partial x}=-\frac{\partial H}{\partial t} \quad \ldots\ldots\ldots \quad (8)$$

and hence

$$\frac{\partial^{2}H}{\partial x^{2}}=\beta\frac{\partial H}{\partial t} \quad \ldots\ldots\ldots \quad (3^{II})$$

We shall consider several problems all of which are similar

mathematically to STEFAN's problem of the propagation of the frost [1]) though for one case a slight extension of his mathematical method will be necessary.

Case 1. The material is supraconducting to start with, the field inside and outside is homogeneous and equal to $\overline{H_1} < H_c$. Suddenly the field outside is increased to a value $\overline{H_2} > H_c$.

We begin counting time from the instant of the sudden change. After the lapse of a time t the non-supraconductive state will have advanced a certain distance x_c into the metal. A moving plane separates the regions having the two values of σ. The low value σ_2 is on the right of this bounding plane while the high value σ_1 is on the left. Corresponding to the two values of σ here are two values of β on the right and left (β_2, β_1 respectively). On both sides of the surface of separation $H = H_c$. Also E must be continuous at the boundary. Letting

$$\Theta(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

we know from the work of STEFAN that it is possible to satisfy all the conditions of the problem by letting H on the left and on the right of the boundary have respectively the expressions:

$$H_1 = A_1 + B_1\, \Theta\left(\frac{x}{2\sqrt{\frac{\beta_1}{t}}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$H_2 = A_2 + B_2\, \Theta\left(\frac{x}{2\sqrt{\frac{\beta_2}{t}}}\right) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In fact these satisfy (3^{II}) and by a proper choice of the constants A_1, B_1, A_2, B_2 the initial and boundary conditions can also be satisfied. The equations are

$$\overline{H_2} = A_1 \qquad \overline{H_1} = A_2 + B_2$$

$$H_c = A_1 + B_1\, \Theta\left(\frac{x_c}{2\sqrt{\frac{\beta_1}{t}}}\right) = A_2 + B_2\, \Theta\left(\frac{x_c}{2\sqrt{\frac{\beta_2}{t}}}\right)$$

$$\frac{1}{\beta_1}\left(\frac{\partial H_1}{\partial x}\right)_{x=x_c} = \frac{1}{\beta_2}\left(\frac{\partial H_2}{\partial x}\right)_{x=x_c}$$

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[1]) WEBER RIEMANN, Differentialgleichungen der Mathematischen Physik, VIEWEG und SOHN, 1919. Vol. II, pp. 117–121.

J. STEFAN, Wiener Monatshefte für Mathematik and Physik, I. Jahrgang, p. 1, 1890.

Sitzungsberichte der Wiener Akademie. Vol. 98, Div. IIa, p. 473, 1890.

It follows from the third of these that $\frac{x_c}{\sqrt{t}} = \alpha$ where α is constant. On account of the constancy of α the fourth equation can also be satisfied. Eliminating the constants A, B we find for α:

$$\frac{\overline{H}_2 - H_c}{H_c - \overline{H}_1} = \sqrt{\frac{\beta_1}{\beta_2}} \frac{e^{\frac{\alpha^2\beta_1}{4}} \Theta\left(\frac{\alpha}{2}\sqrt{\beta_1}\right)}{e^{\frac{\alpha^2\beta_2}{4}} \left[1 - \Theta\left(\frac{\alpha}{2}\sqrt{\beta_2}\right)\right]} \quad . \quad . \quad . \quad (11)$$

The expression

$$\frac{e^{\frac{\alpha^2\beta_1}{4}} \Theta\left(\frac{\alpha}{2}\sqrt{\beta_1}\right)}{e^{\frac{\alpha^2\beta_2}{4}} \left[1 - \Theta\left(\frac{\alpha}{2}\sqrt{\beta_2}\right)\right]}$$

increases from 0 to ∞ as α increases from 0 to ∞. We can deduce from this that whatever the values of $\overline{H}_1$, $\overline{H}_2$, H_c may be (provided H_c is between $\overline{H}_1$ and $\overline{H}_2$) there is always one and only one value of α which satisfies (11). An increase in $|\overline{H}_2 - H_c|$ leads to an increase in α. An increase in $|H_c - \overline{H}_1|$ gives a decrease in α. Since β_2 is very large we are concerned with

$$lim\ \sqrt{\beta_2}\, e^{\frac{\alpha^2\beta_2}{4}} \left[1 - \Theta\left(\frac{\alpha}{2}\sqrt{\beta_2}\right)\right] = \frac{2}{\sqrt{\pi}\,\alpha}$$

whence by (11)

$$\sqrt{\pi}\,\frac{H_c - \overline{H}_1}{\overline{H}_2 - H_c} = \frac{e^{-\frac{\alpha^2\beta_1}{4}}}{\frac{\alpha\sqrt{\beta_1}}{2}\,\Theta\left(\frac{\alpha\sqrt{\beta_1}}{2}\right)} \quad . \quad . \quad . \quad . \quad (12)$$

If $\overline{H}_1 = 0$ and if $\frac{\overline{H}_2 - H_c}{H_c}$ is $\sqrt{\pi}$ i. e. if the externally applied field is 2.77 H_c both sides of (12) are unity and hence $\frac{\alpha\sqrt{\beta_1}}{2}$ is the value of x that makes $xe^{x^2}\,\Theta(x) = 1$.

This value is about 0.77. Since β_1 is roughly 1 for tin the constant α is of the order of magnitude of 1.6 and the law of penetration of the boundary is $x_c = 1.6\sqrt{t}$.

It may be shown that the field is unchanged in the bulk of the supra-conductor and that only a surface current is induced. From

this point of view the problem could be solved without reference to medium (2) by introducing a boundary condition in the medium (1) which is to express the fact that the flow $-\frac{1}{\beta_1}\left(\frac{\partial H}{dx}\right)_{x=x_c}$ is spent in supplying the quantity H to new regions of the material having initially $H = \overline{H}_1$ and converted to $H = H_c$. The length of the region converted per second is $\frac{dx_c}{dt}$ and thus the boundary condition is:

$$\left(\frac{\partial H}{\partial x}\right)_{x=x_c} + \beta_1 (H_c - \overline{H}_1) \frac{dx_c}{dt} = 0.$$

The problem can be also solved from this point of view.

This direct solution for the case $\beta_2 = \infty$ naturally leads to the same result which we have obtained by passing to the limit of $\beta_2 \to \infty$. It may be, however, that other problems may be more easily solved for the case of $\beta_2 = \infty$ by this method than by passing to the limit.

Case II. Penetration of supraconductivity into a non-supraconductor.

We next pass to the case of a material in which the supra-conductivity has been destroyed by a magnetic field, we diminish the field from the outside so as to reestablish supra-conductivity. The supra-conductivity is reestablished first in the external layer of the metal and propagates inward as time goes on.

Fixing our attention again on Fig. 2 we suppose that just before $t = 0$ the magnetic field H has a uniform value $\overline{H}_1$ throughout $x > 0$ and $x > 0$. This value $\overline{H}_1$ is greater than the critical field H_c. At $t = 0$ the value of H at the left of AB is dropped to $\overline{H}_2 < H_c$.

After the lapse of a time t the boundary between the two conducting states will have advanced a distance x_c. For $x < x_c$ the metal is microresidually conducting and $\beta = \beta_2$. For $x > x_c$ the metal has its ordinary conductivity and $\beta = \beta_1$. The expression for H for $x < x_c$ will be written as H_1. As in the first case we are induced to try to satisfy our equations by expressions of the form:

$$H_1 = A_1 + B_1 \Theta\left(\frac{x}{2\sqrt{\frac{\overline{\beta}_1}{t}}}\right)$$

$$H_2 = A_2 + B_2 \Theta\left(\frac{x}{2\sqrt{\frac{\overline{\beta}_2}{t}}}\right)$$

The initial and boundary conditions are:

$$\overline{H}_1 = A_1 + B_1 \qquad \overline{H}_2 = A_2$$

$$H_c = A_1 + B_1\,\Theta\left(\frac{x_c}{2}\sqrt{\frac{\overline{\beta}_1}{t}}\right) = A_2 + B_2\,\Theta\left(\frac{x_c}{2}\sqrt{\frac{\overline{\beta}_2}{t}}\right)$$

$$\frac{1}{\beta_1}\left(\frac{\partial H_1}{\partial x}\right)_{x=x_c} = \frac{1}{\beta_2}\left(\frac{\partial H_2}{\partial x}\right)_{x=x_c}$$

The third of the four above written lines shows that $\frac{x_c}{\sqrt{t}} = \alpha$ where α is constant. Eliminating the constants A, B from the above equations the result for α is:

$$\frac{\overline{H}_1 - H_c}{H_c - \overline{H}_2} = \frac{\sqrt{\overline{\beta}_1}\, e^{\frac{\alpha^2\overline{\beta}_1}{4}}\left[1 - \Theta\left(\frac{\alpha\sqrt{\overline{\beta}_1}}{2}\right)\right]}{\sqrt{\overline{\beta}_2}\, e^{\frac{\alpha^2\overline{\beta}_2}{4}}\,\Theta\left(\frac{\alpha\sqrt{\overline{\beta}_2}}{2}\right)} \quad . \quad . \quad . \quad . \quad (13)$$

We are particularly interested in the meaning of this equation for the case of a very large β_2. The values of α which satisfy (13) for this case must be very small because neither finite nor infinitely large values of α satisfy (13) in the case of an infinite β_2. If α is very small the numerator of the right hand member of (13) reduces to $\sqrt{\overline{\beta}_1}$. Therefore α must vanish at least as $\frac{1}{\sqrt{\overline{\beta}_2}}$ for otherwise the factor $e^{\frac{\alpha^2\overline{\beta}_2}{4}}\,\Theta\left(\frac{\alpha\sqrt{\overline{\beta}_2}}{2}\right)$ will yield an infinite result. The presence of $\sqrt{\overline{\beta}_2}$ in the denominator of (13) assures us moreover that α vanishes to a still higher order than $\frac{1}{\sqrt{\overline{\beta}_2}}$. Therefore $\alpha\sqrt{\overline{\beta}_2}$ is infinitesimal and

$$\Theta\left(\frac{\alpha\sqrt{\overline{\beta}_2}}{2}\right) = \frac{\alpha\sqrt{\overline{\beta}_2}}{\sqrt{\pi}}.$$

Hence (13) leads to:

$$\alpha = \sqrt{\pi}\,\frac{H_c - \overline{H}_2}{\overline{H}_1 - H_c}\,\frac{\sqrt{\overline{\beta}_1}}{\beta_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

if β_2 is very large. Thus the propagation of the boundary between the two states is infinitely slow if the microresidual resistance is infinitesimal. It should take an infinite time for the conductor to become entirely supra-conducting under the conditions just considered.

This result is of course a quite natural one from a purely non-mathematical point of view. The shielding power of the micro-residually conducting layer at AB (Fig. 2) is extremely great on account of its high conductivity. Thus in a finite thickness it transmits practically no magnetic field and as long as the magnetic field is transmitted the thickness of the microresidually conducting layer must be very small.

It is of interest to point out that even though the thickness of the microresidually conducting layer is very small the resistance of a square centimeter of this layer is finite and in the limit independent of σ_2. In fact this resistance is:

$$\frac{\sigma_2}{a\sqrt{t}} = 4\sqrt{\frac{\pi}{\beta_1 t}}\frac{\overline{H}_1 - H_c}{H_c - \overline{H}_2} = 2\sqrt{\frac{\sigma_1}{t}}\frac{\overline{H}_1 - H_c}{H_c - \overline{H}_2}$$

The formula (14) can be made clear also in the following manner. The microresidually conducting layer has two boundaries: one at $x = 0$ and one at $x = x_c$. The value of H at the first is $\overline{H}_2$ and at the second it is H_c. The drop in H in the thickness x_c is $\overline{H}_2 - H_c$. Let us suppose that this drop takes place uniformly throughout the thickness x_c. Then the drop in H per unit length is $\frac{\overline{H}_2 - H_c}{x_c}$ throughout. This quantity divided by β_2 is by (7) the electric intensity E which must be continuous at the passage through $x = x_c$. To the right of $x = x_c$ the conditions for H are determined by the facts that $H = H_c$ for $x = x_c$ and $H = \overline{H}_1$ for $x = \infty$. Since x_c is practically zero we commit no sensible error by replacing the first of these conditions by $H = H_c$ for $x = 0$. For this case it is clear that:

$$H = H_c + (\overline{H}_1 - H_c)\,\Theta\left(\frac{x}{2}\sqrt{\frac{\beta_1}{t}}\right)$$

and

$$-\frac{1}{\beta_1}\left(\frac{\partial H}{\partial x}\right)_{x=x_c} = \frac{1}{2\sqrt{\beta_1 t}}\frac{2}{\sqrt{\pi}}(-\overline{H}_1 + H_c).$$

Since this is the same as $\frac{\overline{H}_2 - H_c}{\beta_2 x_c}$ the equation (14) follows. Thus the assumption of uniform drop of H in the microresidually conducting layer leads to a correct result.

Since x_c is very small it appears legitimate to generalize this conclusion and to assume generally that H drops off uniformly throughout the microresidually conducting layer even in the general

cylindrical case because the curvature of the surface can have no influence in the thickness x_c. Thus if the solution analogous to

$$H = H_c + (\overline{H_1} - H_c)\, \Theta\left(\frac{x}{2}\sqrt{\frac{\beta_1}{t}}\right)$$

can be written down for the cylinder in question the solution for x_c offers no difficulty.

As an example let us consider a thin sheet of metal to which the magnetic field is applied from both sides tangentially to the surface. Let the thickness of the sheet be c. The solution given in WEBER RIEMANN (l.c.) Vol. 2, p. 112 formula II applies here. The constant a^2 is in our notation $\frac{1}{\beta}$. Thus according to this formula if H is suddenly changed by an amount $H_c - \overline{H_1}$ on both sides of the sheet the change in the value of H at a point having a distance x from one of the sides and considered at the time t is:

$$(H_c - \overline{H_1}) \left\{ \frac{x}{c} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-)^n}{n} e^{-\frac{1}{\beta_1}\left(\frac{n\pi}{c}\right)^2 t} \sin \frac{n\pi x}{c} + \right.$$

$$\left. + \frac{c-x}{c} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-)^n}{n} e^{-\frac{1}{\beta_1}\left(\frac{n\pi}{c}\right)^2 t} \sin \frac{n\pi (c-x)}{c} \right\}$$

This expression must now be differentiated with respect to x the value of the derivative with reversed sign at $x = 0$ must be divided by β_1 and equated to $\frac{\overline{H_2} - H_c}{\beta_2 x_c}$. This leads to

$$\beta_2 x_2 = \frac{(\overline{H_2} - H_c)\, c\beta_1}{2(H_c - \overline{H_1})\, \vartheta_2\left(0,\ e^{-\frac{4\pi^2 t}{\beta_1 c^2}}\right)} \qquad \ldots \quad (15)$$

where

$$\vartheta_2(0, q) = 2\left[q^{\frac{1}{4}} + q^{\frac{q}{4}} + \ldots\right].$$

It may be shown that (15) degenerates into (14) if $c \to \infty$.

The essential difference between (15) and (14) is that according to (15) for sufficiently high values of t the quantity $\beta_2 x_c$ is of the order of $e^{\frac{\pi^2 t}{\beta_1 c^2}}$ while according to (14) $\beta_2 x_c$ is always of the order $\sqrt{t}$. The increase in conductivity after a sufficient lapse of time becomes therefore very much more rapid than (14) would suggest.

The agreement between (14) and (15) is good as long as $\frac{4\pi^2 t}{\beta_1 c^2}$ is small because this assures the approximation of $\vartheta_2(0,q)$ by a probability integral. If $\beta_1 = 1$ and $c = 0.01$ cm. the quantity $\frac{4\pi^2 t}{\beta_1 c^2} = 3.9 \times 10^5\, t$. The series for $\vartheta_2(0,q)$ is then approximately $2[e^{-10^5 t} + (e^{-10^5 t})^9 + \ldots]$. Thus t must be considerably less than 10^{-5} sec. if (14) is to be a good approximation.

If now we should deal with a magnetic field which is periodically applied and removed from the cylinder the above calculation must enable one to form an idea as to the average electrical resistance of the cylinder used with a current passing longitudinally. In fact the method of calculation which we used last applies not only in the case of a uniform initial state but also if this state is variable. The solution of any specific case would be connected of course with further calculations.

Case III. Sudden reversal of field.

Fixing our attention again on Fig. 2 let us suppose that just before $t = 0$ the field has the uniform value $H_1 > H_c$. At $t = 0$ the field at $x = 0$ is suddenly changed to $-H_2$ where $H_2 > H_c$.

After the lapse of a time t we may expect to find three regions in the metal. These will be separated by two critical values of x, say x_{c_1}, x_{c_2}, $(x_{c_1} x_{c_2})$. In the intervals $(0, x_{c_1})$, (x_{c_1}, x_{c_2}), (x_{c_2}, ∞) β has the values $\beta_1, \beta_2, \beta_3$ respectively.

We shall try to satisfy the conditions of the problem by letting the magnetic field in these three intervals have the following expressions:

$$H = H_1 = A_1 + B_1\, \Theta\left(\frac{x}{2}\sqrt{\frac{\beta_1}{t}}\right) \qquad 0 < x < x_{c_1}$$

$$H = H_2 = A_2 + B_2\, \Theta\left(\frac{x}{2}\sqrt{\frac{\beta_2}{t}}\right) \qquad x_{c_1} < x < x_{c_2}$$

$$H = H_3 = A_3 + B_3\, \Theta\left(\frac{x}{2}\sqrt{\frac{\beta_1}{t}}\right) \qquad x_{c_2} < x < \infty$$

The equality between H_1 and H_2 at x_{c_1} and the equality between H_2 and H_3 at x_{c_2} leads to the conclusion that

$$x_{c_1} = \alpha_1 \sqrt{t} \quad , \quad x_{c_2} = \alpha_2 \sqrt{t}$$

where α_1, α_2 are constants. Thus the boundary and initial conditions become:

$$-H_c = A_1 + B_1\,\Theta\left(\frac{\alpha_1\sqrt{\beta_1}}{2}\right) = A_2 + B_2\,\Theta\left(\frac{\alpha_1\sqrt{\beta_2}}{2}\right)$$

$$+H_c = A_2 + B_2\,\Theta\left(\frac{\alpha_2\sqrt{\beta_2}}{2}\right) = A_3 + B_3\,\Theta\left(\frac{\alpha_2\sqrt{\beta_1}}{2}\right)$$

$$\frac{B_1}{\sqrt{\beta_1}}\,e^{-\frac{\alpha_1^2\beta_1}{4}} = \frac{B_2}{\sqrt{\beta_2}}\,e^{-\frac{\alpha_1^2\beta_2}{4}}$$

$$\frac{B_2}{\sqrt{\beta_2}}\,e^{-\frac{\alpha_2^2\beta_2}{4}} = \frac{B_3}{\sqrt{\beta_1}}\,e^{-\frac{\alpha_2^2\beta_1}{4}}$$

$$\overline{H_1} = A_2 + B_3 \quad ; \quad -\overline{H_2} = A_1$$

Eliminating the constants A, B two equations in α_1, α_2 are obtained. These may be written in the form:

$$\frac{\overline{H_2}-H_c}{e^{1/4\,(\alpha_1^2\beta_1+\alpha_2^2\beta_2)}\,\Theta\left(\frac{\alpha_1\sqrt{\beta_1}}{2}\right)} = \frac{\overline{H_1}-H_c}{e^{1/4(\alpha_1^2\beta_1+\alpha_2^2\beta_2)}\left[1-\Theta\left(\frac{\alpha_2\sqrt{\beta_1}}{2}\right)\right]} =$$

$$= \frac{2\,H_c}{\sqrt{\frac{\beta_2}{\beta_1}}\,e^{1/4\,(\alpha_1^2+\alpha_2^2)\beta_2}\left[\Theta\left(\frac{\alpha_2\sqrt{\beta_2}}{2}\right)-\Theta\left(\frac{\alpha_2\sqrt{\beta_2}}{2}\right)\right]}.$$

Since β_2 is very large the comparison of the first two expressions with each other shows that $(\alpha_2^2-\alpha_1^2)\beta_2$ must remain finite. Thus writing

$$\beta_2\,(\alpha_2-\alpha_1) = \gamma \quad , \quad (\alpha_1+\alpha_2) = 2\alpha \quad . \quad . \quad . \quad . \quad (16)$$

and considering only the case of very large values of β_2 we have

$$\left.\begin{aligned}\frac{\overline{H_2}-H_c}{e^{\frac{\alpha\gamma}{2}+\frac{\alpha^2\beta_1}{4}}\,\Theta\left(\frac{\alpha\sqrt{\beta_1}}{2}\right)} &= \frac{\overline{H_1}-H_c}{e^{\frac{\alpha^2\beta_1}{4}}\left[1-\Theta\left(\frac{\alpha\sqrt{\beta_1}}{2}\right)\right]} = \\ &= \frac{2\,H_c}{\lim\limits_{\beta_2\to\infty}\sqrt{\frac{\beta_2}{\beta_1}}\,e^{\frac{\alpha_2^2\beta_2}{4}}\left[\Theta\left(\frac{\alpha_2\sqrt{\beta_2}}{2}\right)-\Theta\left(\frac{\alpha_1\sqrt{\beta_2}}{2}\right)\right]}\end{aligned}\right\} \quad (16')$$

The limit last written is taken under the conditions (16), the quantities α,γ being kept constant. It is easily found that the *Lim* in question is $\frac{2}{\sqrt{\pi\beta_1}}\,\frac{e^{\frac{\alpha\gamma}{2}}-1}{\alpha}$.

Eliminating γ and letting

$$a = \frac{\alpha \sqrt{\beta_1}}{2} \quad , \quad \tau = \frac{\overline{H}_1 - H_c}{\overline{H}_2 - H_c} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

the resultant equation for α becomes

$$\frac{2 H_c}{\overline{H}_2 - H_c} = \frac{1}{\sqrt{\pi}} \frac{1 - (\tau + 1)\, \Theta(a)}{\Theta(a)\,[1 - \Theta(a)]\, a\, e^{a^2}} \quad . \quad . \quad . \quad . \quad (18)$$

Solving (16′) for γ we obtain

$$\gamma = \frac{2}{a} \log \frac{1 - \Theta}{\tau \Theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

If $\tau = 0$ (18) becomes $\frac{2 H_c}{\overline{H}_2 - H_c} = \frac{1}{\sqrt{\pi}\, a e^{a^2} \Theta(a)}$. This formula is readily seen to be in agreement with (12) if in the latter $\overline{H}_1 = -H_c$.

Thickness of Supra-Conductive Layers.

Formulas (14) (19) enable us to make an estimate of the thickness of supraconductive layers produced by the suppression or reversal of a strong magnetic field. Thus according to (14) the quantity a is of the order of $\frac{1}{\beta_1}$. Since β_1 is approximately 1, the thickness of the layer reached in 1 sec. measured in centimeters is of the order of magnitude of the ratio of the conductivities just above and just below the transition point. This ratio may be 10^{-8} and thus if formula (14) applies supra-conductive layers the thickness of which is of molecular dimensions are dealt with.

If the thickness of the slab discussed in (15) is 1 cm., the first term of the series $\vartheta_2\left(0, e^{-\frac{4\pi^2 t}{\beta_1 c^2}}\right)$ is $2\, e^{-4\pi^2 t}$ (β_1 being set $= 1$). Thus if $t = 10^{-4}$ sec. (14) and (15) are nearly in agreement and the effect of finite dimensions is not sufficient to throw off the conclusion just drawn because 10^{-4} sec. is a comparatively easily measurable interval of time.

The thickness of the supra-conductive layer brought about by the reversal of the field is according to (19) and (16)

$$\frac{\gamma \sqrt{t}}{\beta_2} = \frac{2 \sqrt{t}}{\alpha \sqrt{\beta_2}} \log \frac{1 - \Theta}{\tau \Theta}$$

and is thus of the same order of magnitude.

It is also of interest to observe that the amount of heat dissipated by the eddy currents in the microresidually conducting layer is

finite. In fact we have shown that the resistance of the layer per cm.^{2} is finite and further the current sheet in the layer has a finite strength being $\frac{1}{4\pi}^{\text{th}}$ of the difference in H on the two sides. Thus for the Case II the amount of energy dissipated per cm.^{3} is

$$\frac{1}{8\pi}\sqrt{\frac{\sigma_1}{t}}\,(\overline{H_1}-H_c)\,(H_c-\overline{H_2})$$

The sudden change in temperature which would have to be produced at the surface in order to supply this amount of heat would be given by

$$\Delta\theta = \frac{\sqrt{\pi}}{8}\sqrt{\frac{\sigma_1}{CK}\,\frac{(\overline{H_1}-H_c)\,(H_c-\overline{H_2})}{4.19\times 10^7}}$$

and is insignificant.

Other considerations for periodic alternating fields indicate that heating may be an important factor, the danger being in eddy currents in the part of the conductor having $\sigma = \sigma_1$.

SUMMARY.

Special cases of the propagation of changes in magnetic field in a supra-conductive metal are discussed. The calculations show that with the assumptions made (treatment of the conductor as a continuous medium) the thickness of the supra-conductive layers involved may be of the order of molecular dimensions during perceptible intervals of time.

The writer wishes to express his gratitude to Professor LORENTZ for his criticism and advice.

Biochemistry. — *"Further Researches on the Antagonism between Citrate and Calcium Salt in Biochemical Processes, Examined by the Aid of Substituted Citrates"*. (First Communication). By Dr. J. R. KATZ. (Communicated by Prof. A. F. HOLLEMAN).

(Communicated at the meeting of May 26, 1923).

I. *Exposition of the Problem.*

In an earlier research [1]) I have tried to analyse the nature of the biological citrate action. After addition of citrate a biological liquid behaves as if it no longer contains any free calcium ions; addition of citrate acts, therefore, in the same way as addition of oxalate or fluoride. With this difference, however, that the action of the latter salts rests on the formation of a very little soluble precipitate, and that a gypsum solution remains perfectly clear after addition of citrate. Complex ions must, therefore, have been formed [2]); it is only the question, how they are constituted.

In order to bring light in this still dark question, I compared at the time the action of the citrates with that of substituted citrates, in which one or more of the groups which possibly can bind the Ca to complexes (the alcohol group and the three carboxyl groups) were made inactive by substitutions (acetylation of the alcohol group; the carboxyl group esterified or converted to acid amide etc.). As typical representative of a biological citrate action the inhibition of the rennet coagulation of milk was investigated.

It then appeared that when either the alcohol group or one of the carboxyl groups is made inactive, the citrate action in a $^1/_{25}$ N. solution is reduced to $^1/_{16}$ of its strength; (i. e. is made equally weak as in a citrate solution of $^1/_{16}$ of the same strength), the removal of two or more groups reducing the action to less than $^1/_{100}$ of the original value. When the alcohol group is made inactive, the action appears to be equally strong as in other tri-or tetra-basic acids of allied structure, but without oxy-group (as tri-carballylic acid, aconitic acid or iso-allylene tetra-carbonic acid. When one

[1]) These Proc. Vol. XV, p. 434.

[2]) SABATANI, Atti della R. Acad. di Torino 36, p. 27—53 and Memorie (2) 52, p. 213—257 was the first to adduce arguments for this theory.
BROMBERG, Diss. Amsterdam.

carboxyl group is made inactive, the action appears to be equally strong as in other bi-basic oxy-acids (as apple acid and tartaric acid). In the same way it appears that when two or three groups are made inactive at the same time, the action has become as great as in the then comparable compounds.

Now the question rises.

1. *is it also possible to prove such a diminution of the number of free Ca-ions in less complicated systems than such biochemical ones by the addition of citrate?*

2. *do the substituted citrates show there a similar diminution of activity as in rennet-coagulation?*

The best way to answer these questions — the determination of the concentration of the free Ca-ions in the original solutions — is unfortunately barred, because we do not know a method as yet to determine the concentration of free Ca-ions potentio-metrically. It is, therefore, necessary to have recourse to indirect methods. The most natural proceeding is to determine how much calcium is held in solution by addition of citrate, when a substance that precipitates the calcium as insoluble compound (e.g. oxalate, fluoride, pyrophosphate, soap etc.) is added to a diluted solution of a calcium salt. The solubility product of this reaction must be chosen so that the action of the citrate manifests itself so as to be easily measured. If this solubility product is known, the percentage of *free* calcium ions is known at least at this small concentration, while it is known how much Ca remains in solution. [1])

The purest results will be obtained by an analytical determination by weight of the quantity of the calcium that has been precipitated or that has remained in the solution, as this can be carried out without appreciably diluting the calcium solution. I shall perform this experiment later on with citrate and with substituted citrates. But in order to get a preliminary rough idea, a titration can also be used, though this has the objection of appreciably diluting the original solution.

Mr. D. P. Ross van Lennep, who assisted me in my experiments on the influence of substituted citrates on rennet coagulation, pointed out to me that the soap-titration of calcium after Clark (as it is used in the determination of the hardness of water) might render us good services here. [2]) He carried out a number of experiments

[1]) The question in how far hydrolytic decomposition complicates the matter, will be treated later.

[2]) A drawback of this method is that the titration does not take place with water, but with 56-volume percentage alcohol, which changes the surroundings

with citrates and substituted citrates, but our experiments were left unpublished. I have again occupied myself with this problem, and performed a number of new determinations as a supplement and check. The results follow.

2. *Experiments.*

The examined Ca-solution, which was strongly split up into ions, was prepared as follows. A saturated solution of $CaSO_4$ (puriss. pro avel.) in distilled water was diluted with the 2,3 fold volume of distilled water. In a narrow-mouthed glass jar of 250 cm³. capacity 50 cm³ of this liquid was pipetted off and mixed with 50 cm³. of distilled water or with cm³ of an aqueous solutions of the substance under consideration. These 100 cm³ were titrated in the same glass jar by CLARK's method (with a solution of soap in alcohol of 56 volume percentages. [1]) In the titration a finely divided precipitate of calcium moleate is formed in the bottle. The endpoint has been reached when by the side of this precipitate so much alkali-oleate remains in the solution that, after shaking, the solution exhibits a not disappearing soap froth. As endpoint was taken the condition at which after a from six to eight times repeated vigorous shaking in the longitudinal axis of the bottle, the soap froth appears at the rim of liquid and bottle, as a white ring, 1 mm. high and from 1 to 2 mm. broad, and remains thus for five minutes. This endpoint can be determined pretty sharply, when the necessary practice has been obtained; when comparing experiments are always carried out in the same way, repeated determinations of the same liquid with a quantity of titration liquid of about 45 cm.³ deviate only some tenths of cm.³ from the mean of the determinations. For our determinations this accuracy is amply sufficient.

Without citrate the 100 cm.³ of calcium sulphate solution require from 45 to 47 cm.³ of titration liquid to reach this end-point; hence the total volume of the liquid at the end of the titration amounts to 145 or 147 cm.³. If in consequence of the addition of citrate the liquid required considerably less titration liquid, I added so much alcohol of 56 volume percentages (spec. gr. 0,921) from a burette to the 100 cm.³ that was to be examined, that at the end of the titration the total volume would again be between 145 and

in which the calcium ions are dissolved If, however, only small differences are measured, in other words if about an equal amount of alcohol is added, this does not prevent us from obtaining *comparable* results.

[1]) I refer for an accurate description, of CLARK's method to Jahresberichte f. Chemie 1850, p. 608; to LUNGE and BERL, 6th edition. Vol. II, p. 232.

147 cm.³; and in this liquid the endpoint was determined. This precaution was omitted, when the total volume was between 140 and 147 cm.³ at the end of the experiment. This measure purposes to prevent that in an inquiry into the titratable calcium in salt solutions of the same molecular concentration, these would have different molecular concentration at the endpoint of the titration, and would no longer be comparable for this reason.

1/10 N neutral solutions of the sodium salts were made from citric acid and its various substitution products (neutral towards litmus; it was verified that they remained neutral towards litmus on dilution with the same volume of the above gypsum solution). As normal solutions were considered those that contained one gramme-molecule per litre (hence not: One gramme-equivalent in multi-basic salts). The mixture of gypsum and of (perhaps substituted) citrate accordingly contained the various salts in the concentration of 1/20 N.

The gypsum solution diluted with the same volume of water consumed on an average 45.7 cm³. This corresponds with 12.2 parts of CaO per 100000 parts of water; or with 8.7 parts of Ca per 100000 parts of water. In citrates etc. it was derived from a table of LUNGE and BERL [1]) (calculated from experiments by FAIST and KNAUSS), how much Ca was *not* found back in the titration, calculated as percentage of the total quantity (8.7).

In the first column is given the consumed quantity of cm³ of titration liquid; in the second column the quantity of calcium that was not found back as percentage of the total quantity.

Thus I found:

a. Citric acid [2])	2.6 cm³	96 %
b. Te alcohol group made inactive.		
Acetylcitric acid	40.9 cm³	12 %
Compared with:		
Aconitic acid	41.3 cm³	11 %
Tricarballylic acid	40.8 cm³	12 %
Isoallylene tetra carbonic acid	39.8 cm³	14 %
c. One carboxyl group made inactive.		
Symmetrical citric acid monoamide	41.1 cm³	11½ %
Compared with:		
Apple acid	40.2 cm³	13½ %
Tartaric acid	40.4 cm³	13 %

[1]) 6th edition, Vol. II, p. 232.

[2]) When so few cm³ of titration liquid are sufficient to reach the limiting value, the limit is much less easy to determine than it is otherwise, and the observations differ much more from each other.

d. One alcohol group and one carboxyl group made inactive.		
Methylene citric acid	43.4 cm³	5½ %
Compared with:		
Ambric acid	43.6 cm³	5 %
Glutaric acid	43.6 cm³	5 %
Acetone dicarbonic acid	43.9 cm³	6 %
e. Two carboxyl groups made inactive.		
Citric acid dimethyl ester	43.35 cm³	5½ %
Citro diamide	43.75 cm³	5 %
f. Three groups made inactive.		
Citramide	44.4 cm³	3 %
Di ethylester of citric acid monoamide	44.8 cm³	3 %

Various indifferent salts of monovalent acids (sodium chloride, cyanide, formiate, acetyl-salicylate etc.) consume 44.6 to 44.9 cm³ of titration liquid; hence also three per cent less than water.

I refer for the structure formulae of the examined compounds to my previous publication [1]), where I have indicated them all.

It appears from these experiments that substitution in the citrates very considerably diminishes the action. If one group is made inactive (it seems to be immaterial whether it is the alcohol group or one of the carboxyl groups), about 11 or 12 % of the calcium is not found back in the titration (instead of 96 %). This quantity is, therefore, bound in complexes in the Ca-ion concentration which corresponds to the solubility of calcium oleate in an alcohol-water-mixture of about 17 volume percentages of alcohol (per 100 volume percentages of liquid mixture). In these solutions the compared citrates and substituted citrates are present in the same molecular concentration.

When two active groups are removed at the same time, about 5 or 5½ % of the calcium appears to be bound in complexes, in three groups only 3 %.

To be able to ascertain to what concentrations of the not-substituted citrate these values correspond, I have carried out some determinations with citrate of much weaker molecular concentration (all this expressed in the same units as in the experiments described before).

1/200 N	citrate	(0.0050 N)	35.9 cm³	24 %
1/400 N	citrate	(0.0025 N)	42.55 cm³	7.7 %
1/800 N	citrate	(0.00125 N)	44.3 cm³	3.4 %
no	citrate		45.7 cm³	—

[1]) These Proceedings. Vol. XV, p. 434.

Through interpolation it is found that 11 or 12 % of not recovered calcium corresponds to 0.0033 N; 5 or $5^1/_2$ % to 0.0019 N citrate, and 3 % to 0.0010 N citrate; hence that the activity is reduced resp. to $^1/_{15}$, $^1/_{26}$, $^1/_{50}$ of its value through the substitution in the unchanged citric acid.

These values show good agreement with the results of the rennet coagulation experiments, where $^1/_{16}$, $^1/_{100}$, $^1/_{100}$ was found. In view of the uncertainty in the determinations with small quantities of complex formation no better agreement can be desired.

We may still point out that also barium and strontium salts are deprived of their free ions by addition of citrate. Thus I found in diluted solutions of barium nitrate, strontium nitrate and calcium sulphate, which required resp

barium	strontium	calcium
23.0 cm³	25.4 cm³	25.4 cm³

of titration liquid, that — when these 100 cm³ contained $^1/_{100}$ resp. $^1/_{50}$ N sodium citrate — they consumed only:

$^1/_{100}$ N	21.6 cm³	6.0 cm³	11.2 cm³
$^1/_{50}$ N	16.85 cm³	1.85 cm³	3.9 cm³

3. *Conclusion.*

a. The biological citrate action rests on the diminution of the concentration of the free calcium ions through formation of complex compounds or ions. This citrate action can also be shown in less complicated systems than biochemical ones, e.g. in the solubility of calcium oleate in citrate.

b. Substituted citrates show there exactly the same diminution of activity as has been observed in a biochemical reaction (as the rennet coagulation). When either the alcohol group, or one of the carboxyl groups is removed, the activity is reduced to $^1/_{16}$ of its value; this diminution is much greater when two groups are removed at the same time.

c. Citric acid owes its strong activity to the fact that it is a multi-basic oxy-acid.

Experiments with other multi-basic oxy-acids are in progress. I refer for the literature to the extensive German publication, which will shortly appear.

Colloidchemistry. — "*Researches on the Nature of the So-Called Adsorptive Power of Finely-Divided Carbon.*" I. *The Binding of Water by Animal Carbon.* By Dr. J. R. KATZ. (Communicated by Prof. A. F. HOLLEMAN).

(Communicated at the meeting of June 30, 1923).

I. *Introduction.*

The power of finely divided carbon to bind all kinds of substances is evidently in connection with the degree of fineness of division; for in not finely divided condition the carbon does not show this property. At present the phenomenon is almost universally considered as a typical example of real surface adsorption, i.e. as the accumulation of a substance in the boundary layer simply in consequence of the surface-forces.

This surface adsorption is generally considered as in sharp contrast with the formation of a solid solution. In the latter case the bound substance is not only found in the boundary layer solid-liquid, but through diffusion it gradually penetrates between the molecules of the solid substance, so that finally the principal quantity of the absorbed substance is not found in the boundary layer, but homogeneously distributed throughout the solid body.

A clear realization of the questions that can be solved by experiments on the nature of this binding to carbon only dates from the time of physical chemistry. BANCROFT [1]) and others have considered the possibility that the substances would have been absorbed by the carbon in solid solution; but the further development of this thought failed on account of the form of the binding-isotherm. If we had to do with a solid solution, — this was the opinion some twenty years ago — the laws of HENRY and NERNST must be valid, hence the quantity of absorbed substance must be in direct ratio to the concentration of the vapour and liquid phase, with which it is in equilibrium. A curve is, however, obtained which is almost horizontal at first, and which then turns its convex side downward. This might be explained by the assumption that the absorbed substance dissociates in the carbon into many (e.g. four

[1]) The Phase Rule.

or ten) molecules. In most of the substances bound by carbon such a hypothesis has no sense. Besides it does not become clear why the carbon works the better as it is more finely divided; this must then be accounted for as a consequence of the easier diffusion.

In 1907 FREUNDLICH showed [1]) that the binding isotherm can be represented by the formula:

$$\frac{x}{m} = \alpha \, . \, c^{\frac{1}{n}}$$

for not too great values of c (m is the quantity of carbon, x the substance bound by it, c the concentration of this substance in the solution which is in equilibrium with the carbon, α and n are constants). He showed that we had to do here with *real* equilibria which are established within a very short time. The degree in which a solid substance binds, varies greatly with the absorbed substance, but is little dependent on the nature of the solid phase. FREUNDLICH demonstrated that these facts become perhaps most easily comprehensible when it is assumed that the binding rests on surface adsorption, on a becoming denser of the surface of the solid phase. But in 1909 he himself does not exclude the possibility that the phenomenon rests on the formation of a dissociable chemical bond or a solid solution; he only calls these explanations "wesentlich unvorteilhafter" [2]).

In course of time, however, in default of new arguments for the other conceptions, this view has gained so many adherers that it often makes the impression as if it were an established fact that the sorption by carbon rests on a real surface-adsorption.

In 1910 I succeeded [3]) in showing that a deviation from the laws of HENRY and NERNST in solid solutions can have another cause than the dissociation of the bound substance into molecules, viz. when the mixing in solid solution is *chiefly* caused by the *attraction* between the molecules of solvent and dissolved substance; whereas in the ordinary diluted solutions the mixing is brought about particularly by the *diffusion impulse* (because mixing is a more probable state, one that takes place with increase of entropy — also when the attraction may be neglected). In this case the decrease of free energy is about equal to the heat effect that takes place in the

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[1]) Zeitschr. f. physik. Chemie **57**, p. 385 (1907).

[2]) Kapillarchemie, 1ste Aufl. p. 289, Akadem. Verlagsgesellschaft Leipzig 1909.

[3]) These Proc. Vol. XIII, p. 958: Address at the Meeting of the Bunsen-Gesellschaft. Kiel, 1911; Gesetze der Quellung, Kolloidchem. Beihefte Bd 9.

binding. If the differential binding heat is great, and if it decreases on absorption of the substance, then follows from the equality of the variations of free energy and of binding-heat that the binding isotherm must have a course as FREUNDLICH must have found, i. e. that it begins pretty well horizontally, and then turns its convex side downwards. This appears to be the case in aqueous solutions of sulphuric acid and phosphoric acid, and in the swelling albumens and polysaccharides. In all these cases FREUNDLICH's formula appears to hold as approximating formula for small concentrations, even particularly well in aqueous solutions of sulphuric acid and phosphoric acid, though we have certainly not to do here with real surface adsorption, but with real mixing.

Hence it is clear that the validity of FREUNDLICH's *formula does not furnish the proof that we have to do with surface adsorption.* Inversely the equality in the variation of free energy and heat-effect is no proof either that there exists an ideal concentrated solution. It does not seem improbable to me that this equality *also* exists with pure surface adsorption, and possibly with many complicated intermediary phenomena called *sorption* at present. I found it confirmed in the absorption of water by cupri ferro cyanide, in which a strong change of colour from violet black to light brown is found[1]). The next step is now in my opinion to test this relation by a number of typical examples of genuine surface adsorption and of sorption. For if it appears to be valid everywhere, this is an important contribution to the knowledge of the sorption phenomena; and if it holds in some cases and not in others, it may be studied on what this depends. But apart from this it leads to a better method of analysis of sorption and adsorption phenomena: *the simultaneous determination of the sorption isotherms and of the sorption heats.* This method gives a much deeper insight than the prevalent one, which is restricted to the determination of the sorption isotherm for small concentrations. That FREUNDLICH's formula is of such universal validity at these small concentrations, will probably appear to mean that (in a system in which the variations of free energy and of heat-effect are equal in approximation) the differential sorption heat is very great at first, and diminishes gradually during the absorption; the longer the (almost) asymptotic horizontal initial part of the isotherm, the longer the differential sorption heat will preserve a great value. What is important in this method of investigation of the sorption phenomena is further that

[1]) Verslag van de gewone vergadering der wis- en natuurk. Afd. Kon. Akad. v. Wet. Dl. XXXI, Nos. 9—10, p. 542.

it can take into account not only the course of the isotherm for small concentrations, but the whole course. And besides it has the advantage that it does not bind itself beforehand by a preconceived opinion on the question which can at present mostly not be decided, of what nature the sorption phenomonon is (solid solution real surface adsorption, dissociable chemical combination, or two or three of these possibilities at the same time). The simultaneous determination of the two curves does, however, supply a collection of facts important for the decision of this question, which every theory has to take into account.

2. *Experiments.*

The purest animal carbon of MERCK was used for the investigation It was placed in air-dry condition in a wide-mouthed glass jar; its water content was determined at 230° C. after 3 hours' drying. It is not impossible that in this way the water percentage is found slightly too high, the weight of the carbon having possibly been slightly diminished by oxidation. As in most hygroscopic substances of this kind it remains somewhat arbitrary *what* is considered to be "dry" substance.

For the determination of the *sorption heats* quantities of from 5 to 12 grammes of carbon were weighed in *air-dry* condition, which can easily be done accurately, as the substance is not particularly hygroscopic in this condition; the carbon cannot be weighed accurately when quite dry. In crystallisation dishes these samples of carbon were brought in exsiccators over sulphuric acid-water mixtures of different strengths; we then waited till equilibrium had been approximately established. In this way samples of carbon were obtained in which the water was very uniformly distributed. Where the water-content of the air-dry carbon was known, the increase or decrease of weight of the sample of carbon yields its water content at the known vapour tension.

This carbon was placed in a glass tube, which was closed with a tight-fitting rubber stopper and placed in a calorimeter vessel filled with water. The experiments were made in a room in which the temperature was particularly constant. After temperature equilibrium had been established, the course of the thermometer was followed; then the contents of the tube were emptied into the water of the calorimeter vessel, after which the temperature was again observed. After from 2—4 minutes the generation of heat did not increase appreciably any longer.

Let us call i the degree of sorption (gr. of water per 1 gr. of dry substance), and W the heat of sorption (generation of heat in cal. when 1 gr. of dry substance absorbs 1 gr. of water). Then I found:

i	Quantity of heat at maximum sorption per 1 gr. of dry carbon	W
0.—	20.91	0.—
0.049	17.66	3.25
0.090	15.34	5.57
0.218	11.79	9.12
0.350	7.90	13.01
0.437	6.05	14.86
0.563	3.12	17.79
0.659	1.59	19.32
0.718	1.09	19.82
0.753	0.29	20.62
sorption-max. 0.93	0.—	20.91

This is the integral heat of sorption. From this I calculate the differential heat of sorption for $i = 0$

$$\left(\frac{dW}{di}\right)_{i=0} = 75 \text{ cal.}$$

This value is considerably smaller than was found in swelling substances (250 to 400 cal.). At the heat of mixing of sulphuric acid (with water) it amounted to 550 cal., of phosphorus 100 cal., of glycerine 20 cal.

The curve of the integral sorption heats is graphed in fig. 1; it starts as the ordinary curve of the heats of imbibition and of mixing, as a hyperbola, then follows a flattened, almost rectilinearly rising part, the end again being a hyperbola. Accordingly it is distinctly different from the curves described by me formerly for bodies that can swell up.

I have not yet succeeded in calculating the differential sorption heat in its full course from these measurements. The curve of the integral sorption heat has so complicated a shape that a formula with a great number of parameters is required to give any description of it. The greater the number of parameters, the more arbitrary is

the calculation of the differential quotient $\frac{dW}{di}$. But this at least may

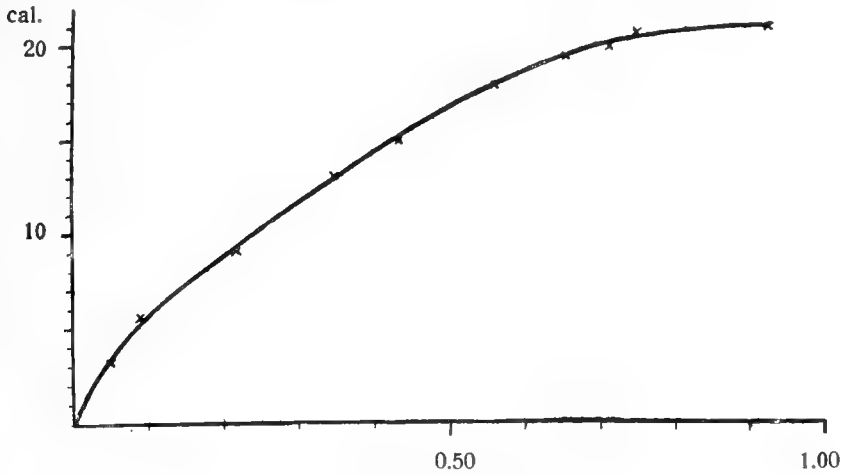

Fig. 1.

be said now that the curve begins with $\frac{dW}{di} = 75$ cal., then decreases pretty rapidly, in a way, which corresponds pretty closely with the course of this quantity in the heats of mixing (of sulphuric acid or phosphoric acid with water). At $i = 0.10$ to $i = 0.15$ it begins to assume a more or less constant (albeit slowly diminishing) value, amounting to about 23 cal., which diminishes again greatly past $i = 0.65$, and converges to zero.

It would be very important also to study the volume contraction at the absorption of water; for, where in expansible and in miscible substances the relation $\left(\frac{c}{W}\right)_{i=0}$ always appeared of the same order of magnitude (between 10 and 30×10^{-4}), it would be important to examine what the order of magnitude of this quotient would be in animal carbon. Unfortunately it is not possible to determine these volume contractions, as carbon probably acts as an adsorbent on *every* pycnometer liquid, at least in anhydrous condition.

The *free energy* at the sorption can most easily be calculated from the vapour tension of the water at different degrees of sorption. These vapour tensions have not been determined directly, but indirectly by the method of Gay Lussac-van Bemmelen (by bringing the substance into equilibrium with sulphuric acid-water mixtures of known strength till constancy of weight is reached). The absorption and loss of water then appeared to be a phenomenon of equilibrium, which presents *hysteresis*. This result is in striking

contrast with FREUNDLICH's experience that the absorption of *dissolved substances*, as iodine, dyestuffs, and organic acids, is an equilibrium, which is readily established independent of the condition from which one starts, and within a few minutes; this observation of FREUNDLICH's was confirmed for *dissolved* substances by many investigators.

In order to obviate the influence of hysteresis, the equilibrium had to be determined from two sides; then the approximative value of the state of equilibrium was calculated by taking the mean of the two values found in this way. Accordingly twice thirteen samples of air-dry carbon, each having a weight of about one gramme, were weighed off in crystallisation dishes. One half of these dishes were dried for one or two weeks in a vacuum exsiccator over sulphuric acid; they then contained no more than 1 or 2 parts of water to 100 parts by weight of dry carbon. The other half was placed over water in a vacuum exsiccator for the same length of time; they then contained about 90 parts of water to 100 parts of dry carbon. Then thirteen small exsiccators were arranged with sulphuric acid-water mixtures of known vapour tension; in every exsiccator there was placed a dry and a moistened carbon. These acids were refreshed a few times. After 40—90 days, when the dishes had become almost quite constant of weight long before, it was assumed that they had reached their onesided equilibrium. All the experiments took place at a temperature of 16—20° C. in a room in which the variations of temperature were particularly small (a room built specially for thermochemistry).

The vapour tension h was expressed as fraction of the maximum tension of water at the same temperature; the sorbed quantity i as grammes of water per one gramme of dry carbon. The free energy at the sorption of one gramme of liquid water is found from the relation $A = \frac{1252}{18} log^{10} h$.

Fig. 2 shows the isotherm. The curve begins as a real adsorption-curve (or as the isotherm of a concentrated solution), but with a *very* short horizontal initial portion [1]), at half its height, ($h = 0{,}40$ to $0{,}65$) it gets, however, an almost horizontal part; at $h = 0{,}65$ and $i = 0{,}57$ there begins a new part of the curve (which, however, issues from the preceding part without any abrupt transition), which again has an S-shape. It is remarkable how great the quantity

[1]) This has probably been drawn too long; has the weight of the carbon not been somewhat diminished by drying at 200° C. through oxidation? The horizontal beginning, *if* it exists, is probably only little pronounced.

of water is which this form of amorphous carbon can absorb; over a sulphuric acid with a $h = 0{,}997$ the substance absorbed 0,929 parts of water per 1 part of dry substance! Accordingly an absorption of water of the same order of magnitude as in *greatly* swelling

h	i			Difference between the two false equilibria
	after moistening	after drying	in equilibrium	
0.010	0.009	**0.022**	**0.016**	—
0.083	0.033	0.021	**0.027**	—
0.176	0.039	0.038	**0.039**	—
0.278	0.062	0.052	**0.057**	0.010
0.410	0.172	0.141	**0.157**	0.031
0.517	0.458	0.266	**0.362**	0.192
0.596	0.570	0.411	**0.491**	0.159
0.721	0.649	0.572	**0.631**	0.077
0.788	0.673	0.631	**0.652**	0.021
0.853	0.698	0.676	**0.687**	0.022
0.914	0.730	0.715	**0.723**	0.015
0.962	0.800	0.814	**0.807**	—
0.997	—	0.929	**0.929**	—

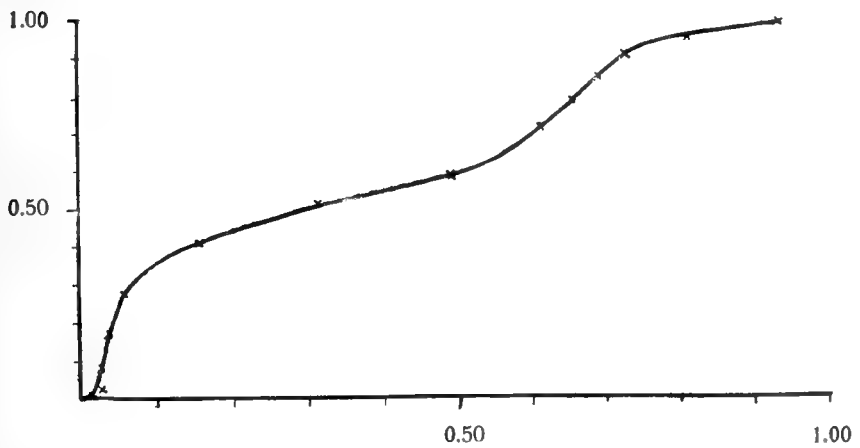

Fig. 2.

substances. BACHMANN [1]), who already determined an isotherm of carbon and water before me, found in cocoanut carbon a maximum water absorption of $i = 0{,}25$. BERL and ANDRESS [2]) also found in their carbon a considerably smaller value than I in mine.

The double-S-shaped curve of the isotherm obtained is practically the same form as that which VAN BEMMELEN has observed in gels of silicic acid and of iron hydroxide. The flat portion there corresponds to the part of the curve in which the gel, which is first transparent, becomes opaque.

3. *Comparison of Free Energy and Heat Effect.*

A simple comparison of the curves fig. 1 and fig. 2 shows that $\frac{dW}{di}$ and $log\ h$ must have an analogous course as function of i. Both curves have an almost horizontal, almost rectilinear (slowly descending) portion between $i = 0.10$ and $i = 0.60$ to 0.65; both curves have before and after this the shape as for liquids which mix with water with strong heat effect. By graphical determination of the differential quotient $\frac{dW}{di}$ this can be estimated for some values of i, for which $log\ h$ is known. Thus I find:

i	h	$\frac{1252}{18} log^{10} \frac{h_1}{h_2}$	$\left(\frac{dW}{di}\right)_1 - \left(\frac{dW}{di}\right)_2$
0.027	0.083	37 cal	11 cal
0.057	0.278	12 „	20 „
0.157	0.410	7 „	8 „
0.362	0.517	4 „	4 „
0.491	0.596	6 „	4 „
0.631	0.721	5 „	5 „
0 687	0.853	3.5 „	6 „
0.867	0.962		

These are only rough estimations; but they show nevertheless with sufficient probability that in the large middle portion of the curve (from $i = 0.05$ to $i = 0.80$) the variation of the free energy

[1]) Zeitschr. f. anorgan. Chemie **100**, p 32 (1917).
[2]) Zeitschr. f. angewandte Chemie 1921. Bd. I.

is of the same order of magnitude as the heat effect. But with small i the heat effect is much smaller than the variation of the free energy. This latter is probably in connection with the small value of the first differential heat of sorption in this substance. Most likely there is no equality in the middle piece either, but only correspondence in the order of magnitude. The experiments are, however, not accurate enough to set forth this difference clearly.

4. *The Analogy of the Curves with those for Newly-made Silicic Acid and* ZSIGMONDY and ANDERSON's *Explication.*

As I already observed, the isotherm has the same typical shape as that found by VAN BEMMELEN and later by ANDERSON for silicic acid gel. The "turn", the point where the second S-shaped curve begins, lies at $i = 0.57$ and $h = 0.65$ for carbon. Also BACHMANN found a curve with a horizontal portion for the cocoanut carbon examined by him (possibly even with two such pieces). And BERL and ANDRESS found a curve of the same shape as mine in the carbon examined by them.

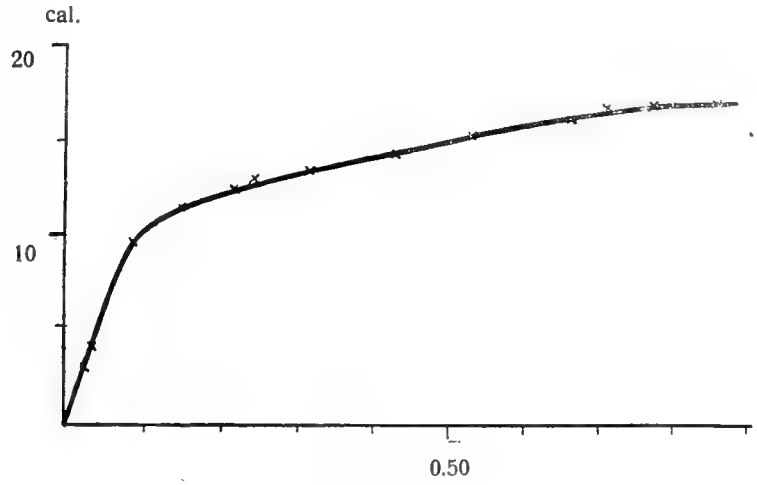

Fig. 3.

That also the curves of the heats of sorption correspond is shown by fig. 3, in which I have represented BELLATI and FINAZZI's results [1]) for newly-made silicic acid (temperature 12°—20° C.). Unfortunately these carefully performed researches have so far escaped the notice of the writers of the books on colloid chemistry, whence they have not met with the recognition they deserve. The curve typically

[1]) M. BELLATI and L. FINAZZI, Atti d. R. Instituto Veneto, Serie VIII, Tomo 4, p. 518.

presents the same course as that found by me for carbon; the initial part as the curve for a heat of mixing, the almost rectilinear middle portion, the end in a curve with the concavity downward. Unfortunately we have no reason to believe that the silicic acid examined by BELLATI and FINAZZI possesses exactly the same constants as that on which VAN BEMMELEN and ANDERSON performed their determinations of the vapour tension, as the properties greatly depend on the preparation. This *is*, however, the case in the experiments with carbon, described above.

In the absorption of water vapour by carbon we have, therefore, to do with a system of which the isotherm and the curve of the heats of sorption are in perfect agreement with the same curves of those silicic acid gels that present a so-called "turn".

In silicic acid it is very probable that in the flat piece very fine capillaries are getting filled with water, for absorption of water causes the opaque substance to become transparent again. ZSIGMONDY and ANDERSON [1]) pointed out that the radius of these fine capillaries can be calculated from the vapour tension of the water in the flat piece; they then arrived at values of the order of magnitude $1.3\ 10^{-6}$ mm. for the initial part, and 2.6×10^{-6} mm. of the end of the flat piece. And they showed further that when the same silicic acid gel is changed into an alcohol or benzene gel, and the radius of the capillary is calculated from the vapour tension of the alcohol or the benzene, values are obtained for this radius of the same magnitude as in water. This pleads very strongly in favour of the view that the flat middle piece is due to the filling of capillaries, which gradually become slightly wider, hence on micro-porosity.

PATRICK [2]) repeated these experiments with liquid carbonic acid and liquid sulphur dioxide with silicic acid gel. Then he found, however, much less concordant values for the size of the capillaries; he tried to explain this by the greater thickness of the capillary layer near the critical point.

BACHMANN [3]), working in ZSIGMONDY's laboratory, also explained the flat middle piece in the isotherm of carbon and water by a system of such fine capillaries. The substance being opaque, it cannot be ascertained if this property becomes stronger in the middle piece.

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[1]) Zeitschr. f. physikal. Chemie, 88, p. 191 (1914); ZSIGMONDY, Lehrbuch der Kolloidchemie, 4th edition, p. 219—234.

[2]) PATRICK, Diss. Göttingen, 1914.

[3]) BACHMANN, loc. cit.

My experiments lead to the following values for this radius:

$i_1 = 0.157$ (beginning of the flat piece)	$h_1 = 0.410$	$r_1 = 1.24 \times 10^{-6}$ mm.
$i_2 = 0.362$	$h_2 = 0.517$	$r_2 = 1.67 \times 10^{-6}$ mm.
$i_3 = 0.491$	$h_3 = 0.596$	$r_3 = 2.13 \times 10^{-6}$ mm.
$i_4 = 0.57$ (end of the flat piece).	$h_4 = 0.65$	$r_4 = 2.57 \times 10^{-6}$ mm.

The values found for the radius of the micro-capillaries, are in such close agreement as regards order of magnitude with the values of ZSIGMONDY and ANDERSON, and with those of BACHMANN, that it is astonishing that always *this* order of magnitude is again met with. (The second system of capillaries which BACHMANN thinks that he can derive from his curves, seems questionable to me).

The agreement in the form of the curves for the heats of sorption with their typically flattened piece corroborates that the flat part of the isotherm for carbon and for silicic acid has the same cause.

It is the more striking under these circumstances that BERL and ANDRESS have found that the same carbon which gives a flat middle piece in the isotherm with water, has *a curve without any flat middle piece*, and with a much longer horizontal initial part (for small l) *with organic liquids* (as benzene or methyl alcohol). If the correctness of these experiments is confirmed, they furnish the proof that ZSIGMONDY's explanation, cannot be the true one, at least for carbon. I am, therefore, occupied with a repetition of these experiments, and also with a determination of the heats of sorption.

Since ZSIGMONDY's explanation is inadequate to account for the flat piece in the isotherm and for the flattened piece in the heats of sorption, it is in my opinion natural to see a connection between the deviating form of the isotherm of water and the fact that *water moistens solid bodies, as carbon, much less easily than organic liquids*, as benzene or methyl alcohol, do. We should then have to do in water and carbon with surface adsorption at a surface that is not easily moistened, a phenomenon of which so far only one example has been studied somewhat more closely [1]), viz. the adsorption of watervapour to glass-wool which has been thoroughly dried beforehand, investigated by TROUTON [2]). The glass-wool had been previously

[1]) FREUNDLICH, Kapillarchemie, 2nd edition, p. 223. Possibly there is solid solution present as a complication in the boundary layer also here.

[2]) FREUNDLICH, loc cit.

treated by drying at 162° over phosphorus pentoxide, and then gave an isotherm with a flat middle piece (possibly even with a faint retrograde piece), which shows a close analogy with the shape of the isotherm for water and carbon. When the glass-wool had been well moistened beforehand, it gave an S-form, as they have been found in mixtures of sulphuric acid and water, and in swelling bodies with water as imbibition-liquid; characteristic is there the beginning with a strongly pronounced horizontal piece for small c, in which region FREUNDLICH's adsorption formula is valid. Similar curves were found by BERL and ANDRESS for the adsorption of those liquids that moisten the carbon well.

This conception might also be able to explain why the adsorption by carbon of water presents such strong hysteresis, whereas that of organic vapours seems to take place without hysteresis. It is, however, possible that solid solution in the boundary layers also plays a part in this[1]).

The experiments are being continued.

5. *Conclusions.*

1. In the investigation of the phenomena of sorption it is insufficient to determine the isotherm of binding; it is necessary to determine at the same time the heat of sorption as function of the quantity of absorbed substance at *the same* material.

2. The examined animal carbon appeared to have an isotherm with an almost flat middle piece, analogous to the isotherm of newly-made silicic acid. The sorption-heat had a course corresponding with this, a flattened middle piece.

3. By assuming that this course is explained by a system of micro-capillaries, I calculate the radius of these capillaries from the isotherm at 1.2 to 2.6 $\mu\mu$ (as for silicic acid). That this dimension agrees so closely with that for silicic acid, is somewhat strange and striking.

4. It is, however, doubtful whether this explanation by the assumption of a system of micro capillaries is the true one. It seems probable to me that the difficult moistening of the carbon by water accounts for it.

5. Very striking is the strong hysteresis in the isotherm[2]).

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[1]) In the search for possible explanations for the deviating behaviour of *water* at carbon much light was thrown on the subject by conversations with Dr. M. POLANYI.

[2]) The complicated results of B. GUSTAVER (Kolloidchem. Beihefte, 1922) and HÄLLSTRONO's experiments (Diss. Helsingfors, 1920) will be discussed in a following paper. Not to lengthen this communication, I confine myself to only mentioning them here.

Chemistry. — "*Volta-Luminescence*". By Dr. J. LIFSCHITZ. (Communicated by Prof. F. M. JAEGER).

(Communicated at the meeting of June 30, 1923).

§ 1. On the passage of electric currents through Voltaic cells phenomena of light are often observed at the electrodes. This "electrolytic", or rather this "electrode" light can appear both at the anode and at the cathode, as well on use of continuous current and of alternate current. The nature of the emitted light has seldom been investigated, and then only unsatisfactorily. Consequently so far only little could be said with certainty about the nature of the process. Some researchers (1, 2, 3) have interpreted some of these phenomena of light as reaction luminescence phenomena — hence as belonging to the phenomena of chemi-luminescence. If this should appear to be true, this would be of importance, because, as is known, ionic reaction is hardly ever attended with luminescence (4, 5). Besides the phenomena in question are of importance spectroscopically and electro-chemically. The light emissions under consideration may certainly not be considered as of an exclusively thermal character. For, as earlier experimenters already observed, the phenomenon of light is as a rule the more intense, as electrode and electrolyte have a lower temperature. Often the luminescence only occurs at very small intensity of the current. The spectrum is mostly discontinuous, or it presents at least a maximum of intensity, as is not possible with purely thermal radiation. At any rate an incandescence of the electrode metal can be distinguished with perfect certainty from the luminescence proper. Hence we are justified in distinguishing the phenomena in what follows as "Volta-luminescences"; and it will appear that inter se these are of very different characters, though on the other hand they resemble each other more or less in the following respects:

1. There is mostly a considerable increase of the resistance of the cells, as long as the electrode emits light.

2. Formation of solid or gaseous layers at the luminescent electrode, which sometimes entirely prevent the passage of the current.

3. Often an abnormal course of the electrolysis can be observed.

1. *Cathodic Luminescence.* (Wehnelt interruptor, Chromoscope of v. Bolton.)

§ 2. The first data about phenomena of light at the anode, as they appear in the Wehnelt-interruptor, were given by Wehnelt (6) himself. Voller and Walther found (7) that much stronger light effects are obtained when the smaller electrode is made cathode, hence when the interruptor is inserted reversely. A very pure spectrum of the electrode metal is then observed, and further some of the hydrogen lines appear. The phenomena also occur when the cell is not inserted as an interruptor, hence without induction coil.

Without taking these observations sufficiently into account, v. Bolton (8) later described an arrangement which was suitable for spectralanalytic purposes and closely resembled the preceding one. He called this arrangement "Chromoscope". As anode served a thick platinum wire or platinum plate; as cathode he used a platinum wire, or a rod of the metal that was to be examined spectroanalytically. The electrolyte (H_2SO_4, or better HNO_3 1 : 4) contained in the first case a small quantity of the substance to be examined. When the current is closed by carefully immersing the cathode, very clear and pure spectra of the metals are obtained, which are present as electrode or in the electrolyte, and besides H-lines (especially H_α) and the Na-D-line. v. Bolton used a potential of 110 Volt; then the strength of the current in his electrolyte-chromoscope amounted to 0,15—0,3 Amp., in his metal chromoscope to 2 Amp.

Morse (9) investigated the light of the Wehnelt-interruptor more closely. He used an alternate current of a pretty considerable strength, and found that cathode and anode give the same spectrum; the cathodic light was, however, much stronger than the anodic light. He did not observe H-lines. The spectra obtained sometimes resembled the arc-spectrum more closely, sometimes the spark spectrum, without his being able to give a satisfactory explanation of this. There are, however, always characteristic differences between Wehnelt- and spark-spectra, resp. Wehnelt- and arc-spectra. We shall come back to further observations of Morse later on.

For the investigation of the cathode spectra the arrangement of v. Bolton is the most suitable; this was still somewhat modified for experiments of longer duration. Fig. 1 represents a simple model of an electrolyte-chromoscope, with which experiments can be made without difficulties. A *U*-tube is placed within a cooling-jacket; the legs of this tube are closed by two rubber stoppers, in which

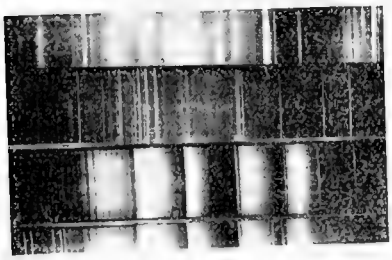
the electrodes are fastened. By means of an intermediate piece the two legs are connected with each other and with the water-jet suction-pump, which immediately removes the oxyhydrogen gas formed in the electrolysis. The luminescence is started by the immersion of the cathode, and at the same time the cell is closed air-tight. In the metal chromoscope the tube drawn in fig. 2 comes in the place of the *U*-tube.

In order to photograph the spectra, the light was thrown on the slit of a HILGER spectrograph by means of a small condenser with small focal distance. When Viridin-Inalo plates were used, the exposure had sometimes to be continued from 40 to 150 minutes, because spraying took place. As electrolyte HNO_3 1 : 4 was generally used; other electrolytes, however, may equally well be used, e.g.

Fig. 1 Fig. 2

diluted or concentrated H_2SO_4, KOH etc.; this brings about no essential difference as to the nature of the phenomena.

Al spectra.



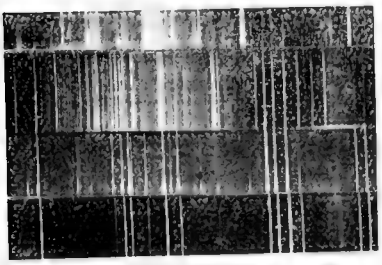
Spark.

Electrolytechromoscope.

Metalchromoscope.

Arc.

Cu-spectra.

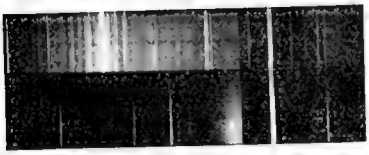


Spark.

Electrolytechromoscope.

Arc.

Metalchromoscope.



Mg-spectrum in d.electrolytechromoscope

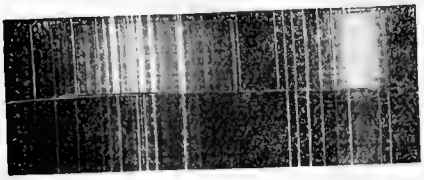
Mg-arcspectrum.

Strontium.

Strong solution.

Dilute solution.

Copper.



160—180 V. 68 min.

< 100 V. 136 min.

Spectra obtained with the electrolytechromoscope.

Fig. 3.

§ 3. In contrast with what was found by Morse, H-lines (especially H_α) were present in the emitted spectrum; further Pt-lines at platinum cathodes. *Apart from this it was stated that electrolyte and metal chromoscope, give totally different spectra*, — a fact which was quite overlooked both by v. Bolton and by Morse. When the metal that is to be detected, only occurs in the electrolyte, the spectrum very closely resembles that of the spark-spectrum of the metal. If this same metal is, however, immersed as cathode in pure acid, a spectrum is obtained which agrees closely with the arc-spectrum. As an illustration of these facts, which I could verify repeatedly, some photographs have been reproduced here (fig. 3).

That we are justified in speaking of a general behaviour here, follows for the rest, besides from our own observations (with Mg, Pb, Fe, Wo, Mo, Ta, Al, Cu etc.), also from the data of v. Bolton and Morse themselves. If the metal is at the same time in electrolyte and electrode, it is to be expected that a superposition of the two spectra is observed. Since, however, the metal chromoscopes produce more intense phenomena, it is easy to understand that Morse observed a strong arc-spectrum that is generally superposed by a weak spark-spectrum.

If the chromoscopes are to function normally, a definite current intensity is required in both cases, which though dependent on the adjustments of the apparatus, always remained within the limits indicated by v. Bolton. With Cu-salt in the electrolyte chromoscope (fig. 1) e. g. 0,4 –0,5 Amp. appeared to be required. A greater intensity of the current caused incandescence of the wire, and the disappearance of the luminescence, whilst a weaker current caused the total light intensity to become smaller. As appears from the adjoined photographs, also a selective weakening takes place: some lines losing much more in intensity than the rest. The same effect may also be reached by greatly diminishing the concentration of the metal salt.

In many cases, especially when earth-alkali salts are used, one has the impression that the whole liquid at the cathode is luminescent. This effect is, however, not always found; besides the spectrum was not changed by this. The co-luminescence seems to be caused by still unknown accessory circumstances.

With regard to the mechanism of the emission process it may be considered as an established fact that the cathode is surrounded by a gas envelope. As already Voller and Walter observed, this may be shown simply as follows: when a well-luminescent chromoscope is

first cut out, and then immediately inserted again, the luminescence quickly continues without it being necessary to take the electrode out of the liquid and immersing it again. If, however, we wait a short time after the cutting out, a hissing sound is heard after about 2 or 3 sec., and now the chromoscope is not at once luminescent again when it is inserted. Moreover the experiments of Riesenfeld and Pfützer (11) have described, plead still more in favour of the existence of a gas layer. There a small of light arc is formed between cathode and liquid, and I could verify that the same spectral phenomena are obtained as in the chromoscope. On use of Pt- or Ir-cathodes, the metal to be detected being present in the liquid, a spark-spectrum is obtained; when, however, the metal is used as cathode with pure acid, and arc-spectrum.

§ 4. Probably the following idea must be formed about the origin of these cathodic luminescence phenomena. Between electrode and electrolyte there is formed a gas envelope containing hydrogen, water-vapour and some oxygen; within this layer lies almost the whole fall of potential of the cell. The cations not being able to traverse this layer, there a current of rapid cathode rays is formed, which discharge these cations. The discharged metal atoms now get into the gas layer, and are excited to the emission of a spark-spectrum by collision with similar flying electrons.

The spraying of the cathode is greatly promoted by the impact with positive particles. If, as in the metal chromoscope, the current density and the strength of the current intensity are relatively high, also uncharged atoms of the electrode metal get into the gas layer — either because the spraying consists primarily in a scattering of molecular particles, or because locally a sufficiently high temperature arises —, and then an light-arc is formed and hence an arc-spectrum is observed.

If the electrolyte at the same time contains a sufficient number of ions of the electrode- or another metal, a spark spectrum of the second metal can of course appear by the side of the arc-spectrum of the first metal. This is, however, not necessary. Depending upon the nature of the electrode metal, the arc-spectrum is more or less apparent. Thus Morse showed already that the spectrum of a platinum cathode is intense in solutions of acids and alkalies, but very faint in solutions of earth alkalies, while a strong aluminium (arc-)spectrum appears with an aluminium electrode in almost any electrolyte. The relations that are valid here must, however, still be examined; possibly the greater or less tendency to spraying of

the electrode material is playing here a prominent part. That melting-point and evaporation point of the metal are not decisive, has already been stated by MORSE.

(II) *Anodic Luminescence.*

§ 5. As might be expected, the phenomena at the anode are much more numerous and much more complicated than those at the cathode. Besides gas layers, also layers of solid substance can establish themselves here between electrolyte and electrode, thus causing luminescence. The sparks which appear in valve cells at the limiting tensions (10), have not been examined in what follows.

According to the nature of the emitted light and the cause of the luminescence at the anode, the following typical cases of luminescence can be distinguished.

1. Line- and band-spectra; to a certain extent these are very similar to those at the cathode, but they are generally much weaker.

2. Arc-spectra, equal to those at the cathode, but which can but rarely be obtained, and then only on definite conditions.

3. Generally a yellowish luminescence — which in so far as this can be ascertained, is spectroscopically continuous, — without formation of a layer of oxide or anything of this kind. The anode metal (or the carbon used as anode) gets shiny or bright.

4. For so far as this can be ascertained a continuous emission, with a maximum of intensity in a definite spectrum region; in this case the formation of solid layers at the anode always takes place.

First of all we will give some instances and some further particulars of the phenomena in each of these four classes.

§ 6. 1. Already VOLLER and WALTER record that at an interruptor anode from platinum in sulphuric acid 1 : 40, they obtained — by the side of the NaD-line — a faint band spectrum. If this cell contained sulphuric acid and also metal salts, the lines of these metals also appeared. The data of these investigators could be fully confirmed; no more than they, did I, however, succeed in determining more accurately the band-spectrum lying in the green [1]). The intensity of the phenomenon was, indeed, too small for spectroscopic investigation, though it was always clearly perceptible, also in aqueous potassium hydroxide 1 : 10, and on use of other anode metals. Special phenomena were obtained on use of platinum anodes in sulphuric acid 1 : 40, containing at once several metallic salts.

In order to obtain anodic metal lines, greater quantities of metallic salt must in general be dissolved in the acid. Even then mostly a

[1]) Very probably these „bands" belong the oxygen spectrum.

few characteristic lines stand out very clearly (e. g. the green Tl-line; the three green Cu-lines). If the acid contains two kinds of metalions, often only one of these kinds of ions can be detected spectroscopically. An example of this is furnished by the following experiment:

A platinum anode was immersed in sulphuric acid 1 : 40, which contained a sufficient quantity of sulphate of sodium and sulphate of copper. First so much current was passed through that the anode wire became incandescent; then gradually resistance was inserted until the incandescence stopped and the characteristic yellow luminescence appeared. Only a very strong Na-D-line was observed then in the spectroscope. When gradually still more resistance was put in, the yellow luminescence and the Na-D-line became fainter and fainter, and the Cu-lines began to appear (in the green). At a definite terminal voltage green sparks were also immediately to be observed by the side of the yellow sparks at the anode.

It is exceedingly difficult to elucidate the nature of these very faintly luminous phenomena experimentally. It can only be stated that the luminescence appears to be caused by numerous sparks, and that there is undoubtedly a gas-envelope present also here, as already Voller and Walter pointed out. Very probably a similar mechanism is to be supposed here as in Lecoq de Boisbaudran's "fulgurator". In this apparatus we have a layer of gas and vapour between anode and electrolyte, through which the sparks penetrate.

§ 7. 2. A beautiful and very intense anodic arc-spectrum can be obtained with an iron rod in hot concentrated or diluted sulphuric acid (sp. gr. 1.80 and H_2SO_4 1 : 4); less easily by means of tungsten anodes in the same medium Then the temperature of the anodes is pretty high; the colour of the emitted light is a brilliant blue. The tension in these experiments was 225 Volts. The emission did not appear until the luminescence described under 3 had been observed for a shorter or longer time. We shall, therefore, have to return to the said phenomenon presently.

3. A very peculiar light phenomenon is observed when the current is closed by immersion of a carbon- or metal-anode in concentrated or diluted sulphuric acid. The carbon then gets covered by a beautiful yellow mantle of light, which continues to persist for a long time; the carbon surface gets smooth, carbon powder and superficial impurites are removed. Metal anodes present an analogous behaviour, as was by observed by v. Bolton (8), to whom we owe a method by this procedure for polishing and cleansing carbon electrodes. (14).

I have been able to corroborate the validity of this experimenter's results in every respect — both on use of concentrated and of diluted sulphuric acid. A digressing behaviour is shown only by typical valve metals (e.g. Al and Ta). These emit a white or bluish light.

For so far as could be ascertained, the spectrum of the yellow light is continuous; often the Na-D-line is still to be observed. After the experiment the electrodes surface is bright and smooth, but the electrode-diameter is mostly slightly diminished. The white light from valve metals is continuous, but on the boundary electrode-electrolyte-air sparks often appear then, which certainly emit line-spectra.

The terminal voltage during the yellow luminescence (in Cu, Fe, Mo, Wo, Ni, C) is about 100 Volts, the intensity of the current some tenths of an Amp., i.e. on use of wire-electrodes of a diameter of some mm., which were immersed 1—2 cm. deep. The temperature of concentrated sulphuric acid then rises very rapidly to the boiling-point, that of diluted sulphuric acid (smaller intensity of current) somewhat more slowly. When once the boiling-point temperature has been reached, the colour suddenly changes from goldish to brilliant blue; at the same time the current is reduced to less than 0,1 Amp., the terminal voltage rising to the total value available (225 Volts). Then the well known arc-spectrum of iron or tungsten is seen in the spectroscope. This experiment is very suitable for demonstration. Analogous phenomena can most probably also be obtained in other metals, though less easily.

§ 8. The appearance of an anodic arc of light particularly at hot anodes, is, indeed comprehensible; the yellow luminescence is, however, less easy to understand. A purely thermal emission of the metal cannot be supposed. Nor can there be any question of a reaction luminescence, since the light always possesses the same colour, no matter what anode material is used. Von Bolton suggested that the anode gets covered with "a yellow incandescent" oxygen mantle. In fact oxygen can be brought to an emission of a yellow continuous light by an electric current at higher pressure (13). At lower pressure a maximum of intensity in the green or yellow green occurs in this continuous spectrum. It may, therefore, be assumed as very probable that our electrodes are surrounded by a mantle of oxygen generated electrolytically, in which the gas is brought electrically to light emission under pretty high pressure. At higher temperatures the pressure in this oxygen layer must diminish, perhaps the layer must become quite unstable, and finally conditions are reached which give rise to a metal arc of light.

§ 9. 4. Anodic light emission has often been observed during electrolyses, when an insoluble or sparingly soluble reaction product is formed at the anode. This product can then form either a solid layer firmly attached to the anode, or a layer that gets more or less easily detached.

The former can often be observed in valve cells. Already below the limiting tension a dullish white light may be seen at the valve anode (10), which becomes pretty intense under definite circumstances, (e. g. with Al-anodes in borax solution, Ta in diluted alkali or carbonate solution). With this emission of light should also be classed the emission of light of magnesium anodes in diluted alkali (15).

In all these cases the potential rises to the maximum value available, the passage of the current is almost entirely prevented. The luminescence begins very soon after the closure of the current, often with periodic oscillations of the intensity during the first minutes, and then continues to persist till the current is broken. The light emission is, however, generally soon prevented, when electrode or electrolyte are heated by the weak current that continues to pass. In prolonged experiments it is, therefore, necessary to ensure good cooling.

The light, which is almost always a dullish white, sometimes more greenish or bluish, appeared to be continuous on spectroscopic investigation.

It is also noteworthy that with magnesium anodes the maximum of light intensity is reached in potassium hydroxide 1 : 100; a very strong luminescence is also obtained by using an ammoniac solution of di-sodium phosphate instead of the hydroxide. In this medium also zinc anodes produce an exceedingly beautiful light emission, a borax solution being the most suitable electrolyte with aluminium. But also with aluminium and with tantalum diluted alkali hydroxide solution etc. can be used.

In these processes the electrolyte is covered by an adhering layer of the oxide or of another insoluble anode product, as this was already shown by other experimenters. The generality of such phenomena is brought out by the fact that always new observations of the kind described are being communicated (cf. e.g. 1*a*).

But also when no direct valve actions are to be observed, such phenomena of light are nearly always found when at the anode a sparingly soluble product is found. To these belong, among others, the following phenomena of luminescence which have partly already been known for some time:

Electrolyte	Phenomenon
KJ aq, saturated	a bright luminescence at anodes of Cd, Hg, Pb.
H_2SO_4 conc.	" " " " " " Pb, Al, Ta; Mg gives a short flash; at Cd anodes there is seen a ring of light, which moves up and down.
KOH aq, strong	Fe (a bright luminescence, but which cannot very easily be examined on account of strong foaming), Ni (very slight intensity of the current).
Na_2HPO_4 NH_3aq	Cu gives a circle of sparks.

Exceedingly intense is the luminescence at an Hg-anode in saturated KI-solution at sufficient density of the current. The bright anode-surface is covered with a thin layer of mercury iodide immediately after the closure of the current, and then begins to emit a golden light. After a short time the intensity of this light reaches a maximum, and then diminishes again. By renewal of the mercury surface, either by stirring or by allowing the mercury to overflow from a funnel-shaped anode vessel, etc. the luminescence can be restored with full intensity.

In agreement with former experimenters (2) the spectrum of the emitted light was found to be continuous, with a maximum of the intensity in a definite spectral region. WILKINSON (2) has pointed out that the colour of this light also agrees with that of the light emitted by the anode product in question, when it is bombarded by cathode rays.

§ 10. It is exactly these kinds of luminescence that are very often considered as reaction luminescence (chemi-luminescence). Formation or decomposition of the anode products were thought to be accompanied by a luminescence which could reach a considerable intensity with sufficient reaction velocity [1]). BANCROFT (1) and his pupils, also WILKINSON (2) have endeavoured to give support to this view. In the course of our own observations on comparison with those of other investigators it appears, however, that this conception is untenable.

In the first place it can be established that all the phenomena described in this chapter, are related. And this not only because they appear to be of the same nature spectroscopically, but also because their occurrence always appears to be bound to the formation of sparingly soluble or unsoluble anode products.

[1]) On this conception compare (5).

Premising this, it may be inferred from a pretty great number of reasons that a conception of these luminescence phenomena as reaction luminescence phenomena must be considered as erroneous.

In the first place with this view of the matter it cannot be explained why only formation of unsoluble products gives rise to luminescence. It can, indeed, be predicted that the probability of anodic luminescence and its intensity will be the greater as the anode-product dissolves with the greater difficulty. For it appears in particular that on formation of readily soluble anode-products, luminescence never seems be observed.

Nor can the considerable increase of intensity of the luminescence at low temperature (hence at smaller reaction velocity) be accounted for on the ground of the said conception. For with valve anodes the luminescence is by no means most pronounced on particularly strong anode-reaction, but only when the excluding layer is as stable and homogeneous as possible, and is attacked as little as possible by the electrolyte. Thus magnesium emits the brightest light in *diluted* KOH, aluminium in borax solution, which would certainly be unaccountable in the case of real "chemi-luminescence". A magnesium anode is particularly strongly attacked by diluted sulphuric acid, though all the same, there is no luminescence at all to be observed.

Moreover it remains inexplicable how anodes which are rapidly covered by an insoluble layer, yet continue to emit light. It might much sooner be expected in this case that the light would cease after the formation of a covering layer. But this is by no means observed in the majority of the cases.

Finally the increase of light intensity after the closure of the current, as is particularly clearly observed with mercury anodes in KI-solution, is unaccountable in a reaction luminescence. For, how a certain quantity of reaction products would be able to increase a direct chemi-luminescence, is not clear. Nor can periodic and rhythmic light emissions (Cd in KI-solution, Mg immediately after the closure of the current) be accounted for in this way.

§ 11. The only conception which can be brought to harmonize with all the experimental facts, is in contrast with the conception discussed just now, in my opinion the following: at once after the closure of the current a layer of reaction products is formed at the anode, which hampers the passage of the ions to the anode, or renders it impossible. Then the electric discharge of these ions takes place (at sufficiently high potential) under the influence of split off

anionic electrons, which fly through the anode layer with strong acceleration. By this the matter in this layer is brought to luminescence in the same way, hence with emission of the same spectrum, as this would happen by means of cathode rays. If the layer becomes too thick, higher potentials will be required to bring about a passage of the current, and finally current could only pass in certain cases when the layer is traversed by sparks (limiting tension with anodes). When on the other hand the layer is attacked by the electrolyte in some way or other, it is very well possible that also the light emission at the anode can vary locally, and in particular the periodic oscillations of the intensity along the anode become possible. Increase of temperature will always hamper the luminescence, either because the solubility of the anode product is in general increased by it, or because the layer is rendered less stable by it in mechanic respect. If the anode layer has little mechanical stability in itself (e.g. mercury iodide), a certain minimum current density will be required to form a coherent layer with sufficient velocity, and to allow this to continue to exist, in spite of continued decomposition.

By this conception also the analogy between the anodic and cathodic luminescences is clearly brought out.

Summarizing we may say that also in these anodic luminescence phenomena, as this was earlier shown for ordinary chemi-luminescence (5), not the anode-*reaction* in itself takes place with light emission.

It must rather be admitted that first reaction products are formed which are brought to emission, in this case by means of the electric energy of a source of current outside the system examined [1]). Hence there is no question of an ion reaction, which takes place with light emission, and of a departure from the general rule that it is just these reactions, which proceed practically with infinite velocity, that are never accompanied by a light-emission.

The above considerations show further that VOLTA-luminescence occurs very frequently, but also that it can be of a very different character. On further investigation of these phenomena it will be necessary to distinguish these kinds of VOLTA-luminescence scrupulously. The present investigation may be considered as a first attempt at reconnoitring the ground in this respect.

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[1]) In cases of common chemi-luminescence the reaction itself furnishes the energy necessary to excite to light emission some of the kinds of molecules present in the system. (see 5).

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Physiology. — *„Is Caesium Radio-active?"* By Prof. H. ZWAARDEMAKER, W. E. RINGER and E. SMITS.

(Communicated at the meeting of June 30, 1923).

Up to the present potassium and rubidium are the only elements in the series Li, Na, K, Rb, Cs, which have been proved to be radio-active. It has often been suspected that caesium also possesses a slight radio-activity, but thus far this is not certainly known. E. RUTHERFORD [1]) simply remarks that caesium is barely radio-active and ST. MEYER and E. VON SCHWINDLER [2]) suggest that radio-activity may possibly exist, but the penetrating power of the rays emitted is so low that it does not reach beyond the limits of the substance. We know for certain that commercial preparations of caesium exert no photographic action, even in exposures for months. Neither could one of us [3]) detect in carefully purified caesium preparations any ionization of the air of a flat ionization-chamber.

It is a fact, however, that biologically caesium exerts in many cases an influence similar to that of potassium and rubidium. This influence was already known to SIDNEY RINGER [4]) and has, moreover, been purposely studied by one of us. [5]) After an unsuccessful effort in winter we succeeded in the summer of 1917 in keeping hearts of coldblooded animals beating on a dosis of caesium-chloride that only slightly differed from the usual potassium-dose. It appeared that potassium-, rubidium- and caesium-chloride could be used promiscuously, but that a much larger quantity of caesium had to be applied for a toxic effect. With regard to uranium, thorium, radium, and radium-emanation it behaved antagonistically, which was afterwards also confirmed by Miss L. KAISER [6]).

Here, then, a contrast manifested itself. Physically well-purified caesium-compounds are to be considered as non-radio-active, whereas

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[1]) E. RUTHERFORD in Marx's Hdb. der Radiol. Bd. II S. 531, 1913.

[2]) ST. MEYER und E. v. SCHWINDLER, Radioaktivität, 1916 S. 428.

[3]) W. E. RINGER, Arch. néerl. de Physiol. t. 7 p. 434, 1922.

[4]) S. RINGER, Journal of Physiol. Vol. 4 p. 370, 1883.

[5]) H. ZWAARDEMAKER en C. DE LIND VAN WIJNGAARDEN, K. Akad. v. Wetensch. 27 Oct. 1917, Proc. vol 20 p. 773.

[6]) L. KAISER. Arch. néerl. de Physiol. t. 3 p. 587, 1919.

biologically a well-proportioned dosis of caesium behaves like the radio-active elements potassium and rubidium.

To elucidate this we undertook an experiment with preparations of various origin. They were carefully purified and examined physically and biologically before and after the purification.

By a physical inquiry we tried to determine the ionization-power of the perfectly dry caesium-salt in a flat, air-tight ionization chamber. The salt had been spread evenly on a copper dish of 30 cm. diameter.

The dish was isolated with amber and charged to constant potential of 500 volts by a battery of small accumulators. $3\frac{1}{2}$ c.m. above the salt layer was a copper disc also of 30 cm. diameter, which was connected with a pair of quadrants of a sensitive electrometer. The "needle" of this electrometer was maintained at 40 volts.

A uranium-unit of Mc Coy of 50 square m.m. showed with this arrangement a deflection of 100 scale-divisions in about 2 minutes; a layer of dried potassium-chlorid in 5 minutes about 50 scale-divisions [1]).

Our caesium-chloride preparations yielded widely differing results, of which a survey is best obtained by a comparison with the ionization power of potassium ceteris paribus.

Activity	Impure	Pure
CsCl of E. de Haën	1/6 of the act. if potassium	1/37 of the act. of potassium
CsCl of Merck	1/10 „ „ „ „	inactive
CsCl of Kahlbaum	1/37 „ „ „ „	1/80 of the act. of potassium
CsCl of Poulenc fr.	1/37 „ „ „ „	1/77 „ „ „ „

The biological examination was carried out with an isolated frog's heart (ventricle + right auricle suspended to a Symes cannula with an overflow, so that the pressure could never exceed 5 cm. of water. Three Mariotte-flasks with a cock-system provided a means of per-

[1]) We see, then, that in this flat ionization chamber the ionization power of potassium (beta-radiator) is 7000 times weaker than that of uranium (alpha-radiator). This ratio will be quite different in a high ionization-chamber. Rutherford (p. 528) estimates the ionization power in the ordinary ionization-chamber at $1/1000$ of the beta-radioactivity of uranium. The beta-radioactivity of uranium in its turn rests on uranium X, with which the uranium of the ordinary preparations is in equilibrium.

fusing the heart alternately with Ringer-solutions of various composition. First we determined the minimum dosis of potassium-chloride that the individual heart required, after which it was perfused with a RINGER-solution, without potassium until it came to a standstill. After ten minutes, in which interval we ascertained that no latent automaticity existed, we proceeded to caesium perfusion. We determined in succession the minimum-, the optimum- and the maximum-doses. The dosage was gradually increased with the greatest care. By means of an air-injector, such as was used by LOCKE and ROSENHEIM, the same $^1/_4$ or 1 Liter of circulating fluid was sent round. The fluid that went through the heart was thus loaded with as much oxygen as is soluble in a weak salt-solution.

The dosis of potassium-chloride and of caesium-chloride that proved just sufficient to make the heart beat regularly, was considered as the minimum dosis; as optimum dosis we took the one which yielded the greatest frequency and maintained it. It was difficult to find the maximum-dosis, because an increase in the caesium dosage brings on an inconvenient negative inotropism.

We then considered as highest practicable dosis the one which produced lytic symptoms of cessation of contractility. Strictly the maximum dosis lies somewhat higher. Meanwhile the caesium has penetrated deep into the heart-cells, for it takes hours before a heart can be deprived of the profusion of caesium and before its action can be arrested by a RINGER-solution that contains neither potassium nor caesium.

A survey of our results can again be best obtained by an intercomparison of caesium and potassium.

Minimum-dosis of the impure preparations.

CsCl of DE HAËN	8.7 × KCl-dosis
CsCl of MERCK	5.3 × KCl-dosis
CsCl of KAHLBAUM	4.9 × KCl dosis
CsCl of POULENC fr.	4.1 × KCl-dosis

The minimum-, optimum-, and highest practicable doses are, for the impure preparations in milligrammes of caesium-chloride per Liter on an average in the ratios of:

1194 : 1538 : 1998.
min. opt. highest pract.

Of the purified doses the quanta must be much larger:

Minimum-doses of the purified preparations.

CsCl of DE HAËN	9.5 × KCl-dosis
CsCl of MERCK	19.4 × KCl-dosis
CsCl of KAHLBAUM	7.2 × KCl-dosis
CsCl of POULENC fr.	12.3 × KCl-dosis

The minimum-, optimum-, and highest practicable doses are for the purified preparations of caesium in milligrammes of caesium-chloride per Liter, on an average in the ratios of:

1678 : 2760 : 4134.
min. opt. highest pract.

When comparing impure and pure caesium we obtained the following mean results:

	Impure CsCl	Pure CsCl
in minimo	5.5 × min. KCl	10.7 × min. KCl
in optimo	7.1 × „ „	17.7 × „ „
highest pract.	9.2 × „ „	26.5 × „ „

So we see that in minimo the quantity of pure caesium must be from 2 to 3 times larger than that of impure caesium.

On the 16th of May 1923 it appeared that of the preparation, purified to such a degree that no radiation whatever could be demonstrated for a given heart, in minimo 39, in optimo 47 and as highest practicable dosis 58 grs. had to be added per Liter of circulating fluid. It is obvious that in such cases the quantities of NaCl had to be largely diminished in order to prevent hyper-isotonia.

One of us has set up for the radio-active substitutes of potassium a working-hypothesis, viz. that in general isolated organs require so much of a radio-active element in their physiological circulating liquid as is necessary to generate per second the emission of a number of ions that is equal for all substitutes. This hypothesis can be expressed in a logarithmic graph.

In such a graph rubidium and caesium cannot be taken up directly, since we do not know how many ions these substances emit per

second, so that no place can be assigned to them on the axis of the abscissae. When, however, the minimum-, and the maximum-

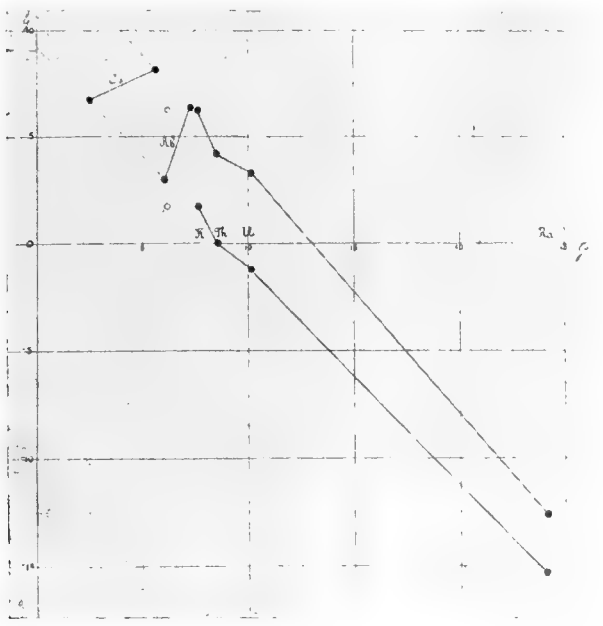


Fig. 1.

dosages are known, a place may be found for them on the axis of the ordinates, and when assuming the law to hold good also for these substances, their hypothetical place on the axis of the abscissae may be determined by erecting a perpendicular. We have plotted the graph accordingly and thus given a value for rubidium as well as for caesium. We know then the presumable number of ions that will be emitted under the given premisses per gram and per second. For our pure preparation of caesium it appears to be 55 per gram and per second. With such a small number of ions we can expect a photographic effect only after 9 years. It is easy to understand, therefore, that up to the present endeavours to produce any effect of caesium upon a sensitive plate, have not been successful.

The 55 ions per gram and per second that, according to the hypothesis of the corpuscular equivalence, should belong to pure caesium, cannot really belong to the caesium as such, but must be due to the impurity of the commercial preparation, which had been removed from the caesium in the following way:

Addition of copper sulphate; perfusion of sulphureted hydrogen for $^1/_2$ hour; after 24 hours removal of the precipitate of copper

sulphide by filtration; removal of the residue of sulphureted hydrogen by boiling the filtrate.

A second precipitate is generated by adding to the filtrate some drops of ferro-chloride solution and afterwards an excess of ammonia; this precipitate of iron hydroxyd is filtered off again after some hours; this process is repeated three times.

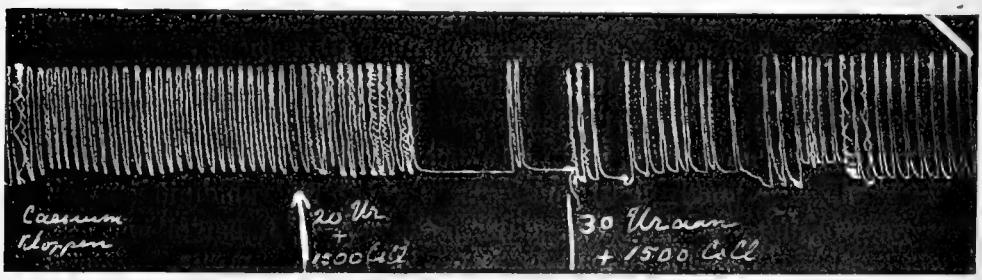
Lastly a third precipitate is generated by adding barium chloride at boiling heat; next day the precipitate is filtered off; this process is repeated under an excess of sulphuric acid, so that all the barium is precipitated; now the filtrate contains a small amount of caesium sulphate over and above all the original caesium chloride.

This procedure serves to remove a heavy radio-active element, which is left behind in the precipitate.

Originally the caesium-salts we used contained some of this impurity. If the dosis is high enough then there will be enough of the impurity to produce a biological action such as we may expect of a radio-active substance.

This biological action has the nature of a beta-radiator as is obvious from the antagonism of our caesium to uranium. Miss L. KAISER has recorded some instances of Cs-U-equilibria.

We annex a recent instance.



kloppen = beat Fig. 2.

A frog's heart beats initially on a Ringer solution, which contains per Liter instead of potassium 10 mgr. of uranyl nitrate. By adding to this solution a quantity of 1500 mgrms of Caesium-chloride a radio-physiological equilibrium is engendered between the alpha-radiator uranium and the beta-radiator caesium. A standstill corresponds with this equilibrium in which there is not even latent automaticity. However, directly when we increase the quantity of caesium, a caesium beat is developed. Another equilibrium will then again be called forth by increasing the quantity of uranium, which now is on a higher level, because more has been taken of the two

components. Finally a larger amount of uranium restores the heart's beat.

Considering that besides radio-active, also non-radio-active paradoxes occur, (Noyons, Busquet) no conclusive value can be ascertained in the easily generated, transient standstills when passing from a uranium liquid to a caesium liquid and vice versa. It is different with the equilibria, which can only be interpreted radio-physiologically. This is most evident with the higher equilibria in which each of the components, co-exist in the mixture in quantities that undoubtedly surpass the threshold-concentration.

Our inquiry, then, is to the following effect:

1°. the impurity that imparts to the commercial preparation of caesium a feeble radiating power, is presumably a heavy radio-active element.

2°. the biological action of the impurity has the nature of a beta-radiator.

Hydrodynamics. — "*On the resistance experienced by a fluid in turbulent motion*". By J. M. BURGERS. (Communicated by Prof. P. EHRENFEST).

(Communicated at the meeting of May 26, 1923).

§ 1. *Introductory remarks.*

The problem which is discussed in the following lines is to search for a method to calculate the resistance experienced by a fluid in turbulent motion. A definite solution has not been arrived at; a first attempt only is given.

As is generally known, in most cases the motion of a fluid through a straight cylindrical tube or channel is not in parallel lines with a constant velocity along each line. On the contrary it is usually very irregular: the velocity of a particle changes its value and its direction continually, and particles situated very near to each other have very different velocities, whereas there seems to be no definite law governing these deviations. This type of motion is called *sinuous* or *turbulent*, as distinguished from the *streamline* or *laminar* motion, which occurs at low velocities only. In studying turbulent flow the conception of the *mean motion* or *principal motion* has been introduced by various authors. This mean motion is obtained if in every point of the space occupied by the fluid the mean value of the true velocity with respect to time is determined, and then the steady motion is imagined the velocities of which are equal to these mean values. The true motion may be described as the resultant of the mean motion and of a fluctuating *relative motion*. The mean velocity of the latter is zero [1]).

A turbulent flow usually experiences a high resistance, which is approximately proportional to the second power of the velocity of the mean motion. If the law of resistance is written:

$$\text{loss of pressure per unit of length } J = C\,\frac{\varrho V^2}{d},$$

[1]) In connection with the distinction between mean motion and relative motion the reader is referred to: H. A. LORENTZ, Turbulente Flüssigkeitsbewegung und Strömung durch Röhren, Abhandl. über theoretische Physik I (1907), p. 58—60.

in which formula V represents the mean velocity (i.e. the volume of fluid which in unit of time flows through a section of the tube, divided by the area of that section), d the diameter of the tube, and ϱ the density of the fluid, then C is called the coefficient of the resistance, and appears to be a function of the characteristic number introduced by REYNOLDS: $R = \frac{Vd\varrho}{\mu}$ (μ is the coefficient of viscosity of the fluid). The value of C for different cases is given in textbooks; as an example may be mentioned:

a. for rough walled tubes C is approximately independent of R; however, it is a function of the roughness;

b. for very smooth tubes of circular diameter:

$$C = 0{,}1582\ R^{-\frac{1}{4}}\ ^{1}).$$

The greater part of the theoretical investigations on the turbulent motion treat the problem: how does it originate? [2]) An explanation of the increase of resistance which accompanies the appearance of the turbulent state of flow has been given by REYNOLDS and LORENTZ [3]). More than once it has been remarked that this problem is one of statistical nature [4]). The resistance experienced by the fluid and indicated by our measuring apparatus is a mean value. It is possible that such a mean value may be calculated sufficiently approximate without an exact knowledge of the fluctuating and never exactly returning relative motions.

In the following lines a preliminary attempt is made to determine the value of the resistance and to explain the quadratic law. In the first part (paragraphs 2 and 3) two equations given by REYNOLDS and LORENTZ are discussed and put into such a form that immediately appears what quantities are wanted in order to calculate the resistance. In the second part (paragraphs 4 and 5) a simple idealized "model" of the turbulent flow is constructed which allows these quantities to be determined.

Instead of the flow through a tube or channel a more simple

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[1]) Comp. f i. R. VON MISES, Elemente der technischen Hydromechanik I (1914) p. 57 and H. BLASIUS, Mitt. über Forschungsarbeiten, herausgeg. vom V. D. I., Heft 131 (1913).

[2]) Cf. F. NOETHER, ZS. für angew. Math. u. Mechanik **1**, p. 125, 1921.

[3]) O. REYNOLDS, Scientific Papers II, p. 575—577;
H. A. LORENTZ, l.c. p. 66—71.

[4]) Among others by TH. VON KÁRMÁN at a lecture at the "Versammlung der Mathematiker und Physiker" in Jena 1921; comp. a remark in the ZS. für angew. Math. u. Mechanik **1**, p. 250, 1921.

type has been chosen: the motion of a fluid between two parallel walls, one of which has a translational motion in its own plane with the velocity V with respect to the other, while the distance between the two walls has the constant value l (comp. fig. 1). To ensure this motion forces of magnitude S per unit of area must be applied

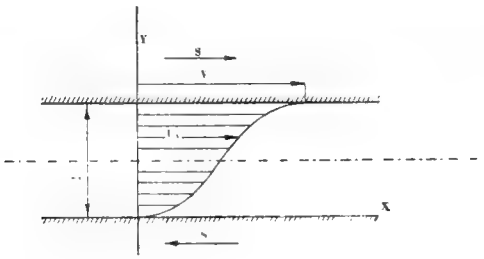

Fig. 1.

to the walls in opposite directions. The tangential force between any two adjacent layers of the fluid has the same value S. The law of resistance will be written:

$$S = C \varrho V^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The coefficient C is a function of REYNOLDS' number:

$$R = \frac{V l \varrho}{\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For small values of R the motion is laminar, and the value of C is easily seen to be:

$$C = \frac{1}{R} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the value of R is high, the motion becomes turbulent, and C decreases much slower. There do not exist any direct measurements for this case of motion; however, the arrangement of the experiments made by COUETTE comes very near to it [1]). According to this author we may expect a formula of the following type:

$$C = c_1 + c_2 R^{-1} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

Investigations by VON KÁRMÁN on the law of decrease of the mean motion in the neighbourhood of a smooth wall [2]) point to:

$$C = 0.008 R^{-1/4} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

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[1]) M. COUETTE, Ann. de Chim. et de Phys. (6) **21**, p. 457, 1890.

[2]) TH. VON KÁRMÁN, ZS. für angew. Math. u. Mechanik, l.c.

In order to simplify the mathematical treatment it has been assumed that the motion is confined to a plane.

Finally in paragraph 7 some results are given for the flow between two fixed parallel walls.

§ 2. *The principal equation.*

In the following lines the mean or principal motion of the fluid will be denoted by U. It is a function of the variable y only; at the wall $y = 0$ it is equal to 0, at the wall $y = l$ it takes the value V. The components of the velocity of the relative motion are written u and v; the vorticity of the relative motion is written:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These latter quantities are functions of the variables x, y and t. The velocities u and v are subjected to the boundary conditions:

$$u = 0, v = 0 \text{ for } y = 0 \text{ and for } y = l \quad . \quad . \quad . \quad . \quad (6)$$

and to the equation of continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now both Reynolds and Lorentz have shown that the peculiar character of turbulent motion is caused by the action of an apparent frictional force, influencing the principal motion, and due to the existence of the relative motions. This is expressed by the formula:

$$\mu \frac{dU}{dy} - \varrho \overline{uv} = S \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The bar over uv indicates that the mean value of this quantity is meant, taken at a certain point during a certain lapse of time, or taken at a certain moment along a line parallel to the axis of x. This mean value is a function of the variable y only (the same remark applies to $\overline{\zeta^2}$ in formula (9)). The quantity $\overline{uv}$ is *negative*, and $S > \mu \frac{dU}{dy}$.

The relative motions, however, are not independent of the mean motion. In order that the relative motions may always retain the same energy, it is necessary that the following equation is fulfilled:

$$-\int_0^l dy \, \varrho \, \overline{uv} \frac{dU}{dy} = \int_0^l dy \, \mu \, \overline{\zeta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The equations (8) and (9) are substantially the same as the formulae (36) and (46) from LORENTZ' paper l.c. above, only simplified according to the conditions of the problem before us.

Now firstly $\frac{dU}{dy}$ will be eliminated from eq. (9) by the aid of (8):

$$-S\int_0^l dy\,\varrho\,\overline{uv} = \int_0^l dy\,\{\varrho^2\,(\overline{uv})^2 + \mu^2\,\overline{\zeta^2}\}. \quad . \quad . \quad . \quad . \quad (10)$$

Secondly by integrating (8):

$$\mu V = Sl + \int_0^l dy\,\varrho\,\overline{uv} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This equation allows the elimination of S from (10):

$$\frac{\mu V}{l} = \left.\frac{\int_0^l dy\,\{\varrho^2\,(\overline{uv})^2 + \mu^2\,\overline{\zeta^2}\} - \frac{1}{l}\left(\int_0^l dy\,\varrho\,\overline{uv}\right)^2}{-\int_0^l dy\,\varrho\,\overline{uv}}\right\} \quad . \quad . \quad (12)$$

In order to simplify the equations we may introduce undimensioned variables by means of the formulae:

$$x = l\,x',\ y = ly';\ u = Vu',\ v = Vv';\ \zeta = \frac{V}{l}\zeta' \quad . \quad . \quad (13)$$

If now in the following equations the accents are omitted again, we obtain:

$$\frac{1}{R} = \left.\frac{\int_0^1 dy\,(\overline{uv})^2 - \left(\int_0^1 dy\,\overline{uv}\right)^2}{-\int_0^1 dy\,\overline{uv}} + \frac{1}{R^2}\,\frac{\int_0^1 dy\,\overline{\zeta^2}}{-\int_0^1 dy\,\overline{uv}}\right\} \quad . \quad . \quad (14)$$

and by the same substitutions, from (11):

$$\frac{S}{\varrho V^2} = -\int_0^1 dy\,\overline{uv} + \frac{1}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The equations take a very simple form if the following abbreviations are used:

$$\left.\begin{aligned} -\int_0^1 dy\,\overline{uv} &= \sigma \\ \int_0^1 dy\,(\overline{uv})^2 &= (1+\tau)\,\sigma^2 \\ \int_0^1 dy\,\overline{\xi^2} &= \varkappa\sigma \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

It will be easily recognized that the three quantities σ, τ and $\varkappa$ are all of them essentially *positive*.

The equations (14) and (15) now reduce to:

$$\underline{\sigma\tau + \frac{\varkappa}{R^2} = \frac{1}{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and:

$$\frac{S}{\varrho V^2} = C = \sigma + \frac{1}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Formula (17) will be denoted as the *principal equation*.

§ 3. *Discussion of the principal equation.*

Equation (17) shows first of all that an increase of the velocity V of the mean motion cannot be accompanied by a proportional change of the relative motion: in this case σ, τ and $\varkappa$ would remain constants, whereas R increases, which would violate equation (17).

If the value of R is given, (17) gives a condition to be fulfilled by the relative motion. If a certain type of relative motion, fulfilling this condition, accompanies the mean motion, the latter will experience a resistance determined by the value of C, calculated from (18). Now the problem arises: can we find admissible values of the quantities τ and $\varkappa$, without an exact knowledge of the true relative motion? If τ and $\varkappa$ are known, (17) gives σ (i.e. in some measure the relative intensity of the relative motions), and (18) gives the resistance coefficient. If we look at the application of statistical methods in the dynamical theory of gases, we should expect that for high values of R (which mean a fully developed state of turbulence), it may be possible to calculate τ and $\varkappa$ in the following manner: firstly we determine all kinds of relative motions which fulfil eqq. (6) and (7); secondly we admit that all these motions may be present independently of each other, their weights being governed

by some law of probability, or by a maximum- or minimum-condition. Then the mean values are calculated for this assembly.

Prof. von Kármán from Aix-la-Chapelle pointed out to me that before trying to find a condition governing the weight of the different types of motions, it would be advisable at first to search for the *maximum value* of S, or of σ. In this way a higher limit for the resistance of turbulent flow would be found.

That a maximum value exists may be shown thus:

From (17) it is deduced that σ may become great (i.e. especially: great as compared to $\frac{1}{R}$) only if $\varkappa < R$ and if τ becomes small. The value of τ is determined by the distribution of the values of $\overline{uv}$ over the interval $0 < y < 1$. Only if $\overline{uv}$ assumes a constant value throughout this interval, τ can attain its minimum value 0. However, $\overline{uv}$ cannot be a constant everywhere, as u and v decrease to 0 in the neighbourhood of the walls. Hence we will obtain the smallest possible value of τ if $\overline{uv}$ has a constant value throughout the whole region with the exception of two very thin layers along the walls, in which layers $\overline{uv}$ decreases to zero. If the thickness of these "boundary" layers is represented by ε, τ will be of the same order of magnitude as ε, hence with a numerical constant c_1:

$$\tau = c_1 \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

In the boundary layers $\frac{\partial u}{\partial y}$ and ζ will be of the order of magnitude ε^{-1}, and so $\overline{\zeta^2}$ will be proportional to ε^{-2}. Hence if this intensive vorticity occurs in the boundary layers only:

$$\varkappa = c_2 \varepsilon^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Now equation (17) gives:

$$\sigma = \frac{1}{c_1 \varepsilon R} - \frac{c_2}{c_1 \varepsilon^2 R^2}$$

This expression attains a maximum value if:

$$\varepsilon = \frac{2 c_2}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The thickness of the boundary layer appears to be inversely proportional to R. The value of σ becomes:

$$\sigma_{max} = \frac{1}{4 c_1 c_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

It appears that σ takes a value which is independent of R;

according to (18) C approximates to the same constant value, and thus according to (1) the *quadratic law* of resistance is obtained.

This reasoning is in many respects vague, and it does not admit of a determination of the values of c_1 and c_2. It only shows that the particles of fluid with high values of the vorticity $|\zeta|$ must be concentrated along the walls. To get a more definite result it is necessary to develop a picture of the structure of the turbulent motion. Two ways may be followed: we may try to analyze the possible motions into a sum of elementary functions (goniometrical or others) in a manner analogous to a series of FOURIER; or we may imagine the motion to be built up from an assembly of individual vortices (vortex filaments with their axes perpendicular to the plane of x-y), distributed in some way or other throughout the fluid. In the calculation of the critical value of R (i.e. the value at which the turbulence occurs for the first time) analogous methods have been used: REYNOLDS, ORR and other writers have directed their attention to disturbances which are propagated in a periodic way through the whole fluid; LORENTZ at the other hand has studied the disturbance caused by a single vortex [1]).

The statistical treatment of such an assembly of elementary motions is very difficult on account of the circumstance that every elementary motion is damped by the action of the viscosity. At the other side the mutual actions between the elementary motions (brought forth by the quadratic terms in the equations of hydrodynamics) and the influence of the mean motion continually generate new motions. From the formula given by LORENTZ it follows that types of motion for which $\iint dx\,dy\,uv$ is negative, are intensified by the action of the mean motion. Hence a mean stationary state can exist, in which every elementary motion changes continually its intensity and its phase (or its position, if it is an individual vortex), but in which every one of these motions has a constant mean intensity. It is obvious that for the greater part, if not exclusive, these will be types of motion for which $\iint dx\,dy\,uv < 0$.

The statistical problem will not be attacked here. On the contrary a simple type of turbulent motion will be studied in the following paragraphs, built up from an assembly of elliptic vortices, all of them having the same configuration, but having different dimensions.

[1]) O. REYNOLDS, l.c. p. 570;
W. Mc. F. ORR, Proc. Roy. Irish Acad. 27, p. 124—128, 1907;
H. A. LORENTZ, l.c. p. 48.

If they are distributed over the fluid in a certain way, with an appropriate distribution of intensities, it will appear that it is possible to make τ very small, without making the value of $\varkappa$ surpass that of R. It further appears that in the choice of the dimensions of the vortices an element remains arbitrarious, which element may be adjusted in such a way that σ takes a maximum value.

§ 4. LORENTZ' *elliptic vortex.*

It has been shown by LORENTZ that we can obtain a simple type of motion which obeys the conditions (6) and (7), and for which $\iint dx\,dy\,uv < 0$, by considering a vortex in which the particles of the fluid describe elliptic paths [1]). Geometrically this motion can be deduced from that in a circular vortex by a lateral compression. In the circular vortex the fluid moves in concentric orbits with the angular velocity ω, which is a function of the radius r of the orbit. At the outer boundary of the vortex ω has the value zero, whereas in its centre ω and $\frac{d\omega}{dr}$ have finite values. LORENTZ takes for ω a BESSEL function of r; in order to obtain simpler formulae in this paper an algebraic function will be taken.

The construction of the elliptic vortex is shown in figure 2. The

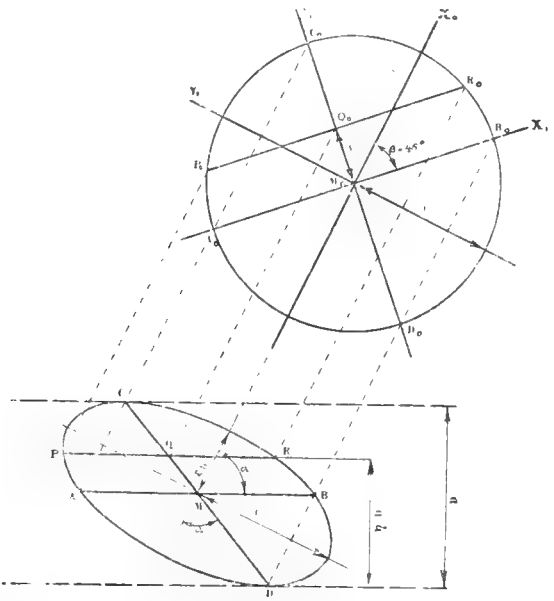

Fig. 2.

[1]) H. A. LORENTZ, l.c. p. 48—52.

axes of the ellipse have the lengths $2b$ and $2\varepsilon b$, in which expression ε has the value $^1/_8(\sqrt{15}-\sqrt{6}) = 0{,}475$; the smaller one makes the angle $arctg\,\frac{1}{\varepsilon} = \alpha$ with the direction of the mean motion. The conjugated diameters $A\,B$ and $C\,D$ correspond to the diameters of the circle $A_0\,B_0$ and $C_0\,D_0$, which make angles of 45° with the directions of the axes of the ellipse. Besides the system of coordinates $x_0\,y_0$ used by LORENTZ, the system $x_1\,y_1$ along $M_0\,B_0$ and $M_0\,C_0$ will be introduced.

From the formulae given by LORENTZ at page 49 we deduce the following expression for the value of uv in a point of the vortex, corresponding to the point $x_0\,y_0$ of the circle:

$$M_0 = -uv = \frac{1}{2}(x_0^2 - \varepsilon^2 y_0^2)\,\omega^2 \sin 2\alpha + \varepsilon x_0 y_0\,\omega^2 \cos 2\alpha = \left.\begin{aligned} &= \frac{\varepsilon}{1+\varepsilon^2}\,\omega^2\{x_1^2(1-\varepsilon^2) + x_1 y_1 (1+\varepsilon^2)\} \end{aligned}\right| \quad (23)$$

For the determination of the mean value of uv along a line parallel to the axis of x, it is necessary to calculate the integral of M_0 along a line $P\,R$ which is parallel to the same axis. This line corresponds to the line $P_0\,R_0$ of the circle; the lengths of these lines are in the constant proportion:

$$\frac{AB}{A_0B_0} = \frac{1}{\sqrt{2}\sin\alpha} = \sqrt{\frac{1+\varepsilon^2}{2}}$$

Hence this integral takes the value:

$$M_1 = \int_{-\sqrt{b^2-y_1^2}}^{+\sqrt{b^2-y_1^2}} dx_1 \frac{\varepsilon\omega^2}{\sqrt{2}\,(1+\varepsilon^2)}\{x_1^2(1-\varepsilon^2) + x_1 y_1 (1+\varepsilon^2)\}. \quad . \quad (24)$$

As has been mentioned already above, ω is a function of $r_0 = \sqrt{x_0^2 + y_0^2} = \sqrt{x_1^2 + y_1^2}$; this function will be taken to be:

$$\omega = c\,(b^2 - r_0^2)^{5/4} = c\,(b_2 - x_1^2 - y_1^2)^{5/4}\ ^1) \quad . \quad . \quad . \quad . \quad (25)$$

The second term of the integral vanishes on account of the symmetry of ω; the first term gives:

$$M_1 = \frac{\sqrt{2}\,\varepsilon(1-\varepsilon^2)}{\sqrt{1+\varepsilon^2}}\,c^2 \int_0^{\sqrt{b^2-y_1^2}} dx_1\, x_1^2\,(b^2 - x_1^2 - y_1^2)^{5/2}$$

---

[1]) In the formulae below everywhere c^2 occurs; the sign of c is of no importance.

or, using the substitution:

$$x_1 = \sqrt{b^2 - y_1^2}\, \sin\chi,$$

$$M_1 = \frac{\sqrt{2}\,\varepsilon\,(1-\varepsilon^2)}{\sqrt{1+\varepsilon^2}}\, c^2\,(b^2-y_1^2)^4 \int_0^{\pi/2} d\chi\, \sin^2\chi\, \cos^6\chi = \left. \right. \quad (26)$$

$$= \frac{5\pi}{256}\, \frac{\sqrt{2}\,\varepsilon\,(1-\varepsilon^2)}{\sqrt{1+\varepsilon^2}}\, c^2\,(b^2-y_1^2)^4 \left. \right.$$

Formula (25) was chosen with a view of obtaining this latter result for M_1, which facilitates the further calculations. If a new variable η is introduced, determined by the formula:

$$\eta = \frac{y_1 + b}{2\,b}$$

(it appears from this formula that η has the value 0 on the tangent at the ellipse at the point D, and takes the value 1 on the tangent at C), then equation (26) can be written:

$$M_1 = A\eta^4\,(1-\eta)^4 = A\varphi\,(\eta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Here A is a factor independent of the variable η.

If we imagine a great number of these vortices to be present, all of them having the same dimensions and lying between the same tangents parallel to the axis of x, (comp. fig. 3), the amount contributed by them to the value of $\overline{uv}$ will be proportional to the function represented by (27) [1]).

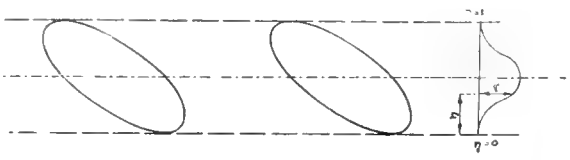

Fig. 3.

The integral of the quantity M_0 taken over the entire area of the vortex amounts to:

$$M_2 = \frac{2\pi\,\varepsilon^2(1-\varepsilon^2)}{63\ \ 1+\varepsilon^2}\, c^2\, b^6 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

[1]) Other types of motion may lead to the same form of the function determining M_1; for instance we may take the motion defined by the current function

$$\Psi = \eta^2\,(1-\eta^2)\,(e^{1-\eta}\cos\alpha x - e^{\eta}\sin\alpha x)$$

for values of η between 0 and 1, so that the components of the velocity have the values:

$$u = -\,\partial\,\Psi/\partial x,\ v = \partial\,\Psi/\partial y .$$

The integral of the square of the vorticity $N_2 = \iint dxdy\, \zeta^2$ extended over the same area becomes according to the formula given by LORENTZ:

$$N_2 = \frac{\pi}{4\varepsilon}(3 + 2\varepsilon^2 + 3\varepsilon^4)\int_0^b dr_0\, r_0^3 \left(\frac{d\omega}{dr_0}\right)^2 = \frac{5\pi}{42}\frac{3 + 2\varepsilon^2 + 3\varepsilon^4}{\varepsilon} c^2 b^7 \quad . \quad (29)$$

From (28) and (29) we deduce:

$$\frac{N_2}{M_2} = \frac{15}{4}\frac{(3 + 2\varepsilon^2 + 3\varepsilon^4)(1+\varepsilon^2)}{\varepsilon^2(1-\varepsilon^2)}\frac{1}{b^2},$$

or, introducing the "thickness" D of the vortex (cf. fig. 2), so that:

$$b = D\frac{\sqrt{2(1+\varepsilon^2)}}{4\varepsilon},$$

we get:

$$\frac{N_2}{M_2} = 30\,\frac{3 + 2\varepsilon^2 + 3\varepsilon^4}{\varepsilon(1-\varepsilon^2)}\frac{1}{D^2} = \frac{294}{D^2} \quad . \quad . \quad . \quad . \quad (30)$$

This fraction surpasses only by a small amount its minimum value, calculated by LORENTZ:

$$14{,}68\,\frac{2(3+2\varepsilon^2+3\varepsilon^4)}{\varepsilon(1-\varepsilon^2)}\frac{1}{D^2} = \frac{288}{D^2}\ ^1)$$

§ 5. *Distribution of the vortices over the fluid.*

It has already been remarked in § 1 and 3 that our object in this paragraph is not to analyse the true distribution of the vorticity of the fluid, but that we will construct an ideal case only, a "model", which affords us an admissible image of the behaviour of the quantities $\overline{uv}$ and $\overline{\zeta^2}$. This model is obtained by distributing a number of elliptic vortices, of the type studied in the foregoing paragraph, over the mean current $U(y)$. In doing this we do not want to pay any attention to the abscissae of the centra of the vortices, if only their mean distribution along lines parallel to the axis of x be uniform. Positively and negatively rotating vortices are distributed uniformly through each other. If two or more vortices may happen to overlap, they may as well strengthen as enfeeble their respective fields; hence in calculating the mean values $\overline{uv}$ and $\overline{\zeta^2}$ it is unnecessary to take account of these overlappings, and the contributions of the different vortices may be simply summed.

If for a moment we direct our attention to a special class of

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1) Comp. a remark made by LORENTZ, l.c. p. 54/55. The function defined by eq. (25) above fulfils the condition: $d\omega/ds = 0$ for $s = 1$ $(s = r_0/b)$.

vortices, the thickness D of which lies between the limits D and $D + dD$, and the lower tangents of which (i.e. the tangent at the point D in fig. 2) are enclosed between the limits $y = \xi$ and $y = \xi + d\xi$, then we may say that all of them are lying between the same lines parallel to the axis of x, and by what has been remarked above all of them will give proportional contributions to the field of $\overline{uv}$-values. As the integral $-\int dx\, uv$ extended over a section PR of a single vortex has been calculated in (26) and (27), we may write the contribution of the whole class:

$$b\,(D,\,\xi)\;\eta_1^{\,4}\,(1-\eta)^4\;dD\;d\xi = b\varphi\,(\eta_1)\;dD\;d\xi.$$

In this expression: $\eta_1 = (y-\xi)/D$, and the factor $b\,(D,\xi)\,dD\,d\xi$ represents the product of the number of these vortices contained in a strip of unit length parallel to the axis of x, their mean intensity (i.e. the mean of c^2), and the other factors which are contained in the letter A of formula (27). If the function $b\,(D,\,\xi)$ is given, the distribution of $\overline{uv}$ can be calculated.

It is not necessary to know the value of the quantity $\overline{\zeta^2}$ at every point of the current, its integral only over the whole breadth being wanted, which integral can be found as the sum of the integrals of ζ^2 over all vortices contained in a strip of the full breadth, and of unit length. With the aid of formula (30) we find as the contribution of the considered class of vortices:

$$\left.\begin{aligned}\iint dx\,dy\;\zeta^2 &= -\frac{294}{D^2}\iint dx\;dy\;uv = \\ &= \frac{294}{D^2}\,b\,dD\,d\xi\int_{\xi}^{\xi+D} dy\;\varphi\left(\frac{y-\xi}{D}\right) = \frac{294}{630}\,\frac{b\,dD\,d\xi}{D}\end{aligned}\right\} \quad . \quad . \quad . \quad (31)$$

A simplification further arises from the fact that the second and third equations (16) which determine τ and $\varkappa$ are homogeneous as regards to the intensity of the vortices. In using these equations it is allowed to multiply b with an arbitrary factor. The true value of σ is found from the principal equation (17). It would be possible to calculate the true value of b afterwards, but this is of no use.

The problem put in paragraph 3: to make σ as great as possible, obliges us to search for a function $b\,(D,\,\xi)$ which gives a value of $-\overline{uv}$ as nearly constant as possible. Two rather simple types of functions will be discussed.

I. We will begin with an investigation of what can be reached if all vortices have the same thickness D. In that case in order to

obtain a constant value of $-\overline{uv}$, it is necessary to make b independent of ξ, in other words to distribute the vortices uniformly over the breadth of the current. However, it is obvious that the vortices cannot pass through the walls of the channel; hence we must take:

$$\left.\begin{array}{l} b = constans, \text{ if } 0 < \xi < 1 - D \\ b = 0 \quad\quad\ , \text{ if } \xi < 0 \text{ or } \xi > 1 - D \end{array}\right\} \quad . \quad . \quad . \quad (32)$$

Consequently the quantity $-\overline{uv}$ will have a constant value in the region defined by: $D < y < 1 - D$ only; in the two remaining strips it decreases to zero.

With the omission of a constant factor, the following expressions for $-\overline{uv}$ are found:

a) if $y < D$:

b) if $D < y < 1 - D$:

$$\left.\begin{array}{l} -\overline{uv} = \int\limits_0^y d\xi\, \varphi\left(\frac{y-\xi}{D}\right) = D \int\limits_0^{y/D} d\eta\, \varphi(\eta) = \\ = \frac{D}{630}\left\{126\left(\frac{y}{D}\right)^5 - 420\left(\frac{y}{D}\right)^6 + 540\left(\frac{y}{D}\right)^7 - 315\left(\frac{y}{D}\right)^8 + 70\left(\frac{y}{D}\right)^9\right\} \\ -\overline{uv} = \int\limits_{y-D}^{y} d\xi\, \varphi\left(\frac{y-\xi}{D}\right) = D \int\limits_0^1 d\eta\, \varphi(\eta) = \frac{D}{630} \end{array}\right\} \quad (33)$$

c) if $y > 1 - D$: in the expression given under *a*) y has to be replaced by $1 - y$.

By means of these formulae we find:

$$-\int\limits_0^1 dy\, \overline{uv} = \frac{D}{630}(1 - D)$$

$$\int\limits_0^1 dy\, (\overline{uv})^2 = \left(\frac{D}{630}\right)^2 (1 - 1{,}172\, D).$$

hence:

$$\tau = 0{,}828\, D + \ldots . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

All vortices being of the same dimensions, equation (30) gives immediately:

$$\varkappa = \frac{294}{D^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Inserting these values into equation (17):

$$\sigma = \frac{1}{0{,}828\, DR} - \frac{294}{0{,}828\, D^3 R^2} - \ldots . \quad . \quad . \quad . \quad . \quad (36)$$

(if the terms of the highest order only are written down). This formula gives a maximum value for σ if the thickness D of the vortices is determined by:

$$D = \frac{29{,}7}{\sqrt{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

which gives:

$$\sigma = \frac{0{,}027}{\sqrt{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The coefficient C of the resistance formula (1) now becomes, according to (18):

$$C = \frac{S}{\varrho V^2} = \frac{0{,}027}{\sqrt{R}} + \text{terms of the order } \frac{1}{R} \quad . \quad . \quad (39)$$

C diminishes proportionally to $\frac{1}{\sqrt{R}}$; hence we do not obtain the quadratic law of resistance, but the resistance appears to be proportional to the $1\frac{1}{2}$-power of the velocity. This does not conform to the result of paragraph 3. In the latter paragraph, however, it was assumed that the most intensive vorticity was concentrated in the neighbourhood of the walls only, whereas in the model considered above it is distributed uniformly over the whole breadth. If all vortices have the same dimensions, it is not possible to distribute them otherwise, without disturbing the field of $\overline{uv}$-values. Hence we must try to obtain a better result by using vortices of different dimensions.

II. If we take vortices of different dimensions, say with thicknesses ranging from $D = 1$ to a lower limit D_0 (to be determined later on), the thickness of the boundary layers in the most favourable case will be of the same order of magnitude as D_0. The same applies to the quantity τ. If now the contribution of the vortices of thickness D to the integral $\int \overline{\zeta^2}\, dy$ becomes asymptotically proportional to $\frac{dD}{D^2}$ for small values of D, the value of this integral will become of the order of magnitude of $\frac{1}{D_0}$. In this case we shall be in the circumstances considered in the deduction of equations (19) and (20). Paying attention to equation (31), it is necessary that $B = \int b\, d\xi$ shall be proportional to $\frac{1}{D}$ for small values of D.

Now it appears that a distribution of vortices fulfilling these

conditions can be found, if all vortices are put against the walls. If this be done, it is of course unnecessary to use the variable ξ introduced in the beginning of this paragraph, as the positions of all vortices are fixed. Only a determination of the function $B(D)$ is wanted. The following form of this function gives the right distribution of $\overline{uv}$-values:

1. the class of vortices whose thicknesses lie between the limits D and $D+dD$ have a total intensity proportional to $B\,dD = 2\frac{dD}{D}$; these vortices are divided into two equal groups, each of them situated along one of the walls;

2. besides the vortices mentioned under 1), there is a number of vortices of thickness $D=1$, which have the total intensity $^1/_4$ (in same unit as used above).

With this determination of $B(D)$, the value of $-\overline{uv}$ appears to be, if $D_0 < y < 1-D_0$:

$$\left.\begin{aligned} -\overline{uv} &= \int_{y}^{1}\frac{dD}{D}\varphi\left(\frac{y}{D}\right)+\int_{1-y}^{1}\frac{dD}{D}\varphi\left(\frac{1-y}{D}\right)+\frac{1}{4}\varphi(y)= \\ &= \int_{y}^{1}\frac{d\eta}{\eta}\varphi(\eta)+\int_{1-y}^{1}\frac{d\eta}{\eta}\varphi(\eta)+\frac{1}{4}\varphi(y)= \\ &= \frac{1}{280} \end{aligned}\right\} \quad (40)$$

The first term represents the contribution of the vortices lying along the wall $y=0$; of these vortices only those are of importance for which $D>y$. The second term represents the contribution of the vortices situated at the other side; here only those for which $D>1-y$ are of importance. The third term represents the contribution of the group of vortices whose thickness D is equal to 1 [1]).

[1]) If we should take the quantity B proportional to D^{-n}, with $n<1$, the integral $\int\overline{\zeta^2}\,dy$ would take a smaller value, but now the first term of equation (40) which gives the contribution of the vortices situated against the wall $y=0$, would become:

$$\int_{y}^{1}\frac{dD}{D^n}\varphi\left(\frac{y}{D}\right)=y^{1-n}\int_{y}^{1}d\eta\,\eta^{2+n}(1-\eta)^4 \quad (\text{for } y>D_0)$$

If y becomes small, this expression approaches to zero. Only if $n=1$ it approaches to a value independent of y, which is necessary in order that a constant value of $-\overline{uv}$ at all points outside of the boundary layer may be obtained.

In the boundary layer defined by $0 < y < D_0$, the value of $-\overline{uv}$ is found to be:

$$\left.\begin{aligned} -\overline{uv} &= \int_{D_0}^{1} \frac{dD}{D}\varphi\left(\frac{y}{D}\right) + \int_{1-y}^{1} \frac{dD}{D}\varphi\left(\frac{1-y}{D}\right) + \frac{1}{4}\varphi(y) = \\ &= \frac{1}{280} - \int_{y}^{D_0} \frac{dD}{D}\varphi\left(\frac{y}{D}\right) = \\ &= \frac{1}{280}\left\{70\left(\frac{y}{D_0}\right)^4 - 224\left(\frac{y}{D_0}\right)^5 + 280\left(\frac{y}{D_0}\right)^6 - 160\left(\frac{y}{D_0}\right)^7 + 35\left(\frac{y}{D_0}\right)^8\right\} \end{aligned}\right\} \quad (41)$$

Using the formulae (40) and (41) we find:

$$-\int_0^1 dy\,\overline{uv} = \frac{1}{280}(1 - 0{,}889\,D_0)$$

$$\int_0^1 dy\,(\overline{uv})^2 = \left(\frac{1}{280}\right)^2 (1 - 1{,}068\,D_0)$$

and by means of the latter there results:

$$\tau = 0{,}710\,D_0 - \ldots\ldots\ldots\ldots \quad (42)$$

The value of $\varkappa$ can be calculated in the following way: The vortices having thicknesses between the limits D and $D + dD$ contribute to the integral $-\int dy\,\overline{uv}$ the amount:

$$2\frac{dD}{D}\int_0^D dy\,\varphi\left(\frac{y}{D}\right) = \frac{dD}{315};$$

hence, according to (30), to the integral $\int dy\,\overline{\zeta^2}$:

$$\frac{294}{315}\frac{dD}{D^2}$$

To this must be added the contribution of the vortices with thickness 1, amounting to:

in $\quad -\int dy\,\overline{uv} \;:\; \frac{1}{2520};$

hence in $\quad \int dy\,\overline{\zeta^2} \;:\; \frac{294}{2520}.$

Adding all parts together, we get:

$$\int dy\,\overline{\zeta^2} = \frac{294}{315}\left(\frac{1}{D_0} - 1\right) + \frac{294}{2520} = \frac{294}{315}\left(\frac{1}{D_0} - \frac{7}{8}\right).$$

Finally the value of $\varkappa$ becomes:

$$\varkappa = \frac{261}{D_0} + \ldots\ldots \quad (43)$$

The values given by (42) and (43) are inserted into the principal equation (17); retaining the terms of the highest order only, we find:

$$\sigma = \frac{1}{0{,}710\, D_0\, R} - \frac{261}{0{,}710\, D_0^{\,3}\, R^2} - \ldots\ldots \quad (44)$$

σ attains its maximum value if the lower limit D_0 of the thickness of the vortices is determined by:

$$D_0 = \frac{522}{R} \quad (45)$$

This is much below the value of D given by equation (37). Using (45) we find:

$$\sigma = 0{,}00135 + \ldots\ldots \quad (46)$$

and the coefficient of the resistance formula becomes:

$$C = \frac{S}{\varrho V^2} = 0{,}00135 + \text{terms of the order } \frac{1}{R} \quad (47)$$

So this arrangement of the vortices leads to the *quadratic* law of resistance.

§ 6. *Discussion.*

In paragraph 5 II we have found the value 0,00135, as a higher limit of the coefficient C of the resistance formula using an idealized model of the distribution of the vorticity in a turbulent current.

If it is possible to calculate C without the use of this special model, using equations (17) and (18) and conditions (6) and (7) only, a still higher limit will probably be found. At the other side if we compare the value of C obtained here to the value given by formula (4*b*), it appears that in the region which is of importance: $R = 10000$ to 1000000, the value of C is *too high.*[1])

Hence we may assert that the true resistance is not the highest possible resistance. In order to determine the true state of affairs, a further condition will be necessary.

From the result that the value of C appears to be too high, we may deduce that the distribution of the value of $-\overline{uv}$ over the current is too uniform. Paying attention to the results of measurements of the distribution of the velocity over the breadth of the

[1]) According to COUETTE's experiments turbulence sets in at $R =$ ca. 1900.

current, we may expect that $-\overline{uv}$ has not a constant value between the boundary layers, but that it is slightly "rounded off". This might be ascribed to slight irregular displacements of the vortices caused by the irregularly distributed velocities which they impart to each other. This "Brownian" movement might give a distribution of the smaller vortices resembling the one determined by the law of BOLTZMANN-MAXWELL for a gas under the influence of gravity, which possibility has been pointed out by VON KÁRMÁN in the lecture mentioned above.

The true distribution of vorticity in the turbulent motion will take some mean position between the two extremes of paragraph 5 (uniform distribution over the whole breadth with C proportional to

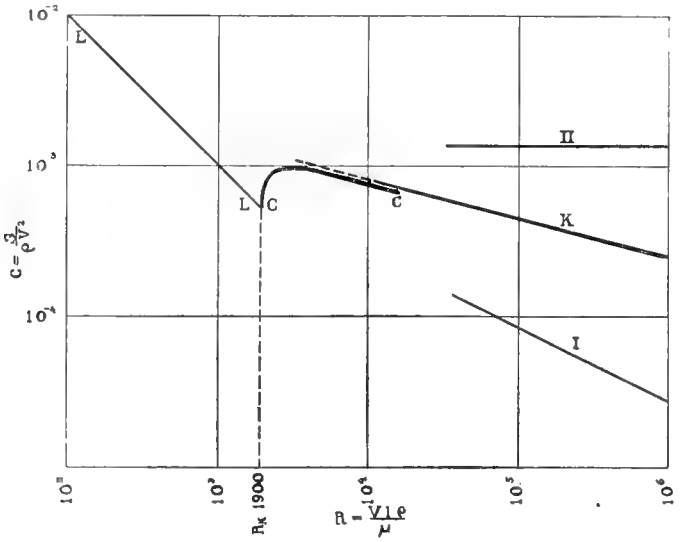

Fig. 4. Logarithmic-scale diagram of the dependence of C on R.

Curve L: laminar region, $C = \frac{1}{R}$ (form. 3).

Curve C: results of COUETTE's experiments (the value of R has been calculated using $\mu = 0{,}01096$, comp. COUETTE, l. c. p. 460).

Curve K: $C = 0{,}008\ R^{-1/4}$ (form. 4*b*), deduced from the investigations by VON KÁRMÁN on the behaviour of $U(y)$.

Curve I: formula (39), deduced from the supposition that all vortices have the same dimensions, and are uniformly distributed over the section.

Curve II: formula (47), deduced from the supposition that the vortices have different dimensions, and are lying against the walls.

$\frac{1}{\sqrt{\overline{R}}}$, or the best ordered arrangement with all vortices along the walls and C equal to a (high) constant value).

For the sake of comparison the formulae (39), (47) and (4*b*) have been represented together in fig. 4 at a logarithmic scale.

§ 7. *Motion of a fluid between two fixed parallel walls.*

The motion of a fluid between two fixed parallel walls may be treated according to the same scheme as has been used for the motion between a fixed and a moving wall. As the former case has somewhat more resemblance to the types of motion occurring usually in practical cases, the principal features of the calculation will be mentioned here.

The distance of the walls will be taken equal to h; the mean velocity of the current is denoted by V; the pressure gradient $-dp/dx$ will be denoted by J. — REYNOLDS' characteristic number becomes: $R = Vh\varrho/\mu$; the coefficient of the resistance formula is written $C = Jh/\varrho V^2$. Equation (8) of paragraph 2 has to be replaced by the following equation governing the principal motion:

$$\mu \frac{d^2U}{dy^2} - \frac{d}{dy}(\varrho \overline{uv}) = -J \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

A first integration of this formula gives:

$$\mu \frac{dU}{dy} - \varrho \overline{uv} = J\left(\frac{h}{2} - y\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

The constant of the integration is determined by observing that on account of the symmetry of the arrangement both quantities dU/dy and $\overline{uv}$ vanish for $y = h/2$. On integrating a second and a third time, and observing that $U = 0$ at both walls, we get:

$$\mu V h = \frac{1}{12} J h^3 - \int_0^h dy\, \varrho y \overline{uv} \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

This equation replaces formula (11). Condition (9) which expresses the dependance of the relative motion on the principal motion, retains its form. Now firstly, using (49), we eliminate dU/dy from (9); then using (50), we eliminate J and we obtain:

$$\frac{\mu V}{h} = \frac{\frac{1}{12}\int_0^h dy \left\{\varrho^2 (\overline{uv})^2 + \mu^2 \overline{\zeta^2}\right\} - \frac{1}{h^2}\left(\int_0^h dy\, \varrho y \overline{uv}\right)^2}{\frac{1}{h}\int_0^h dy\, \varrho y \overline{uv}} \quad . \quad . \quad (51)$$

After the introduction of undimensioned variables, we make use of the abbreviations:

$$\left.\begin{aligned} &\int_0^1 dy\, y\, \overline{uv} = \sigma \\ &\frac{1}{12}\int_0^1 dy\, (\overline{uv})^2 = (1+\tau)\,\sigma^2 \\ &\frac{1}{12}\int_0^1 dy\, \bar{\zeta}^2 = \varkappa\sigma \end{aligned}\right\} \quad \ldots\ldots\ (52)$$

The equations (50) and (51) now reduce to:

$$\sigma\tau + \frac{\varkappa}{R^2} = \frac{1}{R} \quad \ldots\ldots\ (53)$$

$$\frac{1}{12}\frac{Jh}{\varrho V^2} = \frac{C}{12} = \sigma + \frac{1}{R}. \quad \ldots\ldots\ (54)$$

Distribution of the vortices over the fluid.

As appears from equation (49) the value of $\mu \dfrac{dU}{dy}$ will be small compared to that of $J\left(\dfrac{h}{2} - y\right)$ (as is the case for the real motion) only if $-\varrho\overline{uv}$ becomes approximately equal to $J\left(\dfrac{h}{2} - y\right)$. Or, using the undimensioned variables introduced above, we may say that $-\overline{uv}$ aught to be proportional to $\frac{1}{2} - y$.

Hence the quantity $\overline{uv}$ must take a negative value in the neighbourhood of the wall $y = 0$, and it must take a positive value at the other wall. This can be obtained if we use two groups of vortices whose positions are symmetrical with respect to each other. In the first place a group of elliptic vortices having the same position as those described in paragraphs 4 and 5 (i.e. with the long axis extended from the second to the fourth quadrant) is put in against the wall $y = 0$. The contribution of these vortices to the field of values of $\overline{uv}$ will be denoted by

$$-(\overline{uv})_{\mathrm{I}} = \psi(y).$$

Then a second group is put in, situated symmetrically against the other wall: the contribution of the latter to $\overline{uv}$ will be:

$$-(\overline{uv})_{\mathrm{II}} = -\psi(1-y).$$

The contributions of both groups to the integral $\int dy\,\overline{\zeta^2}$ are of course equal and of equal signs.

If we now take vortices having thicknesses ranging from 1 to a minimum value D_0, and we take their intensities proportional to:

$$B\,d\,D = \left(\frac{1}{D} - \frac{3}{4}\right) dD \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

(this expression has a positive value for all values of D), then we obtain for values of y lying between D_0 and $1 - D_0$ the following expression of $\psi(y)$ (with the omission of a constant factor):

$$\psi(y) - \int_y^1 dD \left(\frac{1}{D} - \frac{3}{4}\right) \varphi\left(\frac{y}{D}\right) =$$

$$= \frac{1}{140}\left\{\frac{1}{2} - y + 7\,y^5 - 14\,y^6 + 10\,y^7 - \frac{5}{2}\,y^8\right\}$$

from which follows:

$$\psi(y) - \psi(1-y) = \frac{1}{140}\left(\frac{1}{2} - y\right) \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Hence between the boundary layers the values of $\overline{uv}$ are correctly distributed.

Within each boundary layer $\overline{uv}$ decreases from 1,280 to zero. The full expression of the value of $\overline{uv}$ having been worked out, we obtain the integrals:

$$\int_0^1 dy\, y\, \overline{uv} = \frac{1}{1680}\left(1 - 2{,}667\, D_0 + \ldots\right)$$

$$\frac{1}{12}\int_0^1 dy\, (\overline{uv})^2 = \left(\frac{1}{1680}\right)^2 (1 - 3{,}204\, D_0 + \ldots)$$

from which:

$$\tau = 2{,}129\, D_0 - \text{terms of the order } D_0^2 \ldots \quad . \quad . \quad . \quad . \quad (57)$$

The value of the integral $\int_0^1 dy\,\overline{\zeta^2}$ becomes:

$$2\int_{D_0}^1 dD\, \frac{294}{630}\,\frac{1}{D}\left(\frac{1}{D} - \frac{3}{4}\right) = \frac{294}{315}\left(\frac{1}{D_0} - \frac{3}{4}\,lg\,\frac{1}{D_0} - \ldots\right).$$

This gives:

$$\varkappa = \frac{131}{D_0}\left(1 - \frac{3D_0}{4}\, lg\, \frac{1}{D_0} + \text{terms of the order } D_0 \ldots\right) \quad . \quad . \quad (58)$$

The results of (57) and (58) are substituted into equation (53), and the maximum value of σ is determined. This maximum occurs if:

$$D_0 = \frac{262}{R}\left(1 - \frac{98}{R}\, lg\, \frac{R}{262} \ldots\right)$$

Finally equation (54) gives:

$$C = 0{,}0108 + \frac{2{,}11}{R}\, lg\, R + \text{terms of the order } \frac{1}{R}\ ^{1}) \quad . \quad . \quad . \quad . \quad (59)$$

Discussion.

In this case too the *quadratic* law of resistance is asymptotically arrived at (for values of R surpassing 100000 the logarithmic term is little more than 2% of the constant term). Just like what occurred in the more simple case the value of the coefficient C is *too high*. For channels with smooth walls VON MISES gives that C ranges from 0,006 to 0,0024 if R ranges from 10000 to the greatest values obtained; the formula derived by VON KÁRMÁN's theory gives:

$$C = \text{ca. } 0{,}07\ R^{-1/4}$$

For channels with rough walls the dependance of the coefficient C on the value of R is usually very small, so that a quadratic resistance formula can be used, the value of C depending, however, on the dimensions of the irregularities of the walls as compared to the diameter of the channel. The value of C is much higher than in the case of smooth walls; it may even surpass that given by (59). So GIBSON mentions values ranging to 0,015 for old cast iron tubes or channels, lightly tuberculated [2]).

Laboratorium voor Aero- en Hydrodynamica der T. H.
Delft, May 1923.

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[1]) The constant term of C in this formula has a value of 8 times that of formula (47) An elementary but superficial comparison of the magnitude of the frictional forces exerted on the walls in both cases leads to the same result.

[2]) R. VON MISES, l.c. p. 63, in connection with the definition of r, given at p. 83/84. In the case of a channel of infinite depth as the one treated here, r is equal to h.

A. H. GIBSON, Hydraulics and its applications (1919), p. 209 (in the formula mentioned at p. 206 is m is ½ time the quantity r introduced by VON MISES; comp. GIBSON, l.c. p. 194).

Comp. also L. SCHILLER, ZS. für angew. Math. u. Mechanik, **3**, p. 2, 1923. and others.

Paléontologie. — *„Sur quelques nouveaux insectes des lignites oligocènes (Aquitanien) de Rott, Siebengebirge (Rhénanie).* Par Fernand Meunier.

(Présenté par Mr. le Prof. K. Martin dans la séance du 29 septembre 1923).

Cette contribution à la faune de Rott, fait suite à des travaux antérieurs, commencés en 1894 et dont la bibliographie complète est donnée ici.

Ces nouvelles espèces ont été rencontrées dans les gisements rhénans par M. H. Bauckhorn. Il s'agit d'abord d'un coléoptère qui semble voisin de Otiorhynchus induratus Heyd. mais dont les yeux, au lieu d'être allongés, sont arrondis et de Varus ignotus Schlechtendal (Brachymycterus curculionoides Heyd.) Il paraît avoir des traits de ressemblance avec Phytonomus firmus Heer des couches sannoisiennes de Provence (France). Si la morphologie de la forme de Rott était moins frustement indiquée, on pourrait la comparer avec Laccopygus nilesii Scudder du miocène de Florissant (Etats-Unis), avec Geralophus saxonus Scudd. qui présente une striation très voisine. Par la présence des articles des antennes (le I^{er} article est malheureusement altéré par la fossilisation), je range ce nouveau fossile de Rott dans le genre Laccopygus Scudder. De nouveaux documents s'imposent avant de préciser les diagnoses de Otiorhynchus induratus Heyd. Brachymycterus curculionoides Heyd. et de Varus ignotus Schlechtendal. Tout porte à croire que la nouvelle espèce rhénane, à antennes si curieuses, est à maintenir dans le genre Laccopygus Scudd. Disons encore, que d'autres espèces de Rott, établies par v. Heyden, devraient être redécrites d'après des fossiles en meilleur état de conservation. Il y aurait aussi lieu de donner de nouveaux dessins au trait des principaux organes de ces coléoptères.

Cette note contient aussi de courtes remarques, relatives à deux hyménoptères terebrantia. Un très minuscle, mais très gracieux Proctotrypide, Archaebelyta superba Meun. ♂. La ♀ a été décrite dans Miscellanea entomologica (t. XXVI, p. 82 pl. I fig. 3, 1922) Dans la famille des Ichneumonidae, du groupe des Pimplinae, il est question d'une nouvelle espèce de Pimpla. Parmi les diptères, mentionnons encore la présence d'un Tipulidae Polyneura, se plaçant avec certitude dans le genre Limnophila Maquart. Cette notice

contient encore des remarques et des reproductions phototypiques [1]) de plusieurs espèces intéressantes notamment d'un coléoptère Nitidulidae du genre Nitidula Fabr. ensuite, un hyménoptère Chalostagastra ou Tenthredinidae se groupant parmi les vrais Tenthredo. Dans le monde des Aculéates, il a été observé un Formicidae, de grande taille, se rangeant parmi des Ponera Latr. (P. elegantissima Meunier). Il est aussi signalé un petit diptère, qui appartient vraisemblablement au genre Phora Linné. M. Bauckhorn a aussi trouvé à Rott un Tipulidae du genre Erioptera Meig. (E. oligocaenica Meun.).

Description des espèces [1])

I. Coleoptera.

I. Curculionidae.

Genre Laccopygus Scudd.

Laccopygus rhenanus Meun.

Long. du corps 5 mm., larg. 3 mm. Par la morphologie du corps, cette espèce est voisine de L. nilesii Scudd. du miocène de Florissant (Etats-Unis). Tête robuste, rostre court, yeux paraissant arrondis; antennes composées de 7 articles plus longs que larges (cylindriques) et terminés par un bouton apical, paraissant formé de 3 divisions; le bouton apical est ovoide, très distinct. Thorax plus large que long la fossilisation empêche de décider s'il était ponctué ou orné d'une ponctuation rugueuse. Elytres, recouvrant entièrement les segments de l'abdomen qui sont très distincts. Pattes robustes.

Fig. 1. Antenne de Laccopygus rhenanus.

II. Nitidulidae.

Genre Nitidula Fabre.

Aucun coléoptère de cette famille n'a encore été rencontré sur les feuillets de Rott. On a signalé quelques formes des couches d'Oeningen et de Radoboy et S. H. Scudder a décrit Nitidula prior des couches miocèniques de Florissant.

Nitidula robusta n.sp.

Long. du corps 6 mm., largeur 2 mm.

L'insecte trouvé à Rott, est malheureusement couché sur le dos, ce qui empêche d'étudier les caractères des élytres et le dessus du thorax. Corps ovale, trapu. Tête robuste, un peu proéminente vers

[1]) A cause des frais considérables d'impression, les planches qui accompagnent ce travail n'ont pu être données actuellement.

le clypeus. Antennes atteignant la moitié de la longueur du thorax et paraissant être composées de 11 articles, courts et saillants, dont les 2 ou 3 derniers constituent une sorte de bouton apical. Thorax sinueux, à la partie antérieure, arrondi aux angles antérieurs; il est trapéziforme, bien dévéloppé. Les segments de l'abdomen, très distincts, arqués; le dernier segment ou pygidium accuminé. Pattes très robustes, fémurs bien dévéloppés; la fossilisation ne permet pas de décrire la morphologie des articles tarsaux. Disons encore que chez cette espèce les articles des antennes sont plus larges que longs et serrés les uns contre les autres. (Chez le seul specimen observé de Rott). L'espèce de Florissant, Nitidula prior Scudd. a plusieurs traits de ressemblance avec celle trouvée par M. Bauckhorn sur les schistes rhénans. Le genre Saronia a de l'affinité, on le sait, avec le genre Nitidula Fabr. De nouveaux documents d'études s'imposent avant de donner la diagnose complète de cette espèce.

II. Hymenoptera.

Terebrantia.

Chalastogastra ou Tenthredinidae.

Des couches de Rott, on connait 2 mouches à scie de cette famille: Pinicolites graciosus Meun. et Tenthredo fasciata Meun. Des plaquettes d'Aix, en Provence, j'ai décrit Hylotomites robusta Meun. D'autres Chalastogastra ont été signalés des couches tertiaires de Florissant par T. D. A. Cockerell. Citons notamment Tenthredella oblita, Palaeotaxonus vetus et Eriocampoides minus. [1])

Genre Tenthredo Linné.

Tenthredo fasciata n.sp.

A Rott, on a observé un Chalastogastra qui se reconnait, a première vue, par la présence de bandes transversales ornant la partie postérieure des segments de l'abdomen; ce dernier organe longuement ovoide. Les parties médiane et latérale du thorax garnies de bandes longitudinales. Tarière bien dévéloppée et offrant la morphologie générale des espèces du genre Tenthredo Linné. Pattes robustes. Deux cellules radiales aux ailes antérieures et 4 cellules cubitales, dont la 2ième et la 3ième reçoivent chacune une nervure recurrente. Cellule anale des ailes postérieures non appendiculée; à cette dernière paire d'ailes, il y a 2 cellules discoïdales fermées. La tête de cette espèce n'est malheureusement pas conservée sur le schiste. La longeur du corps (présumée) de cet hyménoptère, y compris la tête, devait être environ de 13 millimètres.

[1]) Proc. U. S. Nat. Mus. vol. 53, pp. 389—390 Washington 1917.

Empreinte et contre empreinte. Coll. H. Bauckhorn de Siegburg.

Observation: Chez les Perineura Hartig, la cellule anale des ailes postérieures est appendiculée.

Proctotrypidae.

Archaebelyta superba Meunier.

(Miscellanea Entomologica t. XXV p. 84 pl. 1 fig. 3, Toulouse 1922).

Ce sexe est plus grêle et plus élancé que la ♀. Les antennes ont des articles de moindre diamètre ce qui donne à leur morphologie générale un aspect plus régulier, de plus, l'extrémité des antennes n'est guère épaissie (chez la ♀, le bout antennaire l'est distinctement). La veination des ailes est pareille à celle de la ♀; les pattes, un peu moins robustes, ne présentent aucun caractère particulier.

Coll. Bauckhorn, *Siegburg.*

Observation: C'est la première fois qu'un hyménoptère, de si petite taille, a été trouvé sur les schistes européens. En son intéressant mémoire, "The parasitic Hymenoptera of the tertiary of Florissant (Colorado) Cambridge 1910", Charles Brues a figuré et donné les diagnoses d'espèces dont la préservation est loin d'être aussi complète que Archaebelyta superba Meun. ♂ et ♀. Pour finir, disons encore que l'ambre de la Baltique et le copal de diverses provenances africaines, sont riches en inclusions de Proctotrypidae. Cette étude à peine esquissée, par Menge attend encore la venue d'un monographe. Autrefois (Ann. de la Soc. scient. de Bruxelles 1901), j'ai signalé la riche faunule que contient l'ambre et le copal en fait de Mymaridae ou „atomes ailés".

Ichneumonidae.

Pimplinae.

On a rarement signalé des Pimplinae des couches fossiles de Rott, toutefois Osw. Heer a décrit un Acoenites des feuillets de Radoboy et Brues les a signalés du miocène de Florissant. Je viens de donner la description d'une nouvelle espèce de l'Aquitanien de Rott „Acoenites Statzi" (Miscellanea Entomologica t. XXVI, p. 85 pl. 1 fig. 4. Toulouse 1922 (23). M. Bauckhorn m'a communiqué un Pimplinae dont malheureusement la tête, le thorax, l'abdomen et les pattes sont trop frustement indiqués pour en donner une minutieuse diagnose, et établir les rapports probables de ce fossile aquitanien avec les Ephialtes Gravenhorst. Toutefois, la conservation des deux paires d'ailes est si parfaite, qu'il y a lieu, dès à présent de le nommer. Je propose de l'appeler Pimpla Morleyi en honneur du distingué

Ichneumonologue M. Morley du Musée de Suffolk, (Angleterre). Chez ce Pimpla, la cellule aréolaire au lieu d'être triangulaire et pétiolée comme c'est souvent le cas chez diverses espèces de Pimpla, n'est pas entièrement pentagonale comme on le remarque chez les espèces du sous-genre Delomerista Foerster. De plus, la 5ième nervure des ailes antérieures (nervus basalis) se raccorde entièrement avec la 6ième nervure ou nervulus de manière à produire une ligne concave. Longueur du corps 5mm.? Longueur de l'aile antérieure 5mm. Largeur de l'aile antérieure $2^1/_2$ mm.

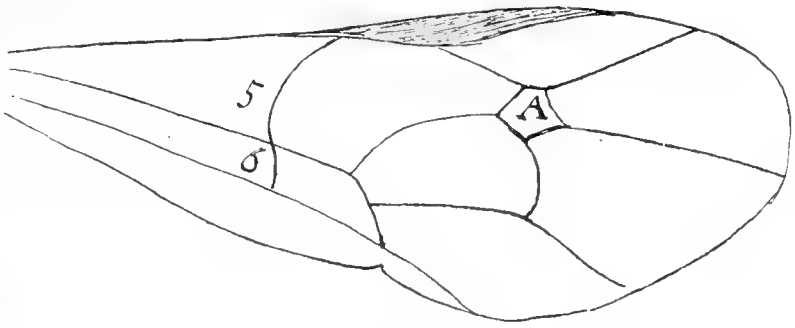

Fig. 2. Aile de Pimple Morleyi n. sp.

Aculeata.

Formicidae.

Von Heyden a signalé à Rott la présence du genre Formica. Naguère, j'ai donné les diagnoses des espèces se classant dans les genres Myrmica, Tapinoma et Formica. La nouvelle espèce, décrite ici, diffère à première vue, par la taille de Ponera rhenana.

Ponera elegantissima n. sp.

Cette espèce se sépare des formes suivantes mentionnées par Osw. Heer: Ponera fuliginosa, oeningensis et radoboyana, P. affinis P. croatica, P. longaeva, P. nitida, P. grassinervis, P. elongatula, P. ventrosa et P. globosa. Par la veination des ailes, elle est voisine de Ponera fuliginosa et oeningensis, par sa grande taille, elle se sépare immédiatement de Ponera rhenana Meun. des couches aquitaniennes de Rott.

Tête robuste ovale mandibulus trapues, paraissant arrondies à leur extrémité. Pétiole de l'abdomen très appréciable. Abdomen formé de 4 segments et nettement ovoïde. Ailes antérieures à nervure transverso-radiale en connection directe avec la cellule limitant la 2ieme cellule cubitale. Chez P. rhenana, ces 2 transversales sont assez éloignées l'une de l'autre[1]).

[1]) Meunier, F., Verhandelingen der K. Akademie van Wetenschappen, tweede Sectie, Deel XX, N. 1, fig. 6, Amsterdam 1917.

Diptera.

Tipulidae.

Eriopterinae.

Les Tipulidae, si fréquents sur les schistes miocèniques de Florissant, sont rares sur les couches des lignites des Sept-Monts (Rhénanie). On sait que l'ambre contient une curieuse faunule de ces intéressants diptères[1]). En 1917[2]), j'ai signalé une aile de vrai Tipula de ce gisement oligocène et fait de courtes remarques relatives aux espèces décrites naguère par v. Heyden. Les diptères de ce groupe n'ont pas encore été signalé dans le copal subfossile de Zanzibar. Les couches d'Aix en Provence (France) ont fournies quelques beaux spécimens de Tipularinae[3]).

On connait actuellement 4 espèces de Tipulidae des schistes tertiaires du Rhin: Cladoneura robusta Meun. Cyttaromyella bastini Meun. et l'espèce décrite ci-dessous. Il faut encore y ajouter une nouvelle espèce de Limnophila Macq, admirable de conservation, dont la diagnose suit.

Erioptera oligocaenica n. sp.

Tête globulaire, robuste. Antennes à articles de la base de plus fort diamètre que ceux de l'extrêmité. Abdomen allongé. Ailes plus longues que le corps; nervure transversale radiale (R)et radio-médial bien distincte. La transversale médio-cubitale part de M_2 et non de la médiane. Secteur du radius (préfourche d'après Osten-Sacken) nettement concave, ce secteur comprend 5 nervures ($R_1 + R_5$). La médiane est longement fourchue ($M_1 + M_2$). Le cubitus (Cu) est simple, il en est de même de la première et de la 2ième nervure anale. Pas de cellulu discoidale.

Longueur du corps: 5 mm. longueur de l'aile 6 mm.

Observation: Par sa forme concave, le secteur du radius rappelle celui des Tipulidae Limnophilinae.

Limnophilinae.

Genre Limnophila Macquart.

Limnophila rhenana n.sp.

C'est la première fois, qu'une espèce de Limnophila a été signalée à Rott. Tête globulaire, petite; antennes à articles subcylindriques,

[1]) Löw H., Ueber den Bernstein und die Bernsteinfauna Meseritz 1850. und Meunier F., Monographie des Tipulidae de l'ambre de la Baltique. Ann. d. Sciences Nat. Paris 1908.

[2]) Verhandelingen d. K. Akademie van Wetenschappen p. 15 (du tiré à part.), Amsterdam 1917.

[3]) Bull. de la Soc. géol de Francet. XIV p. 196, Paris 1914.

assez grêles et paraissant être plus courtes que la longueur du thorax ce dernier est gibbeux, et parait avoir été orné de bandes ou facies longitudinales, les ailes de parfaite conservation permettent de donner les détails topographiques de la veination. Le radius est relié à la première nervure du secteur par une petite transversale (radius cross-vein), il y a 4 nervules qui sortent du secteur du radius; la médiane est simple, toutefois son secteur comprend 3 cellules, dont la première (M) n'est autre chose que la cellule discoidale des anciens auteurs. En définitive, de la dite cellule discoidale partent 3 nervures simples (chez la plupart des espèces de Limnophila la première de ces nervures est fourchue). La cellule discoidale est reliée par une transversale (radius-mediane cross vein) au cubitus qui est déjà fourchu à peu de distance de la base de l'aile. La champ anal comprend 2 nervures simples.

Longueur du corps 8 mm. Longueur de l'aile 6 mm. Largeur de l'aile 2 mm.

L'abdomen assez allongé est composé de segments très distincts, malheureusement l'oviducte de la ♀ est peu appréciable.

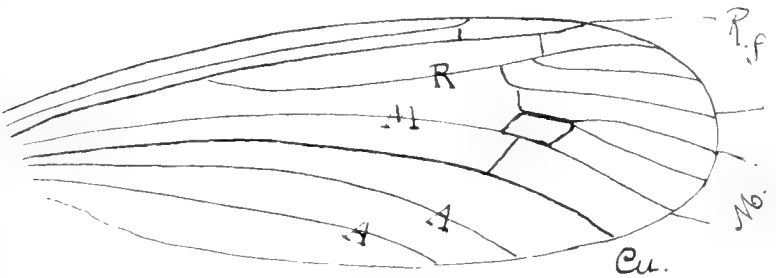

Fig. 3. Aile de Limnophila rhenana n. sp.

Phoridae.

Les diptères de cette famille, bien représentés, dans le succin de la Baltique[1]) dans les schistes miocèniques de Florissant et dans le copal subfossile de Zanzibar n'ont pas encore été signalés des feuillets ligniteux de Rott. Le genre Phora n'a pas encore été remarqué dans l'ambre sicilien.

Genre Phora, Meigen.

Phora sp?

La fossilisation empêche de donner un nom spécifique à ce diptère. On constate toutefois que le 3ième article des antennes est disciforme

[1]) Löw., H. Ueber den Bernstein und die Bernsteinfauna Meseritz 1850. Meunier, F. Monographie d. Leptiden u. Phoriden des Bernsteins Jahrb. d. k.k. preuss. geol. Landesanstalt Berlin 1909.

et qu'il paraît cilié et que le chète est aminci à l'extrémité. Les pattes sont robustes, leurs caractères sont trop noircis pour décider, si ce diptère appartient au genre Aphiochaeta Brues.

Longueur du corps 2 mm.

Coll. Bauckhorn, Siegburg.

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1920. Meunier F., Quelques insectes de l'Aquitanien de Rott Sept-Monts (Prusse rhénane). K. Akademie v. Wetenschappen te Amsterdam p. 727—737. 2 planches Vol. XXII.

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Physics. — "*Magnetic Researches.* XXII. *On the determination of the magnetisation at very low temperatures and on the susceptibility of gadolinium sulphate in the region of temperatures obtainable with liquid hydrogen*". By H. R. WOLTJER. (Communication N°. 167*b* from the Physical Laboratory at Leiden). (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated at the meeting of September 29, 1923).

§ 1. *Introduction.* The significance of extending the investigation on the magnetisation of paramagnetic substances to the temperatures obtainable with liquid helium, that might be expected a priori, has been confirmed in a convincing way by the preliminary research on the magnetisation of gadolinium sulphate in liquid helium carried out by KAMERLINGH ONNES in 1914 [1]). The results then obtained showed the interest of continuing the research on gadolinium sulphate and completing the preliminary qualitative results by more accurate quantitative ones. Other substances, such as the paramagnetic chlorides [2]) presented themselves also for investigation in helium. However, closer inspection of the work of 1914 showed, that it was of little use repeating the work without detailed investigation

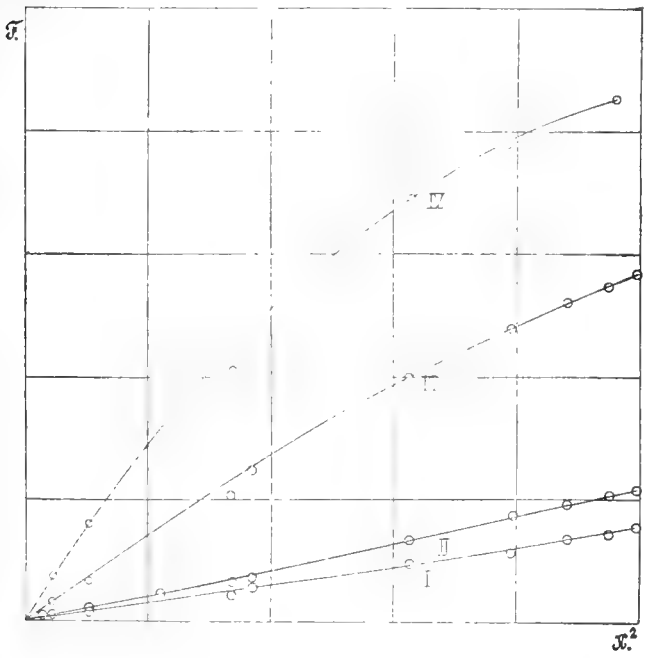

Fig. 1.

[1]) H. KAMERLINGH ONNES, these Proceedings, 17 p. 283; Leiden Comm. N°. 140d. Cf. also idem, Rapport Solvay 1921, p. 131. Leiden Comm. Suppl. N°. 44a.

[2]) l. c. p. 154, resp. p. 25.

of the method, more accurate calibrations and study of the corrections. E. g. concerning the direct results of the observations the considerable deviations [1]) from the proportionality between the force (F) and the square of the magnetic force between the poles of the electromagnet occurring at hydrogen temperatures (cf. fig. 1, taken from the paper mentioned) are particularly striking and, if the results are given in terms of LANGEVIN's theory of paramagnetic gases (cf. fig. 2, taken from Leiden Comm. Suppl. N°. 44a) it may be asked whether no systematic errors occurred.

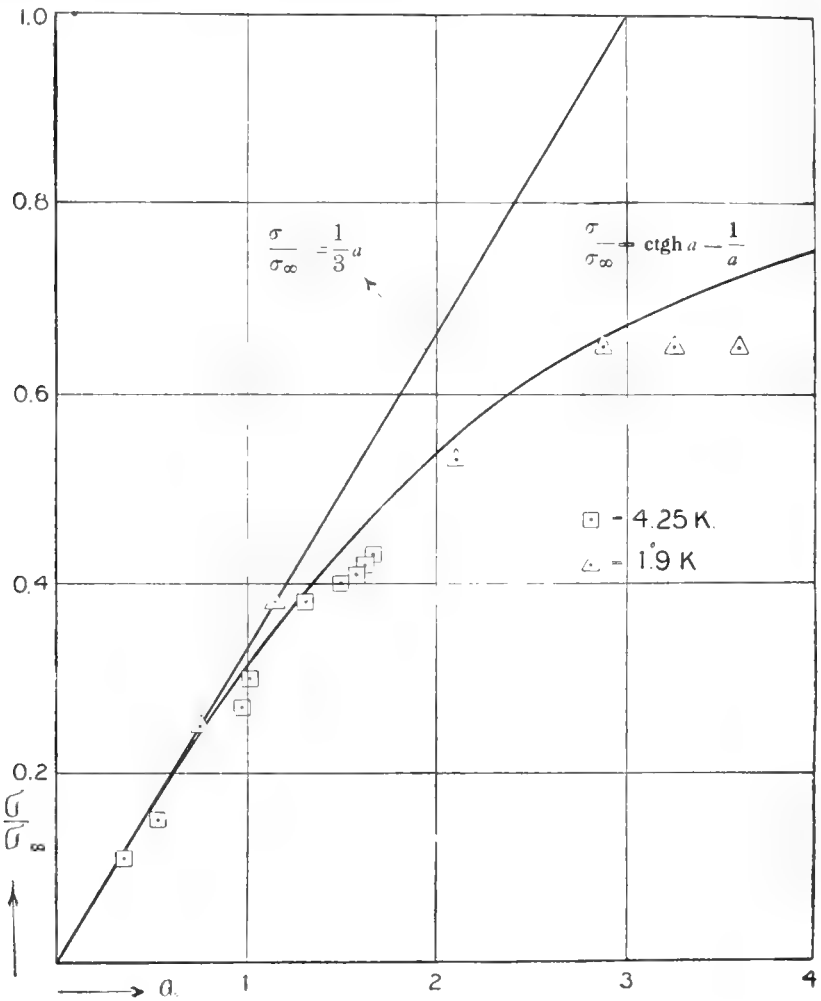

Fig. 2.

The method for measuring the magnetisation, it sources of errors

[1]) In the paper these deviations have been mentioned as probably due to inaccuracy in the topography.

and its corrections form the subject of the following paragraphs; we will consider more especially the topographic calibration of the electromagnet. It was carried out partially by means of the investigation of gadolinium sulphate in liquid hydrogen and so it furnished new material for the knowledge of the susceptibility of this substance, confirming old results. This new material will be communicated at the same time.

§ 2. *Apparatus and method.* The magnetisation was calculated from the force exerted by an inhomogeneous magnetic field on a small quantity of the material. For the measurement of the force the same apparatus was used as in the investigation of gadolinium sulphate in 1914, except a small alteration in connecting the tube containing the substance under consideration. At that time no description was given, so now some details may be mentioned. The apparatus was constructed by Mr. G. J. Flim, chief of the Technical Department of the Cryogenic Laboratory, mainly on the same principles as the apparatus of Kamerlingh Onnes and Perrier [1]) for the investigation of paramagnetic substances. The substance to be investigated is placed at the bottom part of a long rod, the "carrier". This carrier is suspended to one or two floats swimming on mercury. The force exerted by the magnetic field on the substance is compensated by a known force and the compensation is checked by means of a telescope and a scale attached to the carrier (*Sc.* fig. 3). Some modifications were required with a view to the special circumstances. The apparatus is introduced at the top of the helium cryostat (*C*) and is supported by the rim *R*. It is counterbalanced by weights acting on the connecting tube between cryostat and liquefactor. The weight of the apparatus has been minimised. Partially for this purpose the ringshaped trough of the apparatus of Kamerlingh Onnes and Perrier has been replaced by a small glass reservoir (*G*) with only one float (*Dr*). The comparatively large forces occurring in the experiments (up to about 200 gr.) induced to prefer magnetic compensation instead of electrodynamic compensation by two coils, though the accuracy was diminished thereby. The compensating force comes from the attraction exerted by a current of suitable intensity passing through a coil *D* at the top of the apparatus on a weak iron rod *S* at the top of the carrier; by putting rings (*Ri*) under the coil *D* its height can be taken such as to exert upward or downward forces, as appears convenient. The distance of the weak iron rod to the interferrum of the electro-

[1]) These Proceedings 16, p. 689 and 786. Leiden Comm. N°. 139a.

magnet has been chosen such that the action of the latter on the former may be neglected.

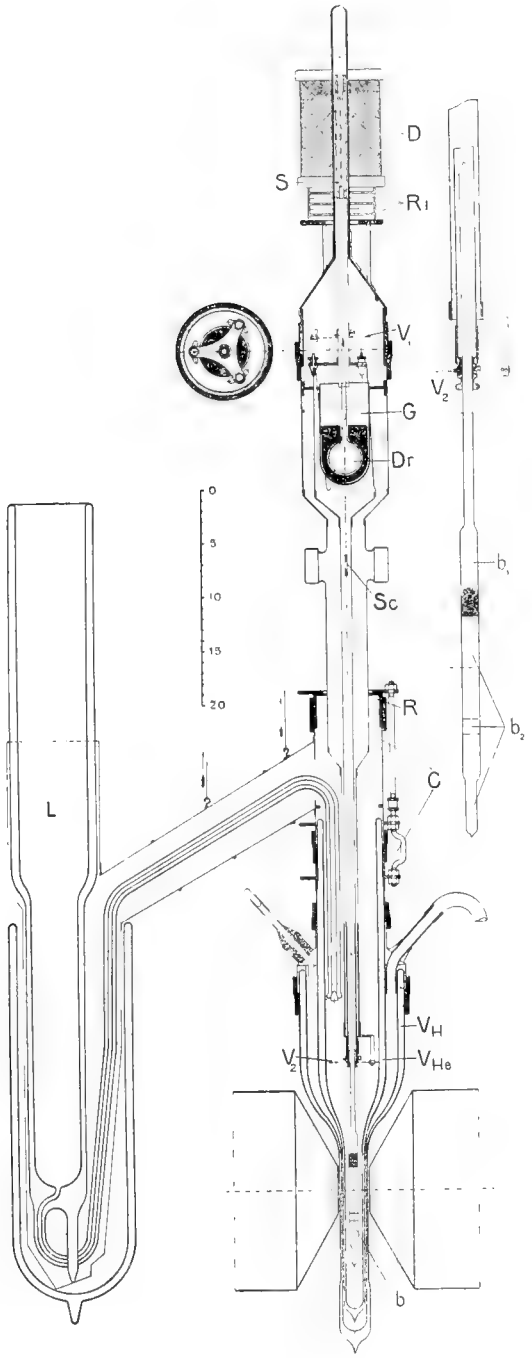

Fig. 3.

The tube (b, cf. the diagram of this detail in fig. 3) containing the magnetic substance, has been made and placed to obtain a symmetrical distribution of glass with respect to a horizontal plane passing through the centres of the poles of the electromagnet. In this way the attraction exerted by the magnet on the glass has been minimised and may be neglected. The dimensions have been chosen such that the sample is at the place of maximum $\frac{\partial H}{\partial z}$ [1], if the tube has been placed symmetrically in the field. The lower part (b_2) of the tube has been evacuated, in the upper part (b_1) a small quantity of helium gas has been introduced in order to improve the temperature equilibrium of the powder and the surroundings and of the particles of the powder mutually. The substance is enclosed between two glass disks, one of which has been melted on the tube, the other is free but is kept in its place by a small plug of cotton wool. Two flattened spiral springs, V_1, V_2 prevent a lateral displacement of the carrier. The lower one has been attached to the carrier and not, as in previous work, to the tube, so that the tubes may be replaced without changing the position of the carrier.

The end faces of the large size Weiss magnet have a diameter of 4 cm and are 26,5 mm apart. The semi-angle of the coneshaped boundary faces is 60°.

The compensating force as function of the intensity of the current in the coil D has often been determined as carefully as possible by suspending weights to the tube. Notwithstanding all precautions unexplained differences subsisted between the different calibrations. The extreme ones differ about 2 %. In calculating a series of observations use was made of the mean of the calibrations "before" and "after".

The specific magnetisation, σ, is calculated from the force measured by means of the relation

$$F\,g = m\sigma\frac{\partial H}{\partial z}, \quad \ldots\ldots\ldots \quad (1)$$

where F represents the force (in grammes) exerted on the mass m. The z-coördinate is measured along a vertical from the middle of the interferrum; H is the magnetic force at the point indicated by z; $g = 981.3$.

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[1]) If the susceptibility does not depend on the field strength, the maximum of $\frac{\partial H^2}{\partial z}$ is preferable. [Note added in the translation].

In every set, i. e. every measurement of the force corresponding to a definite value of the magnetic field and a definite temperature, the intensity of the current in the coil D necessary to bring the carrier into a chosen zero position was read the magnetic field being "of" and "on". These readings were taken for both directions of the currents in the coil and in the magnet.

§ 3. *Corrections, auxiliary measurements and sources of errors.* *a. Forces on the carrier without sample.* These forces appeared to be not quite negligible and they increased with decreasing temperature. Investigation of the different parts of the apparatus showed that those forces were caused especially by a small screw at the bottom of the carrier (near V_2). The comparatively large increase of these forces when the temperature falls from 20° to 14° K. is very striking, e. g. 70 amp. passing through the electromagnet the attraction amounts to

0,259 gr. at atmospheric temp.
0,326 ,, 20° K.
0,350 ,, 14° ,,

This is not what would be expected if the brass of the screw mentioned contained iron as an impurity. Further, such a comparatively very large increase in the liquid hydrogen region would give reason of suspecting much larger forces in the range of helium temperatures. However, they are then not large as appears from there being no systematic difference between the observations in which the mentioned parts of the carrier were certainly below and those in which they were certainly at some distance above the liquid helium level [1]). Particular circumstances prevented determining those forces (whose comparatively large increase in the hydrogen region appeared firstly afterwards) at helium temperatures and in the light of the foregoing remark it seemed not absolutely necessary. In the following observations the correction for the forces on the carrier without sample has been applied for the hydrogen temperatures only.

b. Correction for demagnetisation. This correction may attain considerable values at the temperatures of liquid helium. In the case of a sphere of a homogeneous substance of density d in a homogeneous field the demagnetising field is $-\frac{4}{3}\pi\sigma d$. In our experiments the cirumstances did not correspond exactly to these conditions. The sample is a *powder* in the shape of a small *cylinder*

[1]) Cf. the following communication § 3 note.

and is placed in an inhomogeneous field. Dr. BREIT [1]) has made a careful investigation in the case of a powder. According to him a first approximation for the demagnetisation is obtained if the formula mentioned is applied, taking for d not the density of the powder itself, but of the substance. If necessary this correction has been applied in that manner.

c. Topographical corrections. $\frac{\partial H}{\partial z}$ is in first approximation proportional to the field strength in the middle of the interferrum: H_0. The factor of proportionality was calculated from a ballistic topographical calibration of the magnet [2]). At currents of 10 and 20 amp. no appreciable difference in the topography was stated and for $z = 2.45$ cm. ($\frac{\partial H}{\partial z}$ being there a maximum) was found:

$$H = 0.815 \,.\, H_0, \qquad \frac{\partial H}{\partial z} = 0.199 \,.\, H_0 \quad . \quad . \quad . \quad (2,\ 3)$$

If however for gadolinium sulphate [3]) the force F is calculated as a function of H_0, no proportionality of F to H_0^2 is found, as might be expected on account of previous measurements [4]) (apart from small corrections if LANGEVIN's formula is followed) but deviations occur up to 20 %. This appears from table I and fig. 4. To the observed value of F, given in the third column now first a correction for the demagnetisation is applied: F is multiplied by $1 + \frac{4}{3}\pi d_0 \chi$; according to the remark b (see above), d_0 is taken equal to 3 [5]), for χ, the specific susceptibility, the value following from the un-corrected measurements has been taken. At 20°.42 K. this correction is 1.2 %, at 13°.98 K. 1.8%. In the column headed L the corrections for the deviations according to LANGEVIN's formula have been given. With those two corrections an apparent Curie-constant $C' = \chi T$ has been calculated.

The values found for C' appear to be strongly dependent on the field strength (cf. fig. 5). This may not be due to errors in the

[1]) These Proceedings 25, p. 293; Leiden Comm. Suppl. N°. 46.

[2]) The calibration really refers to a pole distance of 26 mm., not to 26.5 mm., the distance occurring in the experiments described.

The parameters of this field do not belong to those for which FORRER has given so much and such important data (J. FORRER, thesis Zürich, 1919).

[3]) The gadolinium sulphate, $Gd_2(SO_4)_3 \,.\, 8H_2O$, originated from the supply previously kindly sent by Prof. URBAIN. Two tubes have been filled with it, Gd I and Gd II, containing resp. 0.4735 en 0.4414 gr. of gadolinium sulphate.

[4]) H. KAMERLINGH ONNES and E. OOSTERHUIS, these Proceedings 15, p. 322 § 6, Leiden Comm. N°. 129*b*, § 6.

[5]) P. GROTH, Chem. Krystallographie II (1908), p. 460.

calibration of the magnetic field. This calibration may be estimated to be accurate to a few thousands. The deviations must be caused by the circumstance that at large and at small values of H_0 the proportionality mentioned may not be expected to hold [1]).

TABLE I.

Nr.	I	F	H_0	L	$10^2\,C'$	q	$10^2\,C$	$\frac{(C-C_m)}{C}$
Gadolinium sulphate II ($m = 0{,}4414$ gr.) $T = 20°.42$ K.								
4	5 amp.	0.81 gr.	3295	0.0	2.100	1.018	2.064	+1.7%
5	5	0.80						
6	10	3.10	6605	0.1	2.015	0.997	2.021	−0.45
3	15	6.98	9875	0.2	2.031	1.000	2.031	0.0
7	20	12.00	12940	0.4	2.038	1.005	2.028	−0.1
2	30	20.66	17320	0.8	1.962	0.963	2.037	+0.3
8	30	20.56						
9	45	25.99	20235	1.2	1.820	0.897	2.029	−0.1
1	60	28.17	21600	1.4	1.729	0.856	2.021	−0.45
10	60	28.00						
$T = 13°.98$ K.								
15	4	0.74	2627	0.1	2.093	1.026	2.040	+0.5
16	5	1.13	3295	0.1	2.032	1.018	1.996	−1.7
14	10	4.52	6605	0.2	2.025	0.997	2.031	0.0
13	20	17.41	12940	1.0	2.046	1.005	2.036	+0.3
17	20	17.39						
18	30	29.32	17320	1.9	1.942	0.963	2.017	−0.7
12	45	37.36	20235	2.6	1.826	0.897	2.036	+0.3
11	60	40.33	21600	3.0	1.739	0.856	2.033	+0.1
19	60	40.44						
20	70	41.77	22230	3.2	1.701	0.835	2.037	+0.3

[1]) In fig. 2 the points for the higher field strengths show the same kind of deviation from the LANGEVIN curve at 4°,25 K. as at 1°,9 K. In my opinion this fact is caused by the absence of proportionality mentioned in the text.

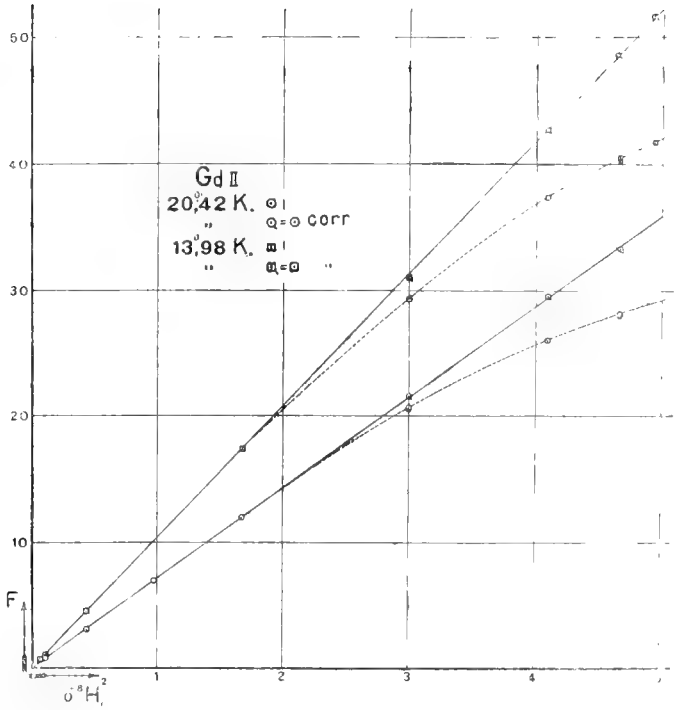

Fig. 4.

We have put:

$$\left.\begin{aligned} H &= s \,.\, 0.815 \,.\, H_0 \\ \frac{\partial H}{\partial z} &= r \,.\, 0{,}199 \; H_0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4)$$

$$q = s \,.\, r \quad \ldots\ldots\ldots \quad (5)$$

and for 15 amp. $q = s = r = 1$.

The quantities q, s and r are called the topographical corrections.

The apparent Curie-constant C' is connected to the true Curie-constant C by the formula: $C' = qC$ and does not depend on the temperature. Fig. 5 shows that within the limits of accuracy of the experiments at both hydrogen temperatures [1]) the same values for C' are found. Only at 5 amp. ($H_0 = 3295$) where the forces are small and the measurements less accurate there exists a larger deviation.

The values for C' have been smoothed graphically and then the topographical correction q has been determined from $q = \dfrac{C'}{C'_{15\text{ amp.}}}$.

In the column $10^2 C$ the value of $10^2 C'$ corrected with q has been

[1]) The circles refer to 20°,42 K., the squares to 13°,98 K.

given and in the last column the difference (in percents) of $10^2 C$ with the mean value 2,030.

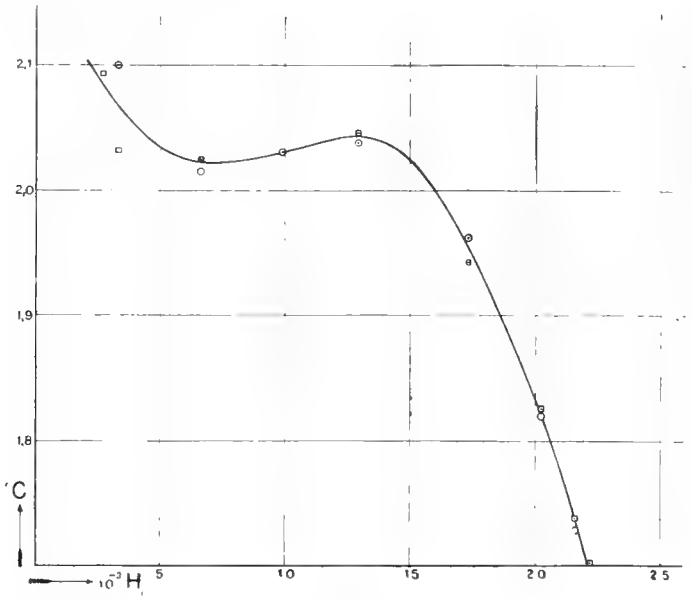

Fig. 5.

r was determined from experiments on the attraction of two small ellipsoids [1]) of Swedish Carbon iron placed as well as possible at the same spot as the substances in the actual experiments. Use was made of the measurements of STEINHAUS und GUMLICH [2]) on the relation between field strength and magnetisation when saturation is nearly reached, the so called law of approach

s was calculated from formula (5). The values found for r and s have also been smoothed graphically [3]).

In these determinations the distribution of magnetism on the pole faces of the magnet has been supposed to be perfectly rigid [4]).

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[1]) Masses 30.0 and 32.0 mg., major axis 6.2 mm., minor axis 1.1 m.m.

[2]) Ber. d. Physik Ges. 17 (1915) p. 271.

[3]) This causes the product of the given r and s to be not exactly equal to q.

[4]) Cf. P. WEISS, J. de Phys. May 1910 and P. WEISS and H. KAMERLINGH ONNES, Leiden Comm. N°. 114, p 16.

Strictly speaking: for a magnet current of 15 amp. the distribution of magnetism on the pole faces of the magnet has been supposed to be perfectly rigid and as regards the other current intensities it has only been supposed to be the same for gadolinium sulphate at hydrogen temperatures and for the S.C. iron ellipsoids. In fact, the magnetic moments are of the same order in both cases (though the volumes on which they are distributed are different); in the case of gadolinium sulphate at *helium* temperatures they are much larger, yet the same values of r and s have been applied (cf. next communication) [Note modified in the translation].

So the values given in table II have been found.

d. Corrections for diamagnetism of the liquid bath and of the anion could be left out of consideration.

e. As regards the *accuracy* and the *sources* of *error* may firstly be pointed out that the *heliumtemperatures* are rather uncertain, especially the lower ones. There was no room for a special stirrer and so the liquid could be stirred only so much as was possible by moving the floating system up and down. Therefore probably the temperature was not always evenly distributed and not perfectly well defined. This is especially important at temperatures below the maximum of density; then the cooling at the surface by evaporation does not give rise to downward convectional currents. However the lower temperatures are not only somewhat indefinite, but the values accepted are not very accurate. They have been determined graphically by means of the total existing material for helium vapour pressures[1]), but this leaves at the temperatures between 1° and 3° K. uncertainties of the order of 0,1 of a degree.

TABLE II.

Pole distance 26.5 mm.; $s = 2.45$ cm.		
I	r	s
3 amp.	0.973	1.062
4	0.983	1.044
5	0.990	1.030
10	0.999	1.003
15	1.000	1.000
20	0.995	1.002
30	0.960	1.010
45	0.893	1.021
60	0.837	1.030
70	0.808	1.035

f. Much care was bestowed on the *adjusting* of the *sample* to the proper place in the magnetic field, or more accurately, of the adjusting of the magnet to the sample, the cryostat not being movable.

[1]) H. Kamerlingh Onnes and Sophus Weber, these Proceedings 18, p. 493; Leiden Comm N°. 147b; H. Kamerlingh Onnes, Leiden Comm. N°. 159 p. 35.

Once the magnet was adjusted in its place, it was marked by means of two plummets suspended to the cryostat and marking two pointers on the yoke of the magnet, for the magnet had temporarily to be removed to afford opportunity of bringing the DEWAR vessels V_{He} and V_H (fig. 3) into place. The large magnet is very heavy and there was no device for moving the magnet sligthly in horizontal direction, so the horizontal adjustment was accompanied by great difficulties and possibilities for inaccuracy.

During the operations with liquid helium and liquid hydrogen the cryostat, forming one whole with the liquefactor, moved slightly in an irregular way as a consequence of the changing temperature circumstances in the different parts. By means of pulling rods the initial position with respect to the magnet was restored.

As far as the adjustment in vertical direction is concerned, it must be pointed out that the distance (at atmospheric temperature) from the centre of the mass to the centre of the field is considered as "place" of the sample in the magnetic field. This place determines the values of the constants in formulae (2)—(5). In the measurements in liquid hydrogen and in liquid helium this place has changed really by the shortening of the carrier in consequence of its cooling. The influence on $\frac{\partial H}{\partial z}$ will be very small as $\frac{\partial H}{\partial z}$ is maximum, but for the same reason the influence on H has to be taken into consideration. In itself there is reason for a correction. In the (rather unfavorable) case that the carrier up to 20 cm above the sample has the temperature of the boiling point of liquid hydrogen and the other part is at atmospheric temperature, a shortening of 0,3 mm would follow from the data of CH. LINDEMANN.[1]) H would be 0,006 H_0 smaller than corresponds to formula (2) i. e. about 0,7 %. Yet no correction has been applied, because it would have required a accurate determination of the place of the substance during the measurements as the sinking of the liquid level changed the temperature distribution along the carrier and thus the place of the sample. Moreover in the measurements in liquid hydrogen and in liquid helium (and the experiments only refer to these temperatures) the correction is nearly equal when the liquid level is on the same height, as the expansion coefficient at these low temperatures rapidly decreases to zero.

g. Finally it must be mentioned that no trace has been observed of the powder particles getting directed or remaining directed by the magnetic forces.

[1]) Physik. Zs. 13, (1912), p. 737.

§ 4. *The Curie constant of gadolinium sulphate.* In § 3*c* it has been mentioned already that for *Gd* II 2,030 $\times 10^{-2}$ has been found.

For the Curie constant of *Gd* I we find:

$T = 20°,31$ K.	$\chi = 1{,}0566 \cdot 10^{-3}$	$C = 2{,}146 \cdot 10^{-2}$
$T = 14°,68$	$\chi = 1{,}4663$ „	$C = 2{,}152$
	mean:	2,149

The measurements on *Gd* I have been considered as less accurate than those on *Gd* II, because (cf. § 3*f* on the difficulties of the adjustment) the tube appeared afterwards for unknown reasons to be not exactly in the middle between the pole faces, but 1,6 mm out of the center. A previous determination of the Curie-constant of *Gd* I quite independent of the present research had given 2,113 $\times 10^{-2}$. So it is not very probable that the large difference between the Curie constants of *Gd* I and *Gd* II is due to inaccurate adjustment of the tube only. Besides it must be remarked that different observers have found values differing more still than the values mentioned: from the results of Mlle Feytis [1]), Kamerlingh Onnes and Perrier [2]), and Kamerlingh Onnes and Oosterhuis [3]) the Curie-constant of gadolinium sulphate is found to be [4]).

Mlle Feytis	$2{,}167 \cdot 10^{-2}$
K. O. and P.	2,086
K. O. and O.	2,016.

These differences are not yet explained.

Finally, I wish to express my sincere thanks to Professor Kamerlingh Onnes for his kind interest in my work.

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[1]) Paris C. R. 153 (1911), p. 668.

[2]) These Proceedings 14, p. 115; Leiden Comm. N°. 122a.

[3]) „ „ 15, p. 322; Leiden Comm. N°. 129b.

[4]) A correction has been applied for the diamagnetism of the crystal water and of the anion. The first correction had been applied already by Mlle Feytis.

Physics. — "*Further experiments with liquid helium. T. Magnetic researches.* XXIII. *On the magnetisation of gadolinium sulphate at temperatures obtainable with liquid helium.*" By H. R. WOLTJER and H. KAMERLINGH ONNES. (Communication N°. 167*c* from the Physical Laboratory at Leiden).

(Communicated at the meeting of September 29, 1923).

§ 1. *Introduction.* Previous[1]) preliminary researches and a detailed discussion[2]) of the results then obtained have shown the importance of a closer investigation of the magnetisation of gadolinium sulphate at very low temperatures: this substance is one of the comparatively few, that follow CURIE's law down to the region of temperatures obtainable with liquid hydrogen. Now in the light of LANGEVIN's theory the CURIE law holds only approximately, viz. as long as the susceptibility may be considered to be independent of the field strength: LANGEVIN gives for the ratio of the specific magnetisation, σ, to the specific saturation magnetisation, σ_∞,

$$\sigma : \sigma_\infty = cotgh\ a - \frac{1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

$$a = \frac{\sigma_{m_\infty}}{R} \cdot \frac{H}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

(σ_{m_∞} being the saturation magnetisation of one gram molecule, R the gas constant per grm. mol., H the magnetic field applied and T the absolute temperature).

For small values of a

$$\sigma : \sigma_\infty = \frac{1}{3} a \quad \text{or} \quad \chi = \frac{\sigma}{H} = \frac{\sigma_\infty \cdot \sigma_{m_\infty}}{3R} \cdot \frac{1}{T} \quad . \quad . \quad . \quad . \quad (2)$$

If T is small and thus a large, χ is no longer independent of H, but the curve $\sigma : \sigma_\infty = f(a)$ deviates from the straight line $\sigma : \sigma_\infty = \frac{1}{3} a$, becomes concave towards the a-axis and approaches asymptotically to $\sigma : \sigma_\infty = 1$ (cf. fig.) The detailed discussion of the preliminary experiments has already made very probable the existence

[1]) H. KAMERLINGH ONNES, these Proceedings 17, p. 283; Leiden Comm. N°. 140d.

[2]) H. KAMERLINGH ONNES, Rapport Solvay 1921, p. 131; Leiden Comm. Suppl. N°. 44a. I.

of deviations of this type. Yet it is not to be expected a priori that LANGEVIN's theory would be followed in this case, for this theory has been deduced for a gas with perfect rotational freedom of the molecules and starts from the assumption of the equipartition of energy in al degrees of freedom. Now the case of powdered gadolinium sulphate at low temperatures does not correspond to either of these assumptions. It is true that LANGEVIN's theory has been extended by WEISS[1]) to powdered crystals, but WEISS confines himself to small values of the parameter a; on the other hand EHRENFEST[2]) has developed a theory in which the relatition (2) is obtained for crystal powders on the assumption of the existence of quanta but then the saturation magnetisation is only half the value corresponding to perfect parallelism of all elementary magnets and in the preliminary experiments a higher value seemed to be reached.

Confirmation and extension of the preliminary results was thus very desirable; the same method has been followed as in the previous work: the specific magnetisation, σ, is calculated from the force F (in grammes) exerted on the mass m by an inhomogeneous magnetic field with aid of the formula $Fg = m\sigma \frac{\partial H}{\partial z}$. A detailed study of the apparatus, the corrections and the sources of error, a comprehensive account of which has been given in the preceding communication[3]), has made it possible to attain a much greater accuracy than in the previous work, at least as far as the magnetic measurements are concerned. The determination of the temperature from the vapour pressure of the bath is still a weak point, especially since the vapour pressure law is as yet not sufficiently well known[4]). The research relates to the same tubes, *Gd I* and *Gd II*, that have served for the research in liquid hydrogen and that have been mentioned in the preceding communication (§ 3*c*).

§ 2. *Observations.* The direct results of the observations may be given first: tables I and II (I being the number of ampères in the magnet coils; H_0 the field strength, in gauss, in the centre; F the force in grammes, on the total mass of substance).

With *Gd II* between the points N°. 15 and N°. 28 points have been left out in which the observations have been taken at increas-

[1]) P. WEISS, Paris C. R. 156 (1913) p. 1674. According to O. STERN (Zs. f. Phys. 1 (1920) p. 147) WEISS' deduction is not sound.

[2]) P. EHRENFEST, these Proceedings 23, p. 989; Leiden Comm. Suppl. N°. 44b.

[3]) H. R. WOLTJER, these Proceedings p. 613; Leiden Comm. No. 167*b*.

[4]) l. c. § 3*e*.

TABLE I.

Gadolinium sulphate I						
Date	Vapour pressure	T	Nr.	I	H_0	F
March 1th, 1923	761 mm.[1])	4°.20 K.	1	30	17320	90.14
"	"	"	2	20	12940	55.26
"	"	"	3	10	6605	15.76
"	"	"	4	5	3295	3.89
"	"	"	5	5	3295	4.01
"	"	"	6	15	9875	33.83
"	"	"	7	30	17320	89.94
"	"	"	8	60	21600	114.76
"	"	"	9	70	22230	117.81
"	"	"	10	45	20235	109.54
"	"	"	11	30	17320	90.96
"	360 mm	3°.53 "	12	70	22230	136.93
"	"	"	13	45	20235	123.42
"	"	"	14	30	17320	103.78
"	"	"	15	20	12940	65.61
"	"	"	16	10	6605	19.04
"	"	"	17	5	3295	4.76
"	"	"	18	5	3295	4.75
"	"	"	19	15	9875	40.26
"	"	"	20	30	17320	102.54
"	"	"	21	60	21600	129.12
"	"	"	[22	70	22230	130.68
"	100 mm.	2°.73 "	[23	70	22230	152.27]
"	"	"	24	45	20235	148.13
"	"	"	25	30	17320	121.71
"	"	"	26	20	12940	79.75
"	"	"	27	10	6605	24.36
"	"	"	[28	5	3295	6.12]
"	763 mm.	4°.20 "	29	30	17320	91.11
"	9.5 mm.	$1°.66_5$ "	30	70	22230	173.70
"	4 mm.	1°.48 "	[31	60	21600	173.41]

[1]) The difference between international and local m.m. mercury (these Proceedings 21 p. 658 note 2; Leiden Comm. No. 152*d* p. 47, note 4)) is here of no importance.

TABLE II.

Gadolinium sulphate II						
Date	Vapour pressure	T	Nr	I	H_0	F
April 13th, 1923	761 mm.	4°.20 K.	1	60	21600	108.27
"	"	"	2	30	17320	85.67
"	"	"	3	15	9875	32.44
"	"	"	4	5	3295	3.74
"	"	"	5	5	3295	3.77
"	"	"	6	10	6605	15.10
"	"	"	7	20	12940	53.00
"	"	"	8	30	17320	85.80
"	"	"	9	45	20235	102.76
"	"	"	10	60	21600	108.04
"	300 mm.	3°.40 "	11	30	17320	98.48
"	39 mm.	2°.30 "	12	30	17320	119.86
"	4 mm.	1°.48 "	13	30	17320	133.53
"	"	"	14	60	21600	152.29
"	"	"	15	30	17320	133 59
"	759 mm.	4°.20 "	28	30	17320	85 48
"	2.9 mm.	1°.41$_5$ "	29	70	22230	156.36
"	"	"	30	45	20235	152.69
"	"	"	31	30	17320	136.30
"	"	"	32	5	3295	10.20
"	"	"	33	4	2627	6.46
"	"	"	34	3	1960	3.68
"	1.7 mm.	1°.31 "	35	70	22230	157.74
"	"	"	36	60	21600	157.72

ing pressure in order to test whether temperature corresponded to pressure, the only stirring possible being made by the moving up and down of the carrier [1]). The magnetisations observed pointed to much lower temperatures than corresponded to the actual pressures and thus to a large temperature lag. Therefore these points have been left out of consideration.

§ 3. *Discussion.* For *Gd II* 0,02024 [2]) has been accepted as Curie-constant and with this value σ_∞ and $\sigma_{m\infty}$ have been calculated according to formula (2). Half the real molecular weight has been used in calculating σ_{m_∞} from σ_∞, as the *atoms* of *Gd* are assumed to have rotational freedom. This is usually done for salts containing more than one metal atom in the molecule [3]); moreover, if the whole molecular weight had been taken, σ_{m_∞} would have become 1/2 larger, σ_∞ 1/2 smaller and thus $\sigma : \sigma_\infty$ again 1/2 larger and one would have found values larger than 1, as for $\sigma : \sigma_\infty$ the value 0.84 has been attained (Cf. table IV).

We find:

$$\sigma_{m\infty} = 434{,}2 \cdot 10^2 \text{ (38.65 WEISS-magnetons).}$$
$$\sigma_\infty = 116{,}25.$$

For the Curie constant of *Gd I* we found [4])

$$C = 0{,}02149$$
$$\sigma_{m\infty} = 447{,}4 \cdot 10^2 \text{ (39.82 WEISS-magnetons)}$$
$$\sigma_\infty = 119{,}79.$$

From the tables I and II $\sigma : \sigma_\infty$ and σ have been calculated for *Gd I* and *Gd II*, with its own particular Curie constant for each substance. The results have been collected in tables III and IV. The values placed in square brackets are a priori less reliable, mostly because during or immediately after the measurement the gadolinium sulphate appeared to be not sufficiently below the liquid helium level [5]). The differences between the observed values of $\sigma : \sigma_\infty$ and

[1]) l.c. § 3*e*.

[2]) Cf. the preceding communication § 4, where on account of a later somewhat modified calculation 0,02030 has been given. The difference is of no importance.

[3]) P. WEISS, Arch. d. Sc. phys. et nat. (4) 31 (1911)

B. CABRERA, J. de Chim. Phys. 6 (1918) p. 442, especially p. 462.

[4]) Cf. the preceding communication § 4, where the difference between both results has been discussed.

[5]) At the points marked with an asterisk the helium level was certainly below the spring V_2 (cf. the preceding communication § 3*a*). Though a general tendency to higher values of $\sigma : \sigma_\infty$ (cf. the diagram) must be acknowledged to exist, there is no systematic difference between the points with and without asterisk.

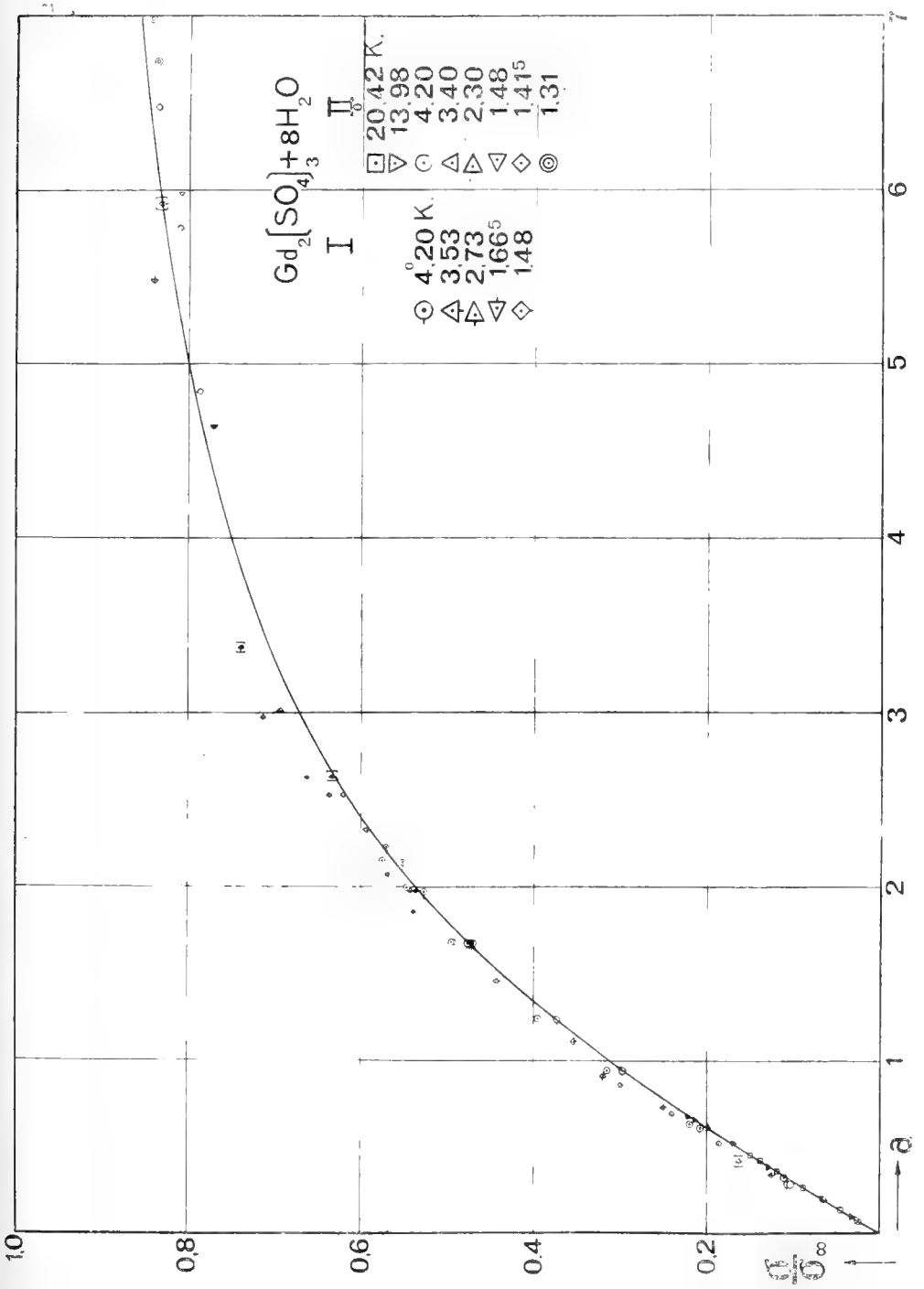
TABLE III. Gadolinium sulphate I.

	4°.20 K.				3°.53 K.				2°.73 K.				1°.66$_5$ K.				1°.48 K.			
I	Nr.	a.	$\left(\frac{\sigma}{\sigma_\infty}\right)_{obs.}$	$100 . \frac{O-C}{O}$	Nr.	a.	$\left(\frac{\sigma}{\sigma_\infty}\right)_{obs.}$	$100 . \frac{O-C}{O}$	Nr.	a.	$\left(\frac{\sigma}{\sigma_\infty}\right)_{obs.}$	$100 . \frac{O-C}{O}$	Nr.	a.	$\left(\frac{\sigma}{\sigma_\infty}\right)_{obs.}$	$100 . \frac{O-C}{O}$	Nr.	a.	$\left(\frac{\sigma}{\sigma_\infty}\right)_{obs.}$	$100 . \frac{O-C}{O}$
5	4	0.2888	0.1036	+ 7.6	17*	0.3384	0.1268	+11.7	[28*	0.4267	0.1630	+13.8]								
"	5	0.2882	0.1068	+ 10.5	18*	0.3384	0.1266	+11.6												
10	3	0.6174	0.2075	+ 3.2	16*	0.7246	0.2508	+ 6.9	27*	0.9162	0.3208	+ 9.8								
15	6	0.9408	0.2977	+ 0.4	19*	1.106	0.3544	+ 3.6												
20	2	1.239	0.3732	— 0.8	15*	1.458	0.4430	+ 3.2	26	1.857	0.5384	+ 5.0								
30	1	1.675	0.4710	— 1.0	14*	1.976	0.5425	+ 1.7	25	2.528	0.6362	+ 3.0								
"	7	1.675	0.4700	— 1.2	20*	1.977	0.5360	+ 0.5												
"	11	1.673	0.4753	0.0																
"	29*	1.673	0.4763	+ 0.2																
45	10	1.970	0.5269	— 1.0	13	2.328	0.5936	+ 0.7	24	2.975	0.7124	+ 6.1					[31*	5.917	0.8339	+ 0.3]
60	8	2.140	0.5516	— 1.7	21*	2.529	0.6209	+ 0.6												
70	9	2.223	0.5702	— 0.6	12	2.624	0.6627	+ 5.0	[23	3.371	0.7369	+ 4.2]	30*	5.475	0.8406	+ 2.8				
"					[22*	2.631	0.6324	+ 0.3]												

TABLE IV. Gadolinium sulphate II

	4°.20 K.				3°.40 K.				2°.30 K.				1°.48 K.				1°.41$_5$ K.				1°.31 K.			
I	Nr	a.	$\left(\frac{\sigma}{\sigma_8}\right)_{obs.}$	$100.\frac{O-C}{O}$	Nr.	a.	$\left(\frac{\sigma}{\sigma_8}\right)_{obs.}$	$100.\frac{O-C}{O}$	Nr	a.	$\left(\frac{\sigma}{\sigma_8}\right)_{obs.}$	$100.\frac{O-C}{O}$	Nr.	a.	$\left(\frac{\sigma}{\sigma_8}\right)_{osb.}$	$100.\frac{O-C}{O}$	Nr.	a.	$\left(\frac{\sigma}{\sigma_8}\right)_{obs.}$	$100.\frac{O-C}{O}$	Nr.	a.	$\left(\frac{\sigma}{\sigma_8}\right)_{obs.}$	$100.\frac{O-C}{O}$
3																	34*	0.5260	0.1854	+ 7.1				
4																	33	0.6952	0.2403	+ 6.5				
5	4	0.3238	0.1102	+ 2.7													32	0.8587	0.3005	+ 9.1				
„	5	0.3238	0.1111	+ 3.5																				
10	6	0.6315	0.2199	+ 6.7																				
15	3	0.9433	0.3157	+ 5.9																				
20	7	1.243	0.3956	+ 4.7																				
30	2	1.682	0.4951	+ 3.6	11	2.063	0.5691	+ 3.7	12	3.009	0.6928	+ 2.9	13	4.634	0.7716	— 1.7	31	4.839	0.7878	— 0.7				
„	8	1.682	0.4959	+ 3.8									15*	4.634	0.7721	— 1.6								
„	28*	1.684	0.4941	+ 3.3																				
45	9	1.995	0.5464	+ 1.8													30	5.777	0.8119	— 1.8				
60	1	2.150	0.5756	+ 2.3									14*	5.973	0.8096	— 2.9					36*	6.738	0.8385	— 1.6
„	10	2.150	0.5744	+ 2.1																				
70																	29	6.467	0.8365	— 1.1	35*	6.982	0.8439	— 1.5

the values calculated according to LANGEVIN's formula, expressed in percents of the observed value, are given in the columns headed $100.\frac{O-C}{O}$.



It cannot be denied that while on the one hand, one gets the strong impression that LANGEVIN's formula is followed (cf. the figure, in which the LANGEVIN curve and the observed points have been drawn), on the other hand the deviations are larger than was anticipated. However they may be explained from the sources of error. Besides all that has been said in the preceding communication as to the accuracy, it must be pointed out that the larger deviations occur especially at the lower field strength values, where the topographical corrections are rather uncertain and also the measurements of the field strength less reliable. Further, the magnetic moment acting at the very low temperatures is so large that the assumption of a rigid distribution of the magnetism on the pole faces (and on this assumption the field measurements and the determination of the topographical corrections are more or less based) certainly holds no longer.

Moreover it must be observed, that errors in σ_∞ and in H_0 exert on the abscissae an influence opposite in direction to that on the ordinates and thus appear greater in the diagram. Taking all these circumstances into account, especially also the uncertainty of the demagnetisation, it may be concluded, that powdered gadolinium sulphate follows LANGEVIN's formula down to about 1°.3 K; thus it seems possible to use the magnetic susceptibility of gadolinium sulphate in thermometry.

§ 4. *Results.* The specific magnetisation of powdered hydrated gadolinium sulphate has been investigated for the temperatures of liquid hydrogen and liquid helium. It appears that though the fundamental assumptions to LANGEVIN's theory do not apply, yet LANGEVIN's formula is followed. For the parameter a of LANGEVIN's theory the value 7 has nearly been reached. The highest magnetisations obtained are about 84 % of the magnetisation corresponding to perfect parallelism of all elementary magnets. This result is independent of the uncertainties in the temperature and the value of the demagnetising field. So it appears that Prof. EHRENFEST's theory is here not applicable without further extension, since this theory (which is based on quanta assumptions and holds, contrary, to LANGEVIN's theory, directly for crystal powders) gives for the saturation magnetisation only 50 % of the value mentioned.

Physiology. — "*The string galvanometer in wireless telegraphy*". By W. F. EINTHOVEN. (Communicated by Prof. W. EINTHOVEN).

(Communicated at the meeting of March 24, 1923).

The string galvanometer, as is well known, consists of a conducting fibre stretched like a string in a strong magnetic field. A current passing through the fibre induces a displacement of it in a plane perpendicular to the lines of magnetic force. The deflection can be observed with a microscope and the magnified image can be photographed.

Many attempts have been made to use this instrument for the reception of wireless signals, but only ordinary models, with a relatively long, not very much stretched string have been tried, and these show great sensitiveness towards disturbing direct currents. The wireless signals were received in such a way that the high frequency oscillations were rectified by means of some device, and the rectified current impulses were passed through the string; this was affected in the same way as when conveying a true direct current.

But, used in this way, the string galvanometer has only brought disappointment in wireless telegraphy, for it reacts to every current of some duration with the same sensitiveness, and even the smallest atmospherics are sufficient to give trouble. Some large Companies, who have tried to use the string galvanometer at their transatlantic stations, have abandoned work with it.

The application here to be described of the instrument is based on a quite different method [1]). The incoming high frequency oscillations are not rectified but are sent through the string immediately. The string is short and stretched so much, that its own period corresponds to the period of the ether waves used in wireless signalling. Choosing the lenght of the string conveniently and adjusting its tension, we can bring it in tune with practically all continuous waves available in radio-telegraphy. If for instance these have a length of 1 kilometer corresponding to 300.000 periods per sec., the string is adjusted so that the proper frequency of its vibrations is also 300.000 per sec.

[1]) Patented.

The length of the string, being about 10 millim. for waves of (for instance) 10 kilom., is only 1 millim. for waves of 1 kilom. We have also experimented with shorter strings showing a still higher frequency of their proper vibrations. Heretofore as far as we know it has not been possible to induce these frequencies in any mechanism.

The string, for which we take a fine quartz fibre, is rendered able to conduct by cathode bombardment, and stretched between two microscopes; one of these serves to concentrate the light, the other to project the image, whilst both microscopes, in order to obtain a sharp definition of the string, must be very near to one another. The objectives, having a numerical aperture of 0,95, are no more than 0,2 millim. away from the string. Since the front lens of such an objective has a diameter larger than the length of the string, a special device is necessary to fix the string; this is done in such a manner that the rays of light are not intercepted, and the full angle of aperture of the objectives is made use of efficiently.

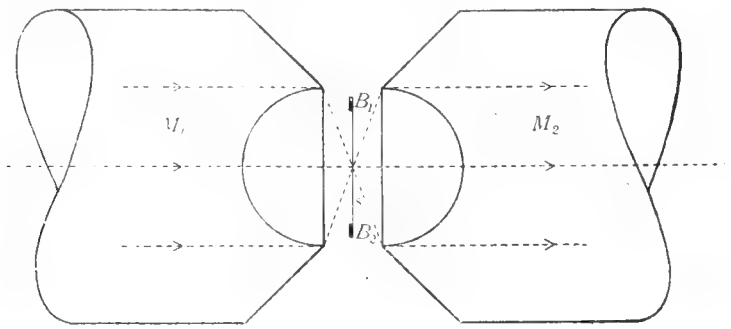


Fig. 1.

Diagram of the string s between both of the microscopes M_1 and M_2. B_1 and B_2, fine metal strips to which the string is soldered. The direction of the rays of light is indicated by the dotted lines and arrows.

The difficulty was overcome by soldering both ends of the string to fine metal strips placed in the optical plane perpendicular to the string, and rigidly attached to the apparatus in order to tighten and slacken the string.

It is important to have the string vibrating as freely as possible. Therefore it has not only to be fine but also strongly stretched like a string of a piano or a violin. Its minute mass per unity of length causes it to suffer a strong damping effect from the air, and this must be avoided. Therefore the space around it is evacuated, and in order to make the vacuum efficient it has to be made high. We

attained vacua of 1 μ Hg and even higher and were able to show, that under such conditions the air damping has practically no more influence on the movement of the string. The vibrations do not die away more slowly when the vacuum is made higher than 1 μ, since the internal friction of the string itself, i. e. the fact that the material of the string has no perfect elasticity is another cause of damping.

It is not to be expected, that the vibrations of a coated quartz fibre stretched like a string would die away as slowly as those of a pure quartz rod which has been fixed at only one end. Experiments of HABER and KERSCHBAUM [1]) have shown that it took more than 12 minutes, before the amplitude of a quartz rod vibrating in vacuo was diminished to one half of the original size. LANGMUIR [2]) succeeded in lowering the pressure in an incandescent bulb lamp so much that the time of halving the amplitude was lengthened to nearly two hours.

But if we cannot make the vibrations of our string die away equally slowly, nevertheless for the purpose aimed at the result is satisfactory. We could for instance show, that a string performing 40.000 vibrations per sec., without the intentional application of a damping factor needed a time $\tau = 0{,}65$ sec. to diminish its amplitude in the proportion of $1 : \frac{1}{e}$, wherefrom it may be inferred, that the logarithmic decrement of the movement amounted to 4×10^{-5}, conf. fig. 2.

Fig. 2.

A string the vibrations of which are dying away freely.
$\lambda = 7{,}5$ km, $\tau = 0{,}65$ sec., $\delta = 4 \times 10{-}5$.

This decrement is of the greatest value for our purpose, for the smaller it is so much the better is the selectivity of the instrument. If the string has been put in tune with a definite wave, it will react to atmospheric disturbances and to currents of different wave lengths coming in from other stations so much the less, the smaller the decrement is. Generally speaking we may say that the efficiency of a receiving apparatus is determined by the amount of its decrement.

For purposes of comparison it may be recalled, that the smallest

[1]) Zeitschr. f. Elektrochemie. Bd. 20, 1914, p. 296.

[2]) Journal of the American Chem. Soc. 35, 107 (1913) cited from HABER u. KERSCHBAUM.

available decrement of an electric circuit is about 0,01 and that in most cases this value is higher. The decrements of all the receiving apparatus known to us, which mechanically register the signals are larger than that of the string galvanometer.

However it is only possible to profit fully by a small decrement,

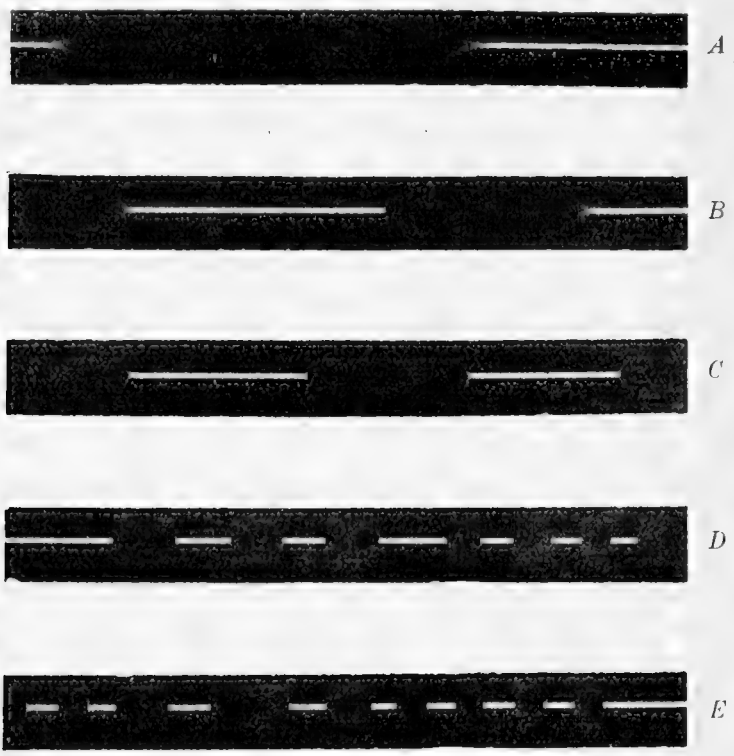


Fig. 3.

	Field magnet current	τ	δ
A	0,5 Amp.	0,27 sec.	$9{,}25 \times 10^{-5}$
B	1,— „	0,1 „	$0{,}25 \times 10^{-3}$
C	2,— „	0,03 „	$0{,}83 \times 10^{-3}$
D	4,— „		$3{,}— \times 10^{-3}$
E	6,— „		$6{,}2 \times 10^{-3}$

when signalling is excessively slow. Signals coming in at the usual speed would be intermingled if the vibrations of the string died away so slowly. Therefore it is necessary to increase the decrement

of the receiver purposely. This is not performed by admitting air around the string. On the contrary the vacuum is kept as high as possible in all experiments, but the strength of the magnetic field is changed. By varying this value from zero to a maximum the amount of the decrement can be adjusted in a simple and at the same time very precise manner.

In the above fig. 3 the photos are reproduced of the same string as that of fig. 2 the strip of paper moving with the same velocity, i. e. 10,75 millim. per sec. Continuous waves the length of which was 7,5 kilom. were coupled inductively with the circuit of the string, and switched on and off repeatedly, whilst in the successive photos the current exciting the field-magnet was increased from 0,5 to 6 Amp. The photos show, that the time in which the vibrations of the string die away is shorter as the exciting current increases. The decrements can only be measured in the photos *A*, *B* and *C* because of the low speed of the paper strip. For *D* and *E* they have been calculated from the intensity of the magnetic field, which amounted tot 7600 and 10.900 GAUSS respectively.

In the calculations of all useful decrements in radio-telegraphy, the proper decrement of the string itself caused by its internal friction is to be neglected, whilst in a high vacuum also the air damping is not to be taken into account. Under these circumstances the relation between the decrement and the field intensity is given by the formula

$$\delta = \frac{4}{\pi_2} \cdot \frac{H^2}{m\, w\, N} \cdot 10^{-9} \quad \ldots\ldots\ldots \quad (1)$$

where δ represents the logarithmic decrement,

H the intensity of the magnetic field in GAUSS,

m the mass of the string in grams per centim.,

w The resistance of the galvanometer circuit in OHMS per centim.,

N the number of periods per sec. of the string when vibrating in resonance with the continuous waves induced.

When receiving a signal the decrement must be adjusted so that the dots and dashes of a signal only just begin to blend, as may

Fig. 4.

Record of signals from an Italian station, made in Leyden. The decrement of the string has been adjusted so that the dots and dashes of a signal only just begin to blend.

be illustrated in the next fig. 4. The greater the speed of the signals so much the greater the intensity of the magnetic field, i. e. so much the larger the decrement has to be made. The maximum speed of still readable signals being about 600 words per minute is obtained when the intensity of the field is maximum, being 22.600 GAUSS in one of our instruments.

If sufficiently strong signals be available, the allowable speed could still be increased by admitting air around the string.

One of the difficulties that had to be overcome on designing the instrument was the adjustment of the tension of the string. This must be secured in an exceedingly precise and punctilious manner.

We stretch the string by extending it. In the figures reproduced above a string has been used of a length of 6 millim., stretched so that it was in tune with a wave of 7,5 kilom. Suppose that it then be extended to an amount of 1 % and thus be lengthened by 60 μ. If the current exciting the electromagnet is 1 Amp. the decrement of the string is 0.25×10^{-3}. From this it can be calculated that an increase of the elongation to an amount of 4,8 $\mu\mu$, suffices to bring the string so much out of tune that the amplitude of its vibrations will decrease in the proportion of $1 : \frac{1}{\sqrt{2}}$ i.e. 30 %. The same effect is produced by changing the wave-length of the signal to an amount of 30 centim. on 7,5 kilom.

For the above calculations formula (2) has been used:

$$\frac{\lambda_r - \lambda_1}{\lambda_r} = \frac{\delta}{2\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where λ_r represents the wave-length of continuous waves inductively coupled with a circuit and being in tune with its proper period, and λ_1 the wave-length that is so much smaller or larger than the former one, that the electric power of the circuit is reduced to one half. Here the movement of the string is substituted for the coupled current, and the amplitude of its vibrations for the square root of the electric power.

It need not be emphasized, that much smaller changes of the amplitude are measurable than the value which is mentioned above for convenience' sake. The apparatus for stretching the string must enable elongation to be effected within certain limits with absolute regularity, and by degrees smaller than 1 $\mu\mu$. Both of our present models comply with this requirement.

The experiments performed with the galvanometer have brought to light some phenomena concerned with the movement of string

in general, which could not be observed heretofore, since no vibrating string with so small a decrement has ever been available. Suppose, that the vibrations of a string of a piano or a violin die away as slowly as they do in the galvanometer, e.g. within about 2 or 3 sec. and that the frequency is a hundred times less, then accordingly the decrement is 100-times greater; for we have

$$\delta = \frac{1}{N\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where δ and N have the same meaning as in form. (1) whereas τ represents the time in seconds necessary for the amplitude to diminish in the proportion of $1 : \frac{1}{e}$. For the time of dying away we may allow 3 to 5-times the value of τ.

The musical string therefore vibrates much less freely, and we cannot tune it as sharply. However this is really unnecessary for the purpose it is used for, since the human ear is unable to discriminate so minute variations of pitch. The increase of tension experienced by the musical string, when moving from the position of equilibrium to that of the maximum displacement, may be left out of account at least when the amplitude is moderate. Every theory of the movement of strings is based on the supposition that in the different phases of a period of the vibration the tension of the string remains constant. [1])

However the conditions of the string of the galvanometer are different. A small amplitude, for instance to an amount of 1 per thousand of the length of the string, may be sufficient under definite conditions to display the influence exerted by the increase of the tension which the string is subjected to by its displacement.

We hope later on to revert to these phenomena, which may be referred to as those of the "jumping point". But it should be noticed here, that the difficulties caused by it are diminished to a large extent and practically overcome when the string which is to be in tune with a certain wave, is made as long as possible and extended to a maximum.

In fig. 5 a record made in Leyden is reproduced, which is of special interest for us in Holland; it represents the signals received from the alternator on the Malabar at Bandoeng. In order not to disclose the secret of the telegram only a few separate words and figures are given, so that the meaning will be understood by none.

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[1]) Conf. RAYLEIGH. The theory of sound, London 1877. Vol. 1, p. 36 and 128.

Although our only receiver was a small, not very favourable aerial, we were able repeatedly to take long telegrams, the signals of which came in absolutely clear.

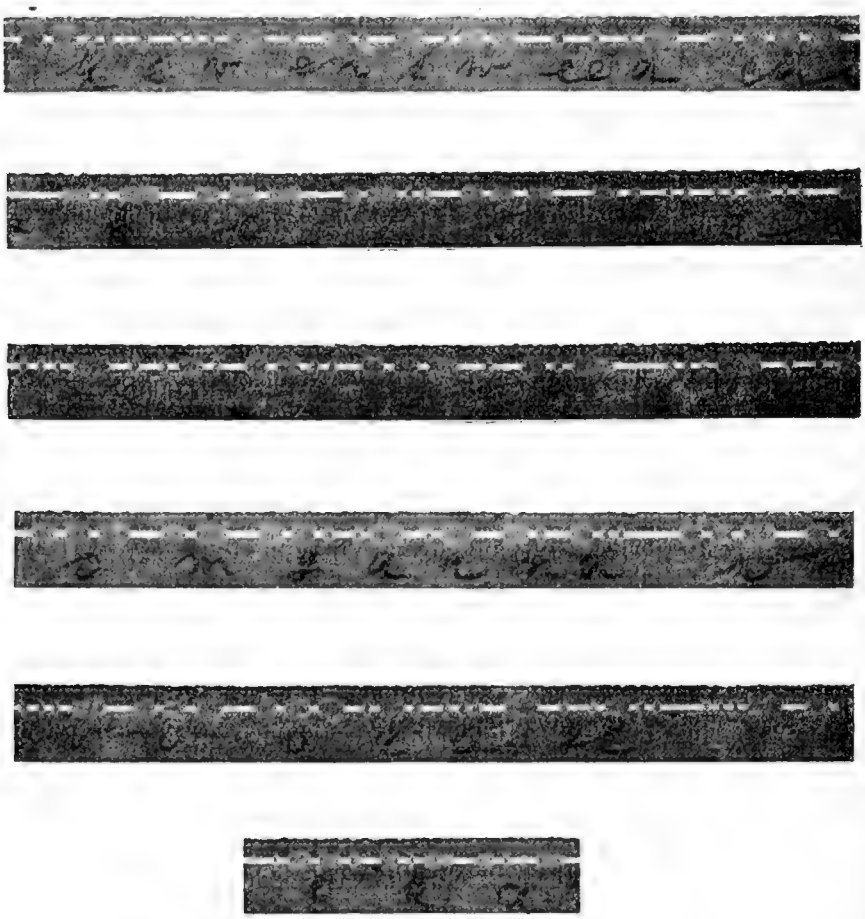

Fig. 5.

Record of signals from the alternator on the Malabar at Bandoeng, made in Leyden January 13th, 1923.

To what extent can the reception by means of the galvanometer stand comparison with the ordinary telephone reception?

In order to answer this question we shall first compare the sensitiveness of the two instruments. The human ear is a very sensitive organ. According to MAX WIEN [1]) it is sufficient to apply to the tympanic membrane an energy amounting to 0.83×10^{-12} ergs

[1]) MAX WIEN. Ueber die Empfindlichkeit des menschlichen Ohres für Töne verschiedener Höhe. PLÜGER's Arch. f. d. ges. Physiol. Bd. 97, S. 1.

per sec. that is a power of $0{,}83 \times 10^{-19}$ watts, in order to produce a sensation of sound; the necessary amplitude of the air waves being a thousand times smaller than the diameter of a molecule. In accordance with this the telephone is capable of responding audibly to very weak currents. The modern telephones, now much in use in wireless telegraphy, which are put in tune with the most favourable note for perception by the human ear, are to be considered among the most sensitive of all existing instruments.

MAX WIEN [1]) states that for the most sensitive telephone under most favourable conditions a power of $3{,}03 \times 10^{-14}$ watts is wanted to produce a just barely audible sound. AUSTIN [2]) indicates a 60 times smaller value viz. $0{,}5 \times 10^{-15}$ watts.

To evaluate the sensitiveness of the galvanometer we suppose that during the reception of a signal a uniform effective electromotive force E be applied to the terminals of the string. At the first moment when the string is still in rest, it will be traversed by a maximum current $I = \frac{E}{W}$, where W represents the ohmic resistance of the string, but gradually the current will decrease by the back-electromotive force which is set up in the string by its movement. If we neglect the internal friction in the string itself and make the vacuum so high, that it may be considered as absolute, the back-electromotive force produced will be equal to E as soon as the end- amplitude is attained. The current flowing through the string then $= 0$. As long as the signal lasts the string goes on oscillating in tune with it without consuming energy.

For evaluating the sensitiveness we have to take the maximum number of watts wanted i.e. $\frac{E^2}{W}$. If the string have a mass of M grams, being in tune with N cycles per sec., and its electromagnetic decrement being δ_{em}, the number of watts wanted to induce an end-amplitude of U cetim. is

$$B = \frac{\pi^2}{2} \times 10^{-7} \times M\, U^2\, N^3\, \delta_{em} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or also

$$B = \frac{4500\, \pi^2\, M U^2}{\lambda^3\, \tau}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

[1]) WIEDEMANN's Annalen 4. IV, 1901, p. 450.

[2]) Jahrbuch der drahtl. Telegr. 11 and 12, 1916. „Conf. also H. O. TAYLOR. Telephone receivers and radio telegraphy. Proceedings of the institute of Radio engineers, 1918, Vol. 6, p. 37.

where λ represents the wave-length in kilometers, and τ the time in seconds necessary to raise the amplitude of the oscillations of the string from 0 to a value $\left(1 - \frac{1}{e}\right)$ of the end amplitude. The speed of transmission admissible is inversely proportional to τ.

The minute amount of energy sufficient to keep the string oscillating with its end-amplitude can easely be evaluated. For the sum of the values neglected previously, can be determined by measuring the decrement of the vibrations when dying away freely.

Denote this decrement with δ_{l+s} then the value to be found is

$$B_1 = \frac{\pi^2}{2} \times 10^{-7} \times MU^2N^2\,\delta_{l+s} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or also

$$B_1 = B \times \frac{\delta_{l+s}}{\delta_{em}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where δ_{l+s} is again supposed to be small in comparison with δ_{em}. This is always the case with a good string, a moderate field and an attainable vacuum. Under the conditions of the figures 2 and 3 we have $\delta_{l+s} = 4 \times 10^{-5}$, whilst δ_{em} with a magnetizing current of 4 amp. attains a value which is 75-times larger viz. 3×10^{-3} and therefore $B_1 = 0{,}0133\,B$.

What value is to be computed for B when use is made of formula (4)? The result depends on the dimensions, especially on the diameter of the string inserted in the galvanometer.

If we take a fine string[1]) with a diameter of $0{,}2\,\mu$, a vibration amplitude of the same dimension will already be visible and suitable to be recorded. We have then $U = 2 \times 10^{-5}$ centim. The mass of a string of the above mentioned diameter and of 1 centim. length may be taken as $M = 2 \times 10^{-9}$ grams. Suppose, moreover, $N = 20.000$, and $\delta_{em} = 0{,}001$, then we find, for the number of watts wanted, that $B = 3{,}2 \times 10^{-15}$. From this we infer, that the sensitiveness of the galvanometer is to be evaluated to an amount of the same order of magnitude as that of the telephone.

The use of such fine strings is attended with certain practical difficulties, so that we prefer to work with strings 5 to 6 times thicker and therefore considerably less sensitive. Moreover the sensitiveness decreases, when the wave-length is shorter and the speed of transmission higher, as may be seen from formula (5).

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[1]) Conf. W. Einthoven, Ueber die Beobachtung und Abbildung dünner Fäden. Pflüger's Archiv. f. d. ges. Physiol. Bd. 191, S. 60.

However, in view of the comparison between string and telephone it may be pointed out, that the maximum sensitivity of the latter named instrument is by no means available in radio practice, for there is a great difference between the intensity of a signal just barely audible and one which is readable.

It will be noticed that we only have compared the power sensitiveness of the galvanometer and of the telephone as such, and that the application of these instruments in combination with the oscillating audion and with low and high frequency amplifiers has been left out of consideration. For the sensitiveness of reception by telephone in combination with the oscillating audion we may refer to the paper of AUSTIN [1]). He mentions that for a just audible signal the absolute sensitiveness of the oscillating audion is $1{,}2 \times 10^{-15}$ watts, that is to say a power, which is about 2,5 times greater than that needed by the telephone as such.

For the practical use of the string galvanometer in radio-telegraphy it is superfluous to try to obtain the greatest possible sensitiveness of the instrument. It is not the sensitiveness which determines its usefulness, since weak signals may be strengthened by means of amplifying vacuumtubes without limit. The efficiency of a receiver is much more determined by its selectivity i.e. its freedom from disturbances.

If we wish to compare the reception by the galvanometer to that by the telephone from the point of view of their selectivities, we must discuss once more the properties of the human ear. As is well known we are able to distinguish by means of hearing many sounds produced simultaneously. If we pay special attention to one of the numerous musical instruments of a complete orchestra, we are able to follow its performance separately. So also the Marconist can distinguish the tone of a signal, although many other sounds or noises of, for instance, extraneous stations or atmospheric disturbances reach him at the same time. This secures for reception by telephone an important advantage over every form of reception which has the object of recording the signal graphically. In the graphical image of a concert of sounds it is extremely difficult to follow the tone which we wish to analyse and often it will be even quite impossible to do so.

But against this disadvantage of the galvanometer there is the

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1) LOUIS W. AUSTIN. The measurement of radio-telegraphic signals with the oscillating audion. Proceedings of the Institute of Radio engineers, 1917, Vol. 5, p. 239.

advantage of a much smaller decrement, and we may ask how far in practice advantage and disadvantage are counterbalanced.

The answer depends on the possibility of deriving the full profit from the small decrement of the receiver. Let us for instance try to receive in Leyden the signals of the present high frequency alternator at Bandoeng. It does not keep its wave of 7,5 kilom. absolutely constant, but according to our measurements the wave varies by amounts of 1 to 2 per thousand. If, by diminishing the field intensity, we decrease the string decrement so much as would be desirable when receiving a constant wave, a signal would only be received now and then, that is to say only at those moments, when the varying wave of the transmitter coincides exactly with the wave to which the string is put in tune. To different wave-lengths the string does not respond, so that the dots and dashes transmitted are not received regularly and the telegram becomes unreadable. We are obliged to increase the string decrement and so to enable the reception of a greater range of variation of the wave-length of the transmitter.

On experimenting we obtained the impression that the reception by telephone of the high frequency alternator of Bandoeng is disturbed by extraneous noises about as much as the reception by the galvanometer. In both cases practically as many signals become unreadable by atmospherics. But we have not yet had the opportunity to carry out exact measurements on this point and it may be noticed, that the difference in skill of the various Marconists, who are carrying on the comparative experiments must also be allowed for.

If the wave transmitted oscillates still more than is mentioned above, the Marconist will obtain the better result, but if it is being kept steady, such as actually is the case with many modern transmitters, then the advantage will pass to the side of the galvanometer. The dots and dashes on the strip of paper will then be like those of fig. 4 and of the upper part of fig. 6.

The slower the rate of transmission so much the smaller the string decrement may be made; the freedom from disturbances becomes improved proportionately and thus the possibility of receiving with the galvanometer increases. On the other hand the Marconist is not able to take advantage of a more constant transmission wave; it is impossible for the human ear [1]) to perceive the minute variations in pitch, to which a string vibrating with a small decrement is capable of responding noticeably.

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[1]) Practically also when the Marconist is applying beat reception.

But it is not only with a slow transmission that the string is superior. If, in relation to the disturbances, the signals are strong enough to make a record of them with a moderate or even a large string decrement, then high speeds of transmission will become possible and soon the Marconist will no more be able to read the signals, while the galvanometer is recording easily a few hundreds of words per minute.

Dr. DE GROOT of Bandoeng, to whom we are much indebted, has suggested a valuable idea. For his enthusiastic collaboration in the difficult experiments carried on at Bandoeng with the galvanometer some time ago, we thank him heartily here.

Dr. DE GROOT has suggested the application of two galvanometers simultaneously when an arc generator be used for transmission; one string may be put in tune with the active wave, the other with the wave of rest. An atmospheric appearing at a given moment may be easily recognized as such, if it influences the registration of only one of the two waves. Thus the possibility of making the signal readable throughout the atmospheric disturbances will become greater.

In fig. 6 a record is reproduced which has been made at Leyden according to the suggestion of Dr. DE GROOT. The string of one

Fig. 6.

Record of the arc generator *FL*, Paris with 2 galvanometers in parallel. The signalling wave is registered by one, the non signalling wave by the other galvanometer.

galvanometer is seen vibrating every time the other is standing still and vice versa. How great the practical value of this method will be has yet to be determined, but the first impressions which we have obtained from the result of a few experiments are favourable.

The idea of using 2 galvanometers simultaneously may find another application when signals are to be received, the wave-length of which is not very constant. Either string may be put in tune with a different wave; one with a wave which is a little longer, the other with a wave, which is a little shorter than the mean length around which the transmission wave is fluctuating. So the admissible range of fluctuation is increased, while the decrement of the vibrations of either string may remain small.

However, rather than applying this, after all, somewhat defective

means, it is better to try to improve the transmitter. As a matter of fact present technique is actually capable of producing transmitters which keep their wave practically constant.

The advantages of the reception by galvanometer in distinction from the reception by telephone are worth mentioning. On transmitting slowly it will be possible to receive signals with the galvanometer, which are not readable by telephone. Every improvement in this direction of the receiving apparatus, which always remains relatively simple, saves, as AUSTIN [1]) observes rightly, large sums needed both for the erection of high power sending stations and for their working expenses. And that, as matters stand at present, improvements are still wanted, is obvious from the many difficulties experienced even with the best installations. To quote an example it may be noticed, that during the whole of July 1921 the communication between two of the Trans-Atlantic stations, which are considered among the most reliable, was so poor that only 23 per cent of the words sent were successfully received [2]).

The high speed reception with the galvanometer makes it possible to take full advantage of the installation at those hours of the day and the night, which are the most favourable for the transmitting of the signals, and to transmit many more words than could be received by telephone. Moreover the secrecy of the telegrams can be better secured since the numerous telephone-receivers will not be able to read the quick signals.

In time of war the interference by a second station will be hindered, when the signalling wave and the non signalling wave of an arc transmitter are received simultaneously with two galvanometers.

Finally we may mention another advantage which bears upon the general use of wireless telegraphy in the world. It is Dr. DE GROOT, who has placed it on the fore-ground. During night and day numerous signals are sent from many hundreds of transmitters. The installations interfere with one another, if they use waves the lengths of which do not greatly differ. The difference in wave-lengths which are applicable for transmitting signals is limited; only these waves are useful, which range neither below nor above a certain length; in other words: the spectrum of the useful waves is comparatively small. Everyone using a part of it takes it away from another man.

[1]) Conf. L. W. AUSTIN, Long distance radio communication. Journal of the Franklin Institute, Vol. 193, Apr. 1922, p. 437 (458).

[2]) Conf. L. W. AUSTIN l.c. p. 443.

The smaller the part of the spectrum he uses, the larger the part which remains for others.

Owing to the small decrement of the galvanometer a wireless installation may be restricted to using a smaller part of the spectrum than heretofore, with the result that it will be possible to increase the number of simultaneously working installations. This increase is badly needed, so we may expect on good grounds, that the galvanometer will be capable of rendering a service to radio communication in general.

We do not finish this paper without rendering our tanks to the many persons who have been ready to help us with our work. Especially we wish to express our gratitude for the interest and the support, which we have received from Mr. Th. B. Pleyte wo was at that time our Colonial Minister.

Anthropology. — "*The Menarche in Dutch Women and its precipitated appearance in the youngest generation*". By Prof. L. BOLK.

(Communicated at the meeting of September 29, 1923).

With the aid of several physicians I have collected a number of data with regard to the menarche in Dutch women, about which nothing was known so far. In collecting these data the greatest accuracy has been observed and in this communication we have only made use of the cases, in which not only the year, but also the month of the first menstruation has been noted. Besides this the colour of hair and eyes of the various subjects had been stated, as I also wished to ascertain through this examination, whether the degree of pigmentation is of influence in the commencement of sexual maturity in the young girl.

Although it is not easy to obtain accurate data, I have succeeded in collecting 1800 reports of non-Jewish women as well as 165 of Jewesses.

On working out this material, several unexpected and surprising results came to light, which I will relate in succession, leaving the data obtained from the Jewesses until the end.

The first question which could be answered with the aid of these reports concerned the age at which the menarche appears in Dutch

TABLE I.

Age	Number	Percentage	Age	Number	Percentage
8 years	2		16 years	121	6.7
9 „	2		17 „	54	3.—
10 „	31	1.7	18 „	25	1.4
11 „	131	7.3	19 „	3	
12 „	302	16.7	20 „	2	
13 „	464	25.7	21 „	2	
14 „	408	22.6	22 „		
15 „	251	13.9	23 „	2	

women in general. It is well known that this age shows great individual variations, and this is also seen in the Table I, in which the actual numbers, as well as the relative percentage, have been stated according to the age.

In fig. I curve *A* shows, in percentages, its appearance at each separate age.

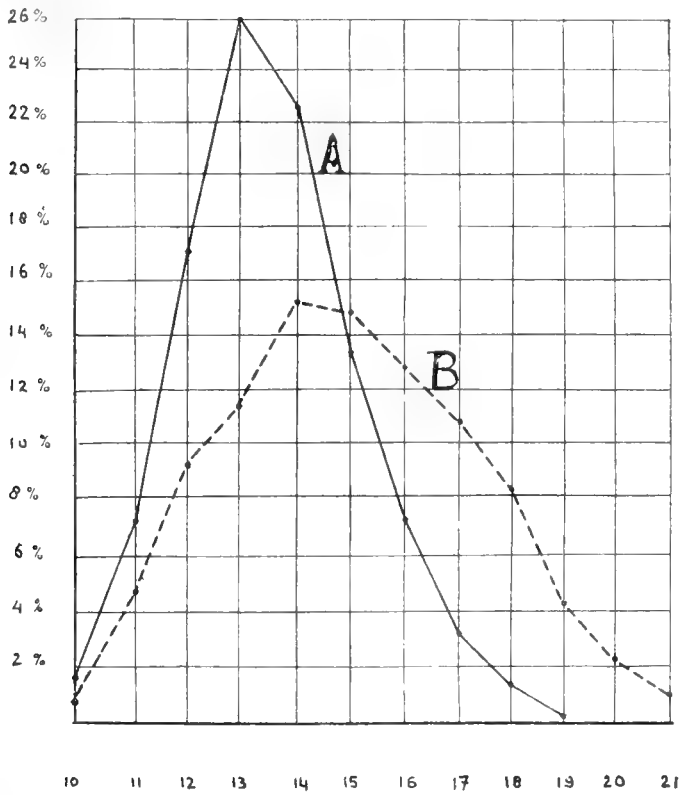

Fig. 1.

From this table and graph it appears that the beginning of the function of the sexual glands varies between the tenth and eighteenth year; it is true that in 4 cases the menses already appeared before the 10th year (8 years 2 mths; 8 y 12 m; 9 y 4 m; and 9 y 12 m), but these cases do not join regularly on to the variability-curve and may be regarded as abnormal precocity.

The varability-curve of the menarche begins as an unbroken line at the age of 10 years and 4 months, mounting continually after this. This mounting during the 10th and 11th year is to be seen in Table II, in which the number of cases per month during these years has been noted. I have inserted this table, as it shows that the earliest age at which the menarche, as a physiological pheno-

menon, begins, is actually the middle of the 10th year, so that when a girl has passed the age of ten-and-a-half years one cannot look upon the beginning of the menstrual process any more as a sign of pathological precocity, at most as a rapid development of the sexual glands.

TABLE II.

Age	Number	Age	Number
10 years 1 month	0	11 years 1 month	3
„ „ 2 months	0	„ „ 2 months	5
„ „ 3 „	0	„ „ 3 „	6
„ „ 4 „	1	„ „ 4 „	7
„ „ 5 „	2	„ „ 5 „	6
„ „ 6 „	3	„ „ 6 „	6
„ „ 7 „	2	„ „ 7 „	9
„ „ 8 „	2	„ „ 8 „	11
„ „ 9 „	3	„ „ 9 „	14
„ „ 10 „	7	„ „ 10 „	19
„ „ 11 „	6	„ „ 11 „	24
„ „ 12 „	6	„ „ 12 „	21

The beginning of the variation-curve in the middle of the tenth year is a sign that sexual maturity in our country can begin at a comparatively early age and the further course of this line confirms this fact, for it mounts rapidly to reach its top in the 13th year.

Sexual maturity made its appearence before the 12th year in 9 % of the girls, before the 13th year in 26 %, and in more than half before the 14th year. The average age of the menarche, taking the months into consideration as well, appears to be 13 years, 9 months and 15 days. If one compares this average with others mentioned in the literature, drawn from the population of Western Europe, then it appears that in our population of the present day the menarche, on an average, begins early.

This commencement, however, is dependent on so many external conditions, that if any conclusions are drawn from a comparison of these averages, this should be done with the greatest care.

As one of the internal influences determining the age of the menarche, the racial factor is usually mentioned. Several authors

deny this influence entirely, others attach great importance to it, which shows how difficult it is to determine whether the race is really of any influence on the menarche, as it also is influenced by other, external, factors, (social surroundings, temperature, soil, etc.).

I do not know of any investigation in which the influence of the race on the commencement of sexual maturity has been actually proved, and this induced me, while collecting the data, to inquire into the degree of pigmentation.

The material was sorted and divided into the women with light and those with dark eyes; these will in future be called "blondes" and "brunettes"; of the former my material contained 1130, of the latter 670.

The appearence of the menarche was worked out statistically for each of these groups separately, the result is seen in Table III and the graphs plotted out from this table have been sketched in fig. 2. in which curve A refers to the Jewesses, curve B to the "blondes", curve C to the "brunettes".

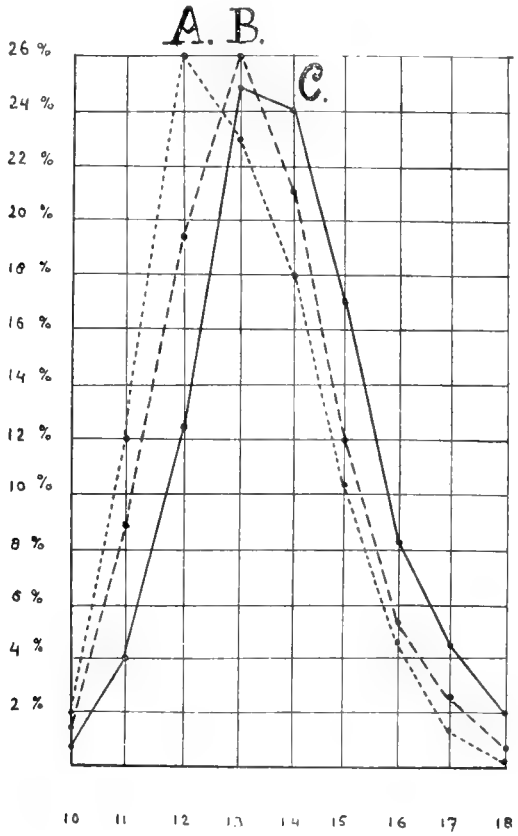

Fig. 2.

The result of this investigation into the relation between menarche and degree of pigmentation was surprising, as it was in contradiction with what one might expect. It is a well known phenomenon that the menarche appears at an earlier age in the dark-coloured races than in the fair ones. Most writers ascribe this to climatic influences, especially to the high temperatures in which the dark-skinned races live.

TABLE III.

Age	Blondes		Brunettes	
8 years	1		1	
9 „	1		1	
10 „	22	1.8 %	9	1.3 %
11 „	104	9.1 „	27	4 „
12 „	220	19.5 „	82	12.3 „
13 „	294	26 „	170	25.2 „
14 „	244	21.5 „	164	24.6 „
15 „	136	12.1 „	115	17.2 „
16 „	65	5.7 „	56	8.5 „
17 „	26	2.3 „	28	4.3 „
18 „	12	1 „	13	2 „
19 „	1		2	
20 „	2		—	
21 „	1		1	
23 „	1		1	

I myself, however, conjectured that the racial factor would be of importance here and that an earlier appearance of the menarche would be a biological characteristic of the more pigmented races.

It appears, however, from the data given in Table III that for the Dutch population the contrary holds good; it is just the fair types which, in comparison to the dark ones, are characterized by an earlier maturity.

The difference is even considerable, for while 56.4 % of the girls of the fair type have come to maturity before their 14th year, this is only the case in 42.8 % of the dark type. As one can, however,

see from the graph and from Table III, the beginning of the variability-curve lies for both types in the 10th year; in the "brunettes" this begins at 10 years 5 months, in the "blondes" at 10 years 4 months; so for both groups what one can designate as the threshold-age of sexual maturity, is the same. After this beginning the curve for the fair type mounts more rapidly than for the "brunettes"; the end of the normal variability, however, is the same for both types, and lies in the 18th year.

The exact difference between both groups appears from the following average figures, which have again been calculated including the months:

Average age of menarche in "blondes": 13 years, 5 months, 17 days; in "brunettes": 14 years, 4 months, 5 days. So this makes a difference of full 10 months between both types.

A difference of this sort, and in a contrary direction to what I had expected, is very remarkable. As we have to do here with two groups of people living in the same circumstances, which excludes external factors which might influence the menarche, this difference must be entirely regarded as the result of an internal factor, and it is only the racial factor which can be taken into consideration here.

The light-eyed component of our population belongs, in general, to the race which peoples the North of Europe, the "Homo nordicus", while the brown-eyed, which constitutes about a third part of the Dutch people, as is proved by a former investigation of mine, belongs to the race inhabiting the centre of Europe, the "Homo alpinus".

It appears, therefore, that a lesser development of the pigment is accompanied by an acceleration of the sexual development. The relation between both phenomena is, however, not so simple; which can be seen from the fact that the average age of the menarche in the more strongly pigmented Jewesses, is earlier than in the "blondes".

The activation of the sexual sphere of the developing individual is dependent on very many factors; and, in considering the difference which has come to light, we must not forget the possibility that the racial factor which is here at work, could be of a psychological instead of a physiological nature. The blonde as well as the brunette girl has reached the threshold-age of maturity on arriving at the age of 10-and-a-half years. (Later on it will appear that this also holds good for the Jewish girl). The time which passes for each individual between this age and the activation of the sexual functions, is determined by a number of external and internal factors, and among the latter we leave room for the special psyche of each race.

Thus far on the average age of the menarche in the Dutch

population in general; I will now proceed to another result of my investigation, which was as surprising as it was unexpected.

It had attracted my notice, while working out my material, that the older people mentioned therein were often characterized by a late appearance of the menarche. This observation gave rise to the question whether the menarche could have undergone some change during the last decades, in such a manner that sexual maturity in the youngest generation begins, on an average, at an earlier age than in the former generations. I have tried to find an answer to this question in two ways. In the first place I collected from my material data referring to persons born before 1880, and calculated from these the average age of the menarche. Secondly I tried to obtain data relating to the menarche in mother and daughters. Especially this last is difficult, considering the fact that only a very few of the women can actually mention the year of the menarche, much less the month. Yet I have succeeded in collecting a number of such data.

Both ways led to the same result, viz. that the menarche in what we may call the youngest generation, as regards sexual maturity, arrives at a considerably earlier period than formerly. I will return to the cause and significance of this phenomenon after communicating the pure facts.

Let us begin with the menarche in women born before 1880. In my material concerning them there were 98 data of the menarche according to year and month, and furthermore I possessed 104 cases in which only the age was mentioned. These 232 cases have been systematically arranged in Table IV, and curve *B* in fig. I gives the direction in percentages for each age.

If one compares Table IV with Table I, the following will be seen: the beginning of the variability-curve lies, for women of the

TABLE IV.

Age	Number	Percentage	Age	Number	Percentage
10 years	2	0.8	17 years	27	11.2
11 „	12	5.—	18 „	19	8.1
12 „	21	9.—	19 „	10	4.3
13 „	27	11.2	20 „	5	2.2
14 „	37	15.9	21 „	4	1.3
15 „	35	15.1	22 „		
16 „	31	13.3	23 „	2	0.8

older generations, also in the 10th year. This fact confirms the opinion already mentioned above, that the middle of the 10th year is the physiological threshold-age of sexual maturity in woman.

Opposite to this very constant starting-point of the variability-curve stands the most changeable ending-point. This falls in the older generations in the 21st year, in contrast to the 18th in the younger generation. The top of the curve, which in the latter individuals lies in the 13th year, has been shifted to a higher age in the older generations and lies in the 14th and 15th year.

From this it already appears that formerly the phase of sexual latency, after crossing the threshold-age, lasted considerably longer in a great many girls than nowadays. This also follows from the fact that, as shown in Table I, more than 50 % of the youngest generation menstruates before the end of the 13th year, while of those born before 1880 this was only the case in 26 %.

During the last 40 years, therefore, the period of the menarche has gradually become earlier, and how much earlier can be learned from both the following averages. The average menarche of the persons worked out in Table I (fig. 1, curve *A*) of whom the greater quantity was born between 1897 and 1906, is 13 years, 9 months, 15 days; while the mean age of the first menstruation in the persons born before 1880 (fig. I, curve *B*) is 15 years, 3 months, and 20 days. From this it follows that in the last decades the menarche arrives a year and a half earlier than formerly.

I must point out, in passing, that the last mentioned average more resembles those found in literature regarding the West-European population, which depend on investigations of an older date.

A second manner in which the earlier appearance of the menarche has been proved, is the comparison of the age of the menarche in mothers and daughters. I arranged these data in two groups; in the first I collected the data in which the age of the menarche was accurately known, even up to the month, for both mothers and daughters. To this group belong 45 mothers and 71 daughters. The second group contains the data in which only the year could be mentioned; here there are 56 mothers and 82 daughters.

It seems to me of interest to discuss the data of the first group more extensively, as one or two remarks must still be made about them; they may be seen in Table V, in which the data have been arranged according to the menarche-age of the mother.

From this table follows, in the first place, that of 71 daughters the menstruation of 52 begins at a younger age than in the mother, though, as remarked already, also in the older generation

it was not a rare thing for the menses to begin at the early age of 11. The average age of the menarche of the mothers was 14 years, 9 months, and 25 days; and of the daughters 13 years, 7 months, and 1 day, which means that in one generation the menarche has precipitated with fourteen-and-a-half months. That the difference found here is not so great as what we find on comparing the menarche of women born before 1880 with those born about the beginning of this century (one-and-a-half years), can perhaps be explained by the fact that among the former there were persons of a much older age, and the process of precipitation of the menarche is presumably already longer at work.

The appearance of the menarche in the youngest generation, $14^1/_2$ months earlier than formerly, as found in Table V, almost coincides with the results of the second group of mothers (56) and daughters (82), of which only the year of the menarche could be mentioned. Here the mothers were, on an average, 15 years, 1 month, and 3 days old, and the daughters 13 years, 10 months, and 15 days; that is again a difference of $14^1/_2$ months.

These results undeniably prove the considerable precipitation of the function of the sexual glands during the last decades; for although the figures of this earlier appearance of the menstration may vary a little, one can fix the average at about 14 months.

This is a fact of great importance, highly interesting as physiological phenomenon, and of not less great significance from a social and paedagogic point of view. For the appearance of the menarche 14 months earlier, means to say a shortening of childhood with this period, an earlier activation of the sexual sphere in the present generation, compared to the former. Much of what the attentive observer and listener sees and hears in modern social life is explained by this earlier awakening of the consciousness of womanhood. This is, however, not the place to enter into this question further.

Extensive speculations as to the cause of this phenomenon will not be given here; I will restrict myself to a few general remarks. In the individual process of development of woman the first menstruation is an event of more than ordinary significance; with the commencement of sexual maturity far-reaching changes take place in the general physiology of her development. And if this process makes its appearance considerably earlier this must be looked upon as the expression of a hastened process in her development. Now in the first place the question arises: have we to do here with a symptom of an accelerated development in general, or is it an independent phenomenon? Without special investigations this question cannot be

answered. One would have to examine whether other signs of development are accelerated in the phase before the menarche, e.g. the growth, changing of teeth and such like. The developmental phenomena after the menarche cannot be counted of course, for

TABLE V.

Mother	Daughter	Mother	Daughter	Mother	Daughter
9.12	10.3	13.7	13.4	15.6	14.2
	10.5		12.8		12.6
11.6	11.9	13.7	12.7	15.9	11.11
11.8	11.6	13.7	13.1		12.8
11.9	11.10		12.3	15.12	11.10
	11.11	13.8	14.11	16.2	13.11
	15.2	13.10	11.10	16.4	12.11
11.10	14.12		12.12	16.6	16.1
11.10	11.10		11.11	17.1	13.6
12.7	14.10	13.11	13.10		12.8
12.9	10.6	14.1	10.8	17.4	14.9
	12.7		11.11	17.7	16.3
13.1	12.8		15.9		16.6
13.1	15.11		16.11	18.1	13.8
13.2	12.5	14.2	13.11		13.3 (12.12) [1]
	14.9	14.3	12.6		13.9
	16.11	14.4	13.11	18.4	14.2
13.3	11.12	14.5	11.6		13.5
	11.7	15.1	15.2	18.10	16.6
13.4	13.2	15.2	13.3	18.11	15.3
	13.1	15.3	14.4		14.11
	13.12		15.12	19.4	17.8
13.5	11.3	15.6	14.10		17.9
13.6	11.11			19.8	15.4

[1]) Grandchild.

then the development also undergoes the influence of the ovarial function. That this latter should have a retarding influence on the growth of the girl is doubtful, considering the fact that the full-grown daughters of the youngest generation generally surpass their mothers in height.

A second question concerns the cause of the phenomenon; is this early appearance of the menarche a reaction on external stimuli, or is it a primary change in the developing process? That we should have to do with a primary biological phenomenon, with the effect of an internal cause, is doubtful. I cannot imagine that an internal factor could, as it were suddenly, so hasten a developmental phenomenon as appears to be the case in the menarche. If this was an individual phenomenon, an exception, this could be possible, but it is a general thing, which makes it necessary to accept some external influences as cause. I will not enter into speculations as to what these are, but will close this part of my communication with a last remark.

The question can be raised whether, in this considerable precipitation of the menarche, one has to do with a phenomenon which falls beyond the limits of normal physiology. I cannot ascribe such a significance to it, and may venture the following idea. I have on purpose often drawn the attention to the fact that in all the groups which I examined (brunettes, blondes, jewesses, older and younger generations), the variability curve of the menarche begins at $10\frac{1}{2}$ years; that is the threshold-age of sexual maturity. In every girl who has passed this age the sexual sphere can be awakened, though in the one it remains latent longer than in the other. The duration of this period of latency is determined by hereditary factors and by external circumstances. While the part determined by the former is an unvariable one, that dependent on external circumstances is on the contrary very variable. It depends on and changes with the external conditions of life, with the mode of living, nature of food, temperature etc. Whether it is advantageous for the individual or not that the sexual sphere is awakened early under the influence of those circumstances, is a question difficult to answer; but its activation after having once crossed the threshold of maturity, falls within the limits of the physiological norm.

The time of activation of the sexual functions is, as just remarked, dependent on hereditary and external factors. The material I have collected enables me to furnish a proof for both influences.

The significance of the heriditary factor has already been shown by comparing the average age of the menarche in blondes (Homo

nordicus) and in brunettes (Homo alpinus). A still more convincing proof can be drawn from Table V, for this table shows that if the menarche appears at an early age in the mother, this is, on an average, also the case in the daughter. I have on purpose arranged the data in this table according to the age of the mother.

A simple calculation shows us the following: the average age of the menarche of those daughters, whose mothers began to menstruate in the 11th, 12th and 13th year, is 12 years and 10 months; of the mothers whose first menses appeared in the 14th, 15th and 16th year, the daughters were, on an average, 13 years 7 months old, and finally this mean age was 14 years and 11 months in those daughters whose mothers first menstruated in their 17th, 18th, or 19th year. These ages prove that a retarded menarche in the mother is inherited by the daughter.

Among the external factors which are of influence on the menarche, the temperature, as has been remarked already, is regarded as being of great significance. This opinion was, up till now, only grounded on the fact that the menarche arrives at an earlier age in the population of a warmer zone than in that of a colder climate. Now I can prove from my investigation that this external influence can be demonstrated even in the population of our country. I put the question whether the menarche appears with equal frequency in the different months of the year; and it became clear that this is not the case. The frequency-curve of the menarche, arranged according to the months of the year, has a most typical direction, as may be seen from Table VI. In this table the frequency for each month is expressed in percentages of the whole.

TABLE VI.

January . . . 8.2 %	May . . . . 10.8 %	September . . 6.9 %
February . . . 5. „	June . . . . 10.— „	October . . . 6.2 „
March . . . . 7.— „	July . . . . 9.5 „	November . . 7.8 „
April . . . . 8.8 „	August . . . 10.9 „	December . . 8.6 „

This table shows that a first menstruation appears more frequently during the warmer months (May, June, July, and Aug.) than during the rest of the year; for the total frequency during these 4 months is 41.3 % to 29.5 % during the first and 29.7 % during the last 4 months of the year.

The monthly course, however, is somewhat more complicated. Besides the greater frequency during the summer months there is

another rise in December and January. I should feel inclined to explain this monthly difference in the following way: Beginning with February I should like to regard the rapid and regular rise up to May as a reaction on the general climatological factor, the influence of awakening nature, and not so much as an influence of temperature, which seems to me in these months not capable of doubling the frequency in May, compared to what it was in February. I would then be inclined to see an influence of the temperature in the fact that during the actual summer months the frequency remains almost equal to what it was in May. The rise of frequency in December and January can perhaps be looked upon as the result of the artificial higher temperatures to which the organism is subjected.

As has been mentioned in the beginning of this communication, I have also been able to collect the data of 165 Jewesses, referring to the age of the menarche. Naturally these almost entirely relate to inhabitants of large towns. The following Table VII gives a survey of the frequency, according to the age of the individual, in absolute figures and in relative percentages, which are made clear by curve *A* in fig. 2.

TABLE VII.

Age	Number	Percentage	Age	Number	Percentage
9 year	1		14 year	30	18.1
10 „	3	1.8	15 „	17	10.3
11 „	20	12.1	16 „	9	5.4
12 „	43	26.—	17 „	2	1.2
13 „	39	23.6	18 „	1	

The following remark must be made with regard to this Table. In the 3 cases arranged under the 10th year, the first menses appeared in the second half of this age (10 years, 7 months; 10 years, 9 months; and 10 years, 11 months). The variation curve of the menarche begins, therefore, in the Jewish girls at the same age as in the non-Jewish. It is true there was one case in which the menarche already began at the age of 9 years, but this case (9 years 1 month), is separated by an interval of a year and a half from the following, and must therefore be regarded as a sign of abnormal precocity. For the Jewish race also, therefore, the middle of the 10th year counts as the threshold of sexual maturity. I would again

emphasize the fact that we have been able to demonstrate this age in different groups. In this manner a criterion has been given to determine in each separate case whether one has to do with a real premature development, or with a normal, though perhaps rapid one. A menarche after the age of ten and a half years is a normal event. As far as the threshold-age of maturity is concerned there is no difference between the Jewish and the non-Jewish girls. And yet there is a difference, viz. the greater frequency of the menarche immediately after the threshold has been crossed, so that before the age of 12 the sexual function has begun in 40 % of the Jewish girls compared to 30 % in the non-Jewish blondes, and 18 % in the brunettes.

It is very curious that after this rapid rise in the variability curve, through which the top is already reached at the age of 12, the variation line descends very slowly. Next to a group with accelerated sexual development comes a second with a retarded one. The result is, of course, that the average age of the menarche in Jewish girls is not much earlier than in non-Jewish individuals; for among the blondes I found a mean age of 13 years, 5 months, and 17 days, while for the Jewish girls the average was 13 years, 3 months, and 24 days.

The averred precocity of the Jewish girls compared with the rest of the population, seems, therefore, not to exist, for the slight difference which can be discerned by the above methods, is sufficiently explained by the fact, that the data of the Jewesses, with the exception of a few, refer to inhabitants of towns. I can, therefore, on the ground of my investigation, agree with FISHBERG's conclusion that precocity is not a characteristic of the Jewish race. [1])

[1]) M. FISHBERG. "Die Rassenmerkmale der Juden." München 1913.

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXVI

Nos. 9 and 10.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vol. XXXII).

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Biochemistry. - "*Researches on the Metabolism of Milch-cows suffering from Acetonemia*". By Prof. B. SJOLLEMA and Miss J. E. VAN DER ZANDE. (Communicated by Prof. C. EYKMAN).

(Communicated at the meeting of September 29, 1923).

It does not unfrequently happen that in milch-cows acetonemia reveals itself a few days after parturition. Then the animals become extremely emaciated within a few days; the milk-yield decreases considerably; they give off a smell of acetone and their appetite is largely diminished. As a rule they recover after a short time, and very soon when put out to grass. The examination of the urine, the blood and the milk of more than twenty milch-cows suffering from typical acetonemia showed us that the urine of these animals often contained from 10 to 13 grms of acetone-bodies per liter. In many cases the blood contained 0.6—1 grm of these substances per liter, while the content in the milk was about half the amount in the blood. These results point to an abnormal fat-metabolism, for the acetone-bodies result mainly from abnormal metabolism of the fats [1]), the alkali-reserve of the blood was in serious cases lowered to $^4/_5$ or $^3/_5$ of the normal value. The determination of the glucose-content of the blood shows that hyperglycemia was absent. Sugar was never found in the urine. So the sugar-metabolism was in no way abnormal. The acidosis, brought about by the acetone-bodies, caused a rise of the calcium- and the ammonia-content of the urine. The disturbed fat-metabolism, was not attended with lipemia. The total content of lipoids and of fat in the blood was not or little higher than normal. This rise was chiefly due to hypercholesterolemia. Instead of about 0,1 % we found namely about 0,2 % of cholesterol in the blood-plasma. The lipoid-phosphoric acid did not seem to have increased.

Basing ourselves on the formula that expresses the border-value

[1]) GEELMUYDEN's hypothesis (Erg. d. Physiol. 1923), that acetone-bodies are normal intermediate products from the conversion of fat into sugar, may be considered highly debatable.

for the relation between ketogenic and antiketogenic substances [1], (SCHAFFER, HUBBARD and WRIGHT) we are in a position to calculate from the obtained data (from which is also deducted that the animals consumed about 375 grams of protein) that a cow must metabolize about 1 k.g. of fat before this border-value is reached. With an ordinary diet normal cows oxidize only little fat. The above relation is then far above the border value. If the animals, as was often the case in our experiments, secrete about 120 grms of acetone-bodies a day, more than a kilogram of fat must be metabolized. So while the animals then ingest little fat with their food, about one kilogram of body-fat is burnt daily. It is evident, therefore, that in the case of acetonemia one of the organs concerned in the fat-metabolism must be seriously interfered with in its function.

The simplest way to account for this is to consider the liver as the etiological factor, as in experiments with ECK's fistula and with the reversed ECK's fistula acetone-bodies are formed in the liver. [2])

This view is supported by different observations on the diminished activity of the liver during pregnancy (N.B. acetonemia in cows manifests itself a few days after parturition) and on the abundance of fat in the liver of cows shortly before parturition.

That the disturbance regards only the function, is proved by the speedy recovery when the animals are sent to grass.

It may also be conceived that abnormally large mobilization of fat is the primary anomaly which is controlled from another organ than the liver.

That milch-cows do not easily secrete such large quantities of acetonebodies as were found with acetonemia, was evident e.g. from our experiments with cows that we allowed to fast after some injections of phloridzin (which engendered glucosuria). Indeed, some acetone occurred in the urine but only little.

Neither were the quantities of acetone-bodies considerable in the urine of cows that, on account of indigestion or for some other reason (foot- and mouth-disease) ingest little or no food.

In a diabetic cow we found the same. Although the urine contained for a considerable time from 3 to 4 % of glucose, the amount of acetone-bodies was normal or scarcely higher.

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[1]) Recent researches have shown that the border value for the healthy organism may also be taken for the organism with disturbed metabolism.

[2]) Of course these experiments do not prove that in no other parts of the organism acetone-bodies may be formed. There is this against them that their conclusiveness is greatly diminished owing to the radical measures taken, and consequently to highly abnormal circumstances.

From the wide ratio between the intake of carbohydrate and that of fat in normal cows it is clear that in milch-cows secretion of acetone takes place only with a very abnormal metabolism.

Our researches go to show that in milch-cows suffering from acetonemia waste of body fat takes place on a large scale, often about 1 kilogram daily. Lipemia, glucosuria and hyperglycemia do not occur. The total quantity of acetone-bodies amounts to about 120 grms. per day. The cholesterol-content of the blood is 50 to 100 % higher than normal sometimes even more. The alkali-reserve has decreased. It is probable that the disturbed fat-metabolism is caused by intoxication of the liver.

From the Chemical Laboratory of the Veterinary University at Utrecht.

Palaeo-botany. — *"Etapteris Bertrandi Scott, a new Etapteris from the Upper Carboniferous (Lower Coal-Measures) from England, and its bearing to stelar-morphological questions."* By O. POSTHUMUS. (Communicated by Prof. J. W. MOLL.)

(Communicated at the meeting of October 27, 1923).

Remains of this plant have been found in a coal-ball from Shore, Lancashire; only the petiole is known, of which a series of transverse sections has been cut by J. LOMAX. Of this series 3 sections are present in the Palaeo-botanical collection of the Mineralogical-Geological Institute of the Groningen University (N°. 140—142); besides I have seen 6 other sections in the collection of Dr. SCOTT in the British Museum (Natural History) in London (N°. 2835—2840). The species has been mentioned by Dr. SCOTT in his catalogue of the collection as Etapteris Bertrandi, and is distinguished, as he remarks, from the other species of the genus by the well developed sinus in the xylem of the vascular bundle of the petiole.

The sections in the Groningen collection, though less in number, show some features which are not present in the British Museum specimens, and enable us to form an opinion of the relation of the species to its nearest allies.

The following description is chiefly derived from the sections present in the Groningen collection.

The order of the sections is 140—141—142; I cannot give with certainty the exact place in the series of the British Museum sections, but of the series the end is in Groningen. They are all transverse sections of the petiole, which is about $2^1/_2$ mm. thick.[1])

The epidermis is wanting; it could not be made out whether assimilating tissue with intercellular spaces had existed under the epidermis, but it is unlikely from analogy with allied species. Under these missing layers we find sclerotic tissue: thick-walled cells with a narrow lumen without intercellular spaces. In its innermost part the thickness of the cell-walls decreases and the lumen is wider. The inner cortex consists of thin-walled parenchymatous tissue without intercellular spaces; it is only preserved at the extremities

[1]) The other dimensions are shown in the microphotographs which are enlarged 45 times.

of the vascular bundle near the pinna-bar; it contains scattered cells, slightly larger than the others and with a black content. In the space caused by the destruction of the inner cortex, the pigment derived from these cells, is also scattered.

The tissue surrounding the vascular bundle has been partially preserved with it. It is thin-walled without intercellular spaces; its elements, though often very indistinct, possess a narrow lumen; they are more clearly shown in some places near the pinna-bar; there the peripheral elements seem to be smaller in size than the inner ones; this tissue may be considered as phloem. It is separated from the inner cortex by a continuous double layer of tangentially elongated cells, the endodermis.

The arrangement of the xylem-tissue of the vascular-bundle in the petiole is characteristic. Its structure is in agreement with the symmetry of the petiole and its appendices. The pinnae are placed in alternating pairs, their position to the petiole is similar to that of a leaf to an erect branch: their upper side is turned towards the petiole.

A pair of pinnae is symmetrical to a plane going through the axis of the petiole and passing between the pinnae.

The vascular bundle is symmetrical to the same plane. The structure at one end of the vascular bundle will be found at a higher or lower level to be on the opposite side. This is caused by the alternation of the pairs of pinnae. It is evident by comparing analogous structures at one end with those at the other side, that the pairs of pinnae had not quite alternated, but approached the subopposite position, often also present in the fronds of existing Ferns.

In section 142 the pinna-bundles are clearly shown, passing the cortex and lying halfway between the periphery and the vascular bundle. They are surrounded by an endodermis. The xylem-tissue is nearly round, with the narrower elements (protoxylem) lying at the inner side. The outer row of trachieds seems not to be fully differentiated yet. When followed in their downward course, the two pinna-bundles fuse, thus forming the pinna-bar, a tangentially elongated reniform bundle, with two protoxylems at its inner side. This bundle is seen at different levels in section 141, 140 and 142. At a somewhat lower level it becomes more flattened, approaches the petiolar bundle and its endodermis fuses with that of the petiolar bundle. The xylem of the latter shows in transverse section the H-form, so characteristic in this genus. From a middle band, the apolar, which is slightly thickened in its middle part and consists

of relatively large elements, two arms, the antennae, are given off at each side; they are slightly recurved and prominent at the outer side at their insertion into the apolar. Thus a more or less well developed sinus is formed. The endodermis but slightly incurves on both sides of the vascular bundle.

When followed in its downward course, the pina-bar fuses with the petiolar bundle; the ends of the xylem of the pinna-bar fuse with the two prominences on both sides of the sinus (N°. 140). Thus an elliptical mass of parenchymatous, or at any rate thin-walled tissue, is enclosed. At a lower level, as seen in section 141, the pinna-bar has wholly fused with the petiolar bundle; the enclosed parenchyma has diminished in size, especially in breadth. The peripheral loop, the downwards prolongation of the pinnabar has diminished in thickness and is but a few elements thick in its middle part.

At a still lower level its continuity is interrupted; now on the surface of the rather flat xylem a deep sinus is seen, which is bordered on both sides by prominent ridges of tracheides. These become more rounded at a lower level, and the original condition is reached again.

The continuity of the peripheral loop which is formed by the fusion of the pinna-bar with the petiolar bundle occurs in 2 of the sections of the Groningen collection. It is not shown in the London specimens. But in these the well developed sinus is clearly shown; in this feature they differ much from the other species of the genus. It is on these grounds that Scott distinguishes in his Catalogue this form from the other species; it is shown here that the deeper sinus is not an independent character but caused by the fusion of the pinna-bar, when still continous, with the petiolar bundle; a feature which is aberrant from that usual in the genus.

If one tries to make a stereometrical model of this structure, the result is shown in fig. 4. In the other species of Etapteris e.g. E. Scotti Bertrand, the pinnae-bundles are also placed in pairs and fuse on their downward course in the cortex. But at a slightly lower level before their fusion with the petiolar bundle, the pinna-bar is split up, and the two bundles resulting from this division fuse independently with the vascular bundle of the petiole. An amount of parenchyma is thus never enclosed by the fusion of the petiolar bundle with the vascular tissue coming from the pinnae. That this difference with the features in E. Bertrandi is but a relative one is shown by comparing the model of the structure of E. Scotti (fig. 5) with that of the former species. Here we see the pinna-bar

fusing with the petiolar bundle. At a somewhat lower level the continuity of the peripheral loop formed by this fusion is disturbed. The interruption thus formed is limited on both sides by the downward continuation of the halfs of this peripheral loop. The xylem of the next pinna-bar fuses with the two ridges at its extremities.

In E. Scotti we see the pinna-bar approaching the petiolar bundle too. But just before its fusion with the latter it is split up in its middle part; thus two separate bundles are formed, which fuse with the petiolar bundle. We see here the same fusion with the petiolar bundle and the same interruption in the pinna-bar; but in E. Bertrandi the highest point of the interruption is below the fusion of the pinna-bar with the petiolar bundle and in E. Scotti it lies above this point.

The interruption, the height of which is different in these two species, is always limited below by the next pinna-bar. It lies above the insertion of the pinna-bar. The relative length of the interruption to the distance between two pairs of pinnae determines the condition of the transverse section. In E. Scotti the distance between two successive pairs of pinnae is but small, often the bundles of two pairs of pinnae are shown on the same side in one and the same transverse section.

Thus the structure of Etapteris Bertrandi Scott enables us to explain the features in other more complicated species of Etapteris. On the other hand it has many points in common with simpler forms, e.g. Diplolabis Römeri (Solms) Bertrand. In this plant an interruption above the insertion of the pinna-bar is present too.

If the petiolar bundle is followed here in its downward course, which Gordon's [1]) researches enable us to do, it can be shown, that the lowest pinna-bar encloses at its inner side an amount of parenchyma by the fusion of the pinna-bar with the two sides of the interruption. At a lower level the two protoxylems which are situated on both sides of the parenchyma fuse. The parenchymatous tissue diminishes in size and ends blind below.

But throughout its course to its lowest point it is in contact with the protoxylem; it seems as if the lowest part of the parenchymatous tissue follows the course of the protoxylem when penetrating into the tracheides of the metaxylem.

It is remarkable that in these plants the protoxylems are always associated with parenchyma except in the lowest part; this parenchyma, or at any rate thin-walled tissue, is situated at the adaxial

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[1]) W. T. Gordon, 1911.

side of the protoxylem. If we assume that the protoxylem was originally wholly immersed in the metaxylem, but that afterwards the development of tracheidal elements has been arrested at the inner side, except in the very lowest part, we can explain the existence of the interruption above the insertion of the pinna-bar. For when the pinna-bar approaches the petiolar bundle and fuses with it, the parenchymatous tissue at its adaxial side is enclosed. The parenchyma associated with the protoxylems of the next pinna-bar approaches in its downwards course the peripheral loop formed by the pinna-bar next above, and as the development of the procambial cells into tracheids has been arrested, a break is formed in the loop. Through this interruption the parenchyma at the inner side of the pinna-bar is connected with that enclosed by the fusion of the pinna-bar next above with the petiolar bundle. The parenchyma which is enclosed and that which lies in the sinus is formed by the fusion of the strands of parenchyma lying adaxially to the protoxylems of successive pinna-traces. These interruptions in the peripheral loop show some resemblance to the leaf-gaps in the stele of many Ferns. Here, too, parenchyma situated adaxially to the protoxylems of the leaf-trace penetrates into the xylem of the stem, either connecting the softer tissue in the interior of the stele with that without, or hollowing the xylem of the stem by the fusion of these parenchymatous formations of successive internodes. In the first case a little strand of parenchyma, ending below blindly, can be found some distance below the insertion of the leaf-trace; in the other case this funnel in the xylem is absent. The parenchyma enclosed inside the peripheral loop may be compared with the pith, formed after the second method, but the connection of the successive parenchyma-strands of successive pinna-traces is not caused by reduction in tissue which was present before (in phylogenetical sense). This structure, caused by the peculiar symmetry of the bundle, is present on both sides.

This species agrees in the form of the antennae with E. Scotti Bertr.,[1]) but differs from it by the simpler structure of the pinnae-bundles, its smaller dimensions, and the more scattered position of the idioblasts in the inner cortex. It differs from E. shorensis Bertr.[2]) by having another form of the apolar. In this species the continuity of the pinnabar is maintained for a rather long distance, but the presence of a peripheral loop has not yet been noted. A continuous

[1]) P. Bertrand, 1909, p. 140—147, 209, pl. XVI, fig. 111, 112.

[2]) P. Bertrand, 1911, p. 30—38, pl. II, fig. 23—31, 34, 35.

peripheral loop however has been found once in E. Tubicaulis Göppert sp.[1]) from Lower Carboniferous strata of Silezia, but in many other respects it is very different from the species under discussion. Perhaps E. Bertrandi may turn out to be really a portion, e.g. the highest portion of the petiole, never before observed, of some species already known, e.g. E. Scotti or E. shorensis. By its aberrant structure however it seemed to me desirable to describe this form.

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EXPLANATION OF THE PLATE.

Fig. 1—3. Etapteris Bertrandi Scott. Transverse section of the petiole; N°. 140, 141, 142 respectively.

Fig. 4. Etapteris Bertrandi Scott. Model of the xyleem tissue of the petiolar bundle (the sides of the sinus are too sharply accentuated).

Fig. 5. Etapteris Scotti Bertrand. Model of the xylem-tissue of the petiolar bundle.

[1]) P. Bertrand, p. 206.

Groningen. *Botanical Laboratory.*

O. POSTHUMUS: "Etapteris Bertrandi Scott, a new Etapteris from the Upper Carboniferous (Lower Coal-Measures) from England, and its bearing to stelar-morphological questions".

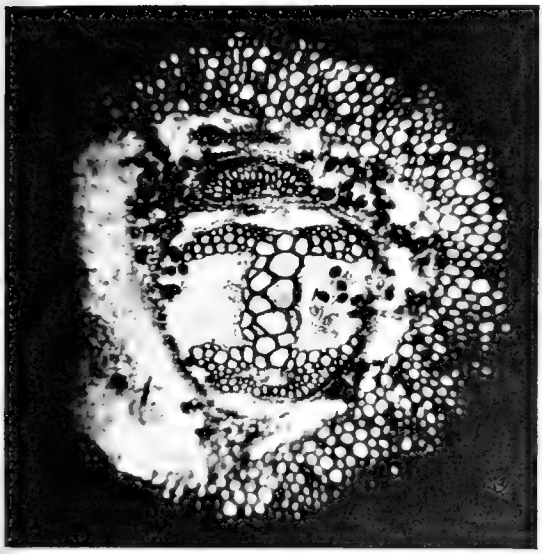

Fig. 1.

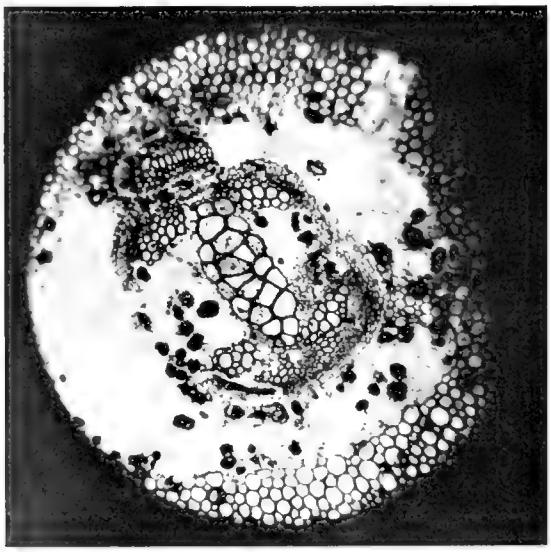

Fig. 2.

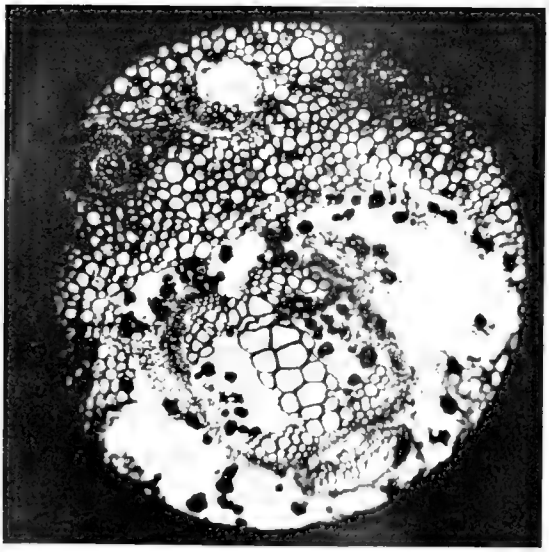

Fig. 3.

Fig. 4.

Fig. 5.

Chemistry : „*The coagulation of Hevea latex*". By Dr. O. DE VRIES. (Communicated by Prof. P. VAN ROMBURGH).

(Communicated at the meeting of January 27, 1923).

I. *Influence of the mixing-proportion of latex, water and acid, irregular series.*

It was known from previous investigations, that the coagulation of Hevea latex with acids shows irregularities. The observations of several investigators, which we intend to discuss shortly in one of the following paragraphs (§ 9), had only been made occasionally, and did not give a sufficient insight into the phenomena; therefore it seemed desirable to us to obtain a total view of the proportions, by a systematical investigation into the complete range of mixing of latex—water—acid.

§ 1. *The Latex.*

Hevea latex is a milky liquid, which, under the microscope, appears to consist of oval globules, $\frac{1}{2}$ to $2\,\mu$ in size, and showing a vivid Brownian movement; particles of less simple form occur now and then. The fact, that one has not to deal with globular particles, shows that latex is not a system liquid : liquid, an emulsion in the sense of Wo. OSTWALD's classification. On the other hand, one should not speak of liquid : solid (suspension); the properties of the coagulum obtained under various circumstances, make it probable that the rubber-particles in latex have a buttery consistency, i.e. between liquid and solid. If we have to look upon this as a more or less liquid nucleus, enclosed in a more solid superficial skin, as some investigators assume, is a matter we do not intend to discuss here. If we apply FREUNDLICH's classification of the colloids to latex, then undoubtedly it is a lyophilic colloid, as shown by the hydrous voluminous gel, obtained on coagulation, and by the behaviour of the latex with regard to dehydrating and salting-out substances; on the other hand, the hydrating power of the rubber-globules is decidedly only limited, and the latex, as regards its behaviour towards mono- bi- and trivalent anions, is strongly reminiscent of lyophabic colloids. So in this classification as well, latex

occupies an intermediate place. Moreover, the rather complicated properties of the latex may be understood, if we bear in mind, that it is a vegetable juice, in which besides the rubber-carbohydrates, also proteins, resins and other colloids play a part, and in which each in its turn may come to the front.

The composition of Hevea latex is not constant. The quantity of rubber and the quantity of secondary constituents depend on several factors, which cause changes in the physiological condition of the tree; moreover the tapping-system has a great influence. Besides we have to bear in mind, that after tapping the acidety of latex principally by bacteriological transformations, increases, even to such an extent, that after twelve hours "spontaneous" coagulation sets in.

If, however, circumstances are carefully chosen, it is an easy matter, to get a regular daily supply of latex of a certain composition. For that purpose one has to be restricted to a certain group of trees, from which, according to a certain tapping-system, latex is gathered daily, which moreover is always treated in the same way. The only remaining changeable factor, the meteorological circumstances, are then immaterial, if one keeps separate the latex of those days, on which in the morning the trunks were still wet, after nocturnal rains, or on which the latex gets drenched by an early shower.

We could, by taking these precautions very carefully, obtain quite sufficiently constant results, in the coagulation-experiments to be described here, with series of observations covering several weeks.

If, however, later on, one reverts to such observations with latex of a different group of trees, or a different tapping-system, the quantitative data do not correspond exactly any more, though the general course of the phenomena remains the same. In § 8 we intend to give a few examples of the differences caused thereby, and also of the influence of the gradually increasing acidity of the latex.

The results to be discussed here, have therefore to be interpreted in such a way, that the principal features of the view are generally available, but that the limits of the different ranges may be moved more or less, according to the composition of the latex with which one operates.

Against this drawback, that one operates with a non-constant, and not arbitrarily reproducible material, we find, as a great advantage, the fact, that Hevea latex is mixable with water in any proportion. So one may easily prepare all percentages of rubber from the original percentage (30—40 %) down to the lowest one, and one may, without great difficulty, traverse and search systematic-

ally in all directions the whole range of the mixing-proportions, by serial determinations with decreasing quantities of more or less diluted latex, and increasing quantities of acid, either diluted or not. The „irregular series" being only found with the lower percentages of rubber, it was possible to determine completely the range where these occur. In most cases, described in literature, the „irregular series" have only be examined with one single or with a few concentrations, the higher or lower concentrations of the colloid not being accessible. The latex, used for most of the observations to be described here, originated from a group of trees, fifteen years old, in the experiment garden at the opposite side of the Tjiliwoeng at Buitenzorg. The trees were tapped daily, with two cuts over $^1/_4$ of the circumference of the trunk, and the latex was used for examination between 10 a. m. and noon. The percentage of rubber (on coagulation) varied from 31,0 to 32,8, and on the average amounted to 31,8 %; the acidity was 0,02—0.04 N. (cf. § 8), the acids present are principally carbonic acid, lactic acid and a little butyric acid [1]).

In 1922 complementary observations were made with latex from a few groups of trees in the Botanical garden.

§ 2. *The phenomena of coagulation.*

With the proportions, as they are chosen in the practice of the preparation of rubber, the coagulation of Hevea latex proceeds slowly. After a quarter of an hour the liquid has become thick, with the consistency of porridge; gradually it begins to cohere, and after one hour a coherent lump is formed, but still with milky serum; only after a few hours the separation into a solid coagulum and a clear serum, is complete. In other cases one causes the coagulation to proceed more rapidly, by adding more acid, so that, after one hour, one obtains a coagulum sustable for working purposes. Or one saves acid, so that only after a few hours the first phenomena occur, and the coagulum can only be worked up next morning. Sometimes the latex is used undiluted, but mostly one dilutes with water to a rubber percentage of 20 or 15 %, on account of which the coagulum becomes softer, and may be worked up more easily. The more the latex is diluted, the softer the coagulum becomes, and the stronger the contraction after the coagula-

[1]) For the composition of Hevea latex in general we may refer to „Estate Rubber, its preparation, properties and testing" by Dr. O. DE VRIES (RUYGROK & Co., 1920), chapter 1 and 2.

tion will be, so that more serum is set free. Only with very strongly diluted latices a flocky coagulum is separated, which does not form a coherent lump, or only gradually coheres after one or more days. If we use less acid, the coagulation sets in slowly; but with decreasing quantity of acid the spontaneous coagulation, caused by bacteria which decompose the sugars and the proteins under formation of acid, begins to play a more and more important part. Ordinary, non-sterilised latex always coagulates, even without any addition of acid, during the first night after tapping; the coagulum is then spongy by the formation of gases, and the surface exposed to the air is covered with a yellow, evil smelling layer of porridge-like separated rubber, mixed with decomposition products of proteins. So in the range of very little acid there are no mixtures, which remain liquid in the long run; the observation "liquid" may be made after a quarter of an hour or after two hours, but after 24 hours one will find the mixture coagulated. The liquid mixtures with more acid, so in what one might call the second liquid range, remain liquid for an unlimited space of time. Sometimes, after being left to themselves for several days, a separation of very thin little flocks, lying on an almost clear or whitish serum, sets in, but in any case one can control and confirm the observation "liquid" after 24 hours. This liquid range passes into the ranges of coagulation by a strip of transition, being broad especially towards the side of the higher acid concentrations, and distinctly showing different stages. The first beginning of coagulation phenomena is the appearance of a thin skin at the surface of the liquid, caused by evaporation in the air, which, on stirring with a glass rod, attaches itself to it as a streak or rolls itself up.

On approaching the range of coagulation a little more, this streak becomes thicker and more cloddy. Advancing further, we get to clotting or curdling of a greater part of the latex; a pap and finally a coherent coagulum is formed. If it is left longer to itself, the coagulation in this range of transition proceeds further; what after two hours was a pap, may after 24 hours have become a coagulum and a mixture which after two hours only showed a thick streak, has changed the next morning into a pap, or even may be coagulated. What is liquid in the middle of the second range, remains liquid even after days, but "liquid" on the limit of "streaky", may have changed into streaky after 24 hours. "Coagulated" of course remains such after one or several days, only the coagulum gradually contracts itself a little and becomes harder.

It may be clear, that with these gradual transitions, we shall

never be able to fix any sharp limits for the different ranges. The ordinary discrimination, by gently shaking or stirring, can only be a rough one. We examined if sharper criteria might be found by means of the microscope, but it appeared, that the formation of little lumps of a few or a great many small rubber-globules also took place quite gradually, without sharp transitions, and that neither the decrease nor the stopping of Brownian movement opened the way for any sharp limitation.

So most of our serial experiments were confined to judging at sight, by means of a stirring rod, only completed occasionally by microscopic oberservations. A short time, about 15 minutes, after the addition of the acid the first observation was made, which in certain ranges is already sufficient. The principal observation followed two hours after the mixture was made, and was controlled the next morning, viz. if then a stage was reached so much further advanced as might be expected from the condition, such as it was two hours after the addition of the acid. In order to be able to sufficiently overlook the whole, we have, in the following paragraphs, interpreted the observations in a somewhat simplified way; therefore, with the classifications "streaks", "curdled", "porridge", and "coagulation" we have to associate the meaning of conditions of separation gradually passing into each other, as described above.

As a rule we worked with 50 cc. of liquid for each determination, the liquid being left open to the air in a small cylindric glass till the next morning, for the last control-observation. With very small quantities of acid the mixture of latex and water was measured with a measuring-cylinder and the acid was added by means of a burette. It was not necessary to measure the diluted latex more exactly than within $\frac{1}{2}$ or 1 c.c., but the acid had to be measured exactly within one drop, especially with the very diluted latices, where the range of coagulation is narrow and sharply limited. With mixtures with larger quantities of acid, the latex, either diluted or not, was always mixed with the diluted or undiluted acid in such quantities that the total volume was 50 c.c.; while the liquid, which occupied over half of the total volume, was poured out first, and the other one added to it.

Especially in the range of a large quantity of acid, or if one uses strong acid, it is necessary to stir vigorously from the beginning, so as to prevent local coagulation, which would cause enclosure of acid, not being set free any more by further stirring. By making one same final mixture, starting from latices of different dilutions

and differently strong acids, one may however control the observations in a satisfactory way.

On account of the increasing acidity of the latex itself it is not advisable to use it more than about two hours before the observations; we only did determinations between 10 a.m. and noon, but during that time one can easily prepare a few series, in total about thirty to fifty mixtures, so that in a rather short time by many hundreds of observations one can search the whole range of mixing in all directions.

Operating in small, open cylindrical glasses, causes a certain evaporation and results in the formation of a small superficial skin of coagulated rubber, which on stirring attaches itself to the stirring-rod.

Apparently this causes an undesired complication; but for distinguishing different liquid mixtures this formation of skin appeared on the other hand an advantage, because it enables us to recognize the liquids inclined to coagulate. By repeating a few series in small Erlenmeyer-flasks, closed with a cork, we have ascertained that really these skins are formed by evaporation at the surface.

§ 3. *Hydrochloric acid.*

The easiest way to summarise the phenomena at different dilutions and different quantities of acid, is to draw these in the wellknown triangle-figure. As angular points (components) we choose water, concentrated hydrochloric acid (9.14 N) and undilated latex, i.e. a liquid with 31.8 % coagulable rubber, about 35 % totally solid substances and about 65 % water, and with an acidity of about 0.03 N. A recalculation of the results, so as to express these as quantity of acid, resp. rubber compared to the whole liquid (water of dilution plus serum) can never be correct by the phenomena of adsorption and, as regards rubber, there is not much sense in it, as coagulable rubber is a substance containing so many secondary substances in small quantities.

In the annexed figure 1 the lines show how the different mixtures are formed by mingling latex and hydrochloric acid, of different dilutions. The mixtures, in which after two hours a well coherent coagulum was formed, are marked with a little cross. As we see this range almost occupies the whole triangle; only in a narrow strip along the latex-water side, we find mixtures, which are represented by an encircled point (pap or curdling) or by a little circle (liquid), and there we can, though indistinctly on account of

the scale-size used, recognize irregular series liquid : coagulation : liquid : coagulation. This narrow strip, the range of small quantities

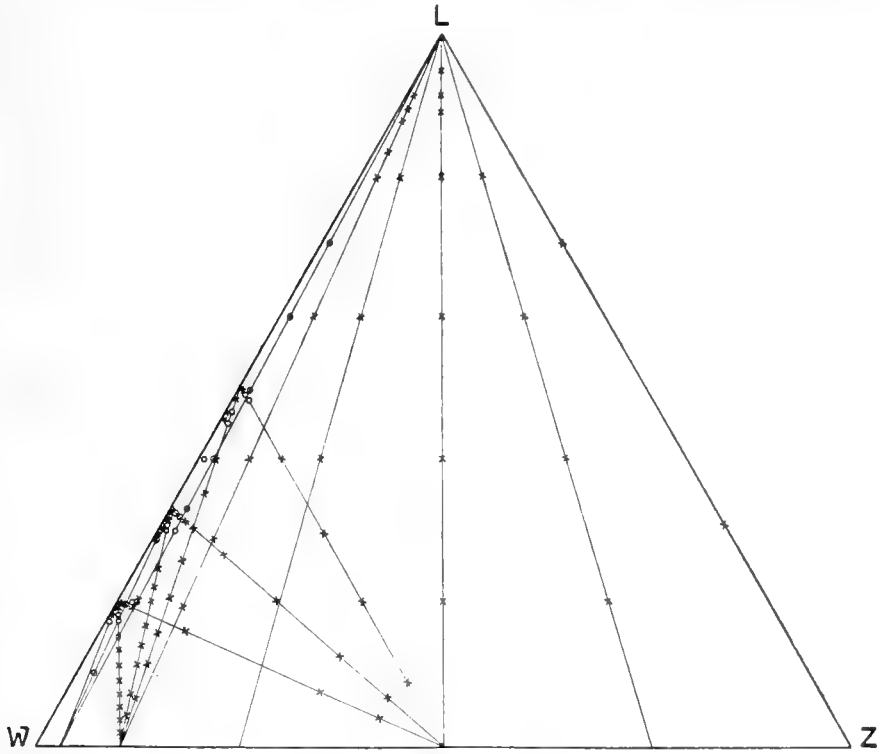

Fig. 1.

of acid, is, with hydrochloric acid, the only interesting item; the remainder of the triangle shows nothing particular, the less water the mixture contains, the harder the coagulum, while in mixtures with little water and much hydrochloric acid the serum assumes a violet tint.

The narrow strip along the latex-water side is represented on a larger scale in fig. 2, where the acid is drawn perpendicularly, as ordinate, and expressed in normality (grammolecules HCl per Liter final mixture).

For quite small concentrations of acid, at any dilution, we first come into the liquid strip, where coagulation has not yet started after two hours. After 24 hours this part shows spontaneous coagulation. At higher acid-concentration (from about 0.007 N) we find after two hours more or less strong curdling or formation of pap, and after 24 hours coagulation. The limit at which after two hours complete coagulation with a clear serum has taken place, is,

found with mixtures beyond 50 % latex, to be fairly constant at 0.012 N. We should bear in mind, that this means the acidity of

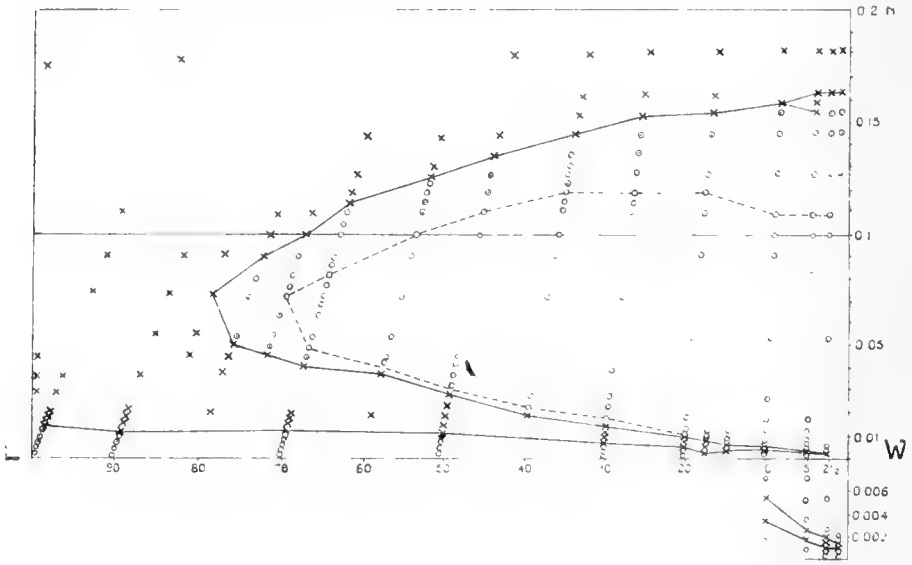

Fig. 2.

the hydrochloric acid added, which has to be increased with the original acidity of the latex, recalculated on the final mixture. For mixtures, containing less than 50 % latex, this bottom-limit of coagulation is regularly lower. Because of reasons mentioned above, the observations could not be made so sharply that the relation between rubber-concentration and limit of acidity appeared quite clearly, but especially with the lower concentrations the small irregularities may be considered to be due to observation errors, and we may assume that the lowering of this limit is inversely proportional to the latex concentration.

With mixtures containing over 80 % latex to which more acid is added we always get a strong coagulum, and so from the beginning we are in the range of coagulation, which fairly occupies the whole triangle of Fig. 1. At 75 % latex we get the first indications that another phenomenon is about to appear, because the coagulum at first is hard, with more acid (about 0.05 N) soft or even like pap, and only with a still larger quantity of acid hard again. A distinctly liquid range only appears with mixtures with 65 % latex and less.

The strip of coagulation between both liquid ranges, the lower range of coagulation, regularly decreases in latitude at lower latex-concentrations, but still remains distinctly perceptible even at the lowest concentrations (1 % latex). In those very diluted liquids the

rubber does not separate itself as a coherent coagulum, but in the form of white flocks. The separation goes much quicker than with higher concentrations and with the liquids with 1 and 2½ % latex, reminds one of a titration of warm nitrate of silver with hydrochloric acid.

At those low concentrations the range of coagulation is so narrow, that, in an acidified but still unchanged liquid, one can see, with a single drop of diluted acid, the white flocks separating themselves, and that one sees the original milky liquid remain unchanged on addition of a few drops more. With mixtures with 5% latex one may get at first, with a small quantity of acid, a flocky separation, cohering fairly quickly as a coagulum; on addition of a little more acid a very soft coagulum may be formed at once. Mixtures with 2½% and 1% latex cause flocky separations, which may remain unchanged for a long time, and with which the coherence as a coagulum is the more difficult, according as the mixture contains less latex.

At higher concentrations, just above 5%, sometimes the liquid separates itself in a remarkable quick way into a coagulum and a clear serum, but the instantaneous coagulation is not found there any more. At still higher concentrations the separation of the coagulum goes slower.

The lower range of coagulation, described here, is limited by a transition to spontaneous coagulation, as discussed above; at the upper part we find a narrow range of transition, where the mixture after two hours is like pap or curdling (after 24 hours mostly coagulated). Only towards the higher latex-concentrations this strip gradually becomes a little broader, and at about 65% latex bends itself in an upper direction, limiting the top of the liquid range, and converging into the broad strip, which separates the liquid range from the upper range of coagulation. Thus the liquid range is perfectly limited, both at the upper- and at the lower side, at least till the lowest concentration, which was examined (1% latex, so 0.3% rubber in the mixture). Whether, at still smaller concentrations, the lower strip of coagulation is continued, or if both the liquid ranges meet there, has not been examined yet. The limits of the various ranges are found at the following normalities of the added acid in the final mixture: (See Table following page).

These figures are illustrated by fig. 2.

We shall now give a short description of the course of the phenomena at a few typical concentrations. To the latex-water-mixture 10% hydrochloric acid (0.914 N) was added from a burette; the

TABLE I.

Latex in the mixture.	Lower limit of coagulation.	Upper limit first range of coagulation.	Upper limit liquid range.	Lower limit second range of coagulation.
65 %	0.012	0.04	0.08	0.10
50 %	0.011	0.029	0.10	0.13
40 %	0.009	0.019	0.11	0.14
30 %	0.007	0.013	0.12	0.15
20 %	0.005	0.009	0.12	0.155
10 %	0.0035	0.0055	0.11	0.16
5 %	0.0018	0.0027	0.11	0.155
2½ %	0.0009	0.0018	0.11	0.16
1 %	0.0008	0.0011	0.14 ?	0.16

quantities were chosen in such a way that the final mixture was always 50 cc., so that the latex-concentration, at larger quantities of hydrochloric acid, decreased a little, and that the serial determinations in fig. 2 are found on slanting lines.

With a mixture with 70% latex the result of the examination two hours after the addition of acid was (cf. fig. 2):

After being left to itself for three hours, the coagulation of course had proceeded further; now 2½ had become a pap, 2¾ a thick liquid with a good many skins, 3—4¼ remained liquid, 5¾ was softly coagulated. The mixtures in the strips of transition show a further advanced coagulation, but the true liquid mixtures remain liquid, even after 24 hours. When it is left in open small cylindric glasses, a skin is formed at the surface, evidently by evaporation, for in closed Erlenmeijer-flasks it was not formed. So the limits of the ranges are somewhat displaced, according to the moment of observation being delayed, but the phenomenon coagulated — liquid coagulated remains. It strikes us, that the transition at the lower side of the liquid range is very acute; at the upper side however much more gradual. The little skins formed on stirring, are partly due to evaporation at the surface, or to latex, drying upon the side of the glass; yet these skins point to a higher inclination for coagulation, as such mixtures after 3 or 24 hours are coagulated further than the purely liquid ones.

c c. 10 % HCl per 50 c c mixture.	DESCRIPTION.
0.1	liquid.
0.3	liquid.
0.4	thick pap; beginning of strip of transition.
0.5	thick liquid, a few little lumps.
0.6	the same.
0.7	a somewhat thick pap, coagulating on stirring; beginning of the range of coagulation.
0.8 and 0.9	strong coagulum, serum whity.
1	coagulated, serum fairly clear, (acid added 0.018 N).
2	the same, serum fairly clear.
2¼	the same, serum white. Upper-limit first range of coagulation.
2½	liquid, a few small lumps on stirring. Therefore sharp transition.
2¾	liquid with some skin.
3	liquid; lower limit liquid range.
3½, 3¾, 4, 4¼	liquid; no skin.
4½	liquid on stirring some skin or streak. Upper-limit liquid range.
4¾	the same, a piece of skin (therefore irregularity).
5, 5¼	the same, more skin.
5½	the same, a fair quantity of skin.
5¾	like pap (at an other time only a fair quantity of streaks).
6	very soft pap, almost coagulated.
6¼	coagulated, but serum quite white, therefore far from complete. Lower-limit second range of coagulation.
6½	coagulated, fairly stiff, serum white.
7	the same, serum white.
8	the same, serum white. The percentage of latex in this mixture is 58.8 %.

An other example with 30 % latex: (See following page).

Of course the coagulum is always soft, because the mixtures only contain 30 % latex, i.e. about 10 % rubber.

Quite typical are the sharp transitions at the first range of coa-

cc 10 % HCl per 50 cc mixture.	DESCRIPTION.
0.1	liquid.
0.2	liquid.
0.3	liquid, somewhat thickish, small lump of coagulum.
0.4	coagulated, rather stiff, serum rather clear, lower limit first range of coagulation.
0.5	coagulated, serum clear.
0.6	coagulated, serum perfectly clear like water.
0.7	well-formed, but soft, jellied coagulum, serum nearly clear. Upper limit first range of coagulation.
1	quite liquid, only somewhat streaky, lower limit liquid range. Sharp transition.
1¼, 1½, 2, 4	quite liquid.
6	quite liquid, somewhat streaky, like 1. (later determination liquid without streak).
6¼	liquid.
6½	liquid, somewhat streaky, upper limit liquid range.
7	liquid, rather streaky.
8	for the greater part liquid, a good deal of streaky soft coagulum.
8½	soft coagulum, serum white. Lower limit second range of coagulation.
9	soft coagulum, serum white.
10	well-formed but soft coagulum, serum quite white.
11	the same — the same.
12	the same — serum almost clear.

gulation of very dilute latices; e.g. at 1 % latex (0.3 % rubber in the mixture), see fig. 2, lower, enlarged part.

The microscopic image of the liquid in the second liquid range, is, e.g. for a mixture with 2 % latex, as follows.

At a small acid-concentration almost all the rubber-globulus are still free from each other, and have a Brownian movement; only very few small lumps are seen, consisting of some little globules touching each other. Starting from an acid-concentration of about 0.02 N to increase somewhat, but by far the greater part of the particles are still free and in vivid Brownian movement. Only at

cc 1% HCl	DESCRIPTION
0.25	liquid, containing a few small flocks.
0.4	liquid, with a few small flocks.
0.45	after about 1/4 hour rising flocks are separative, so that after 1 hour the serum is almost clear.
0.	coagulates almost momentarily in flocks, rising to the surface in a layer, serum almost clear.
0.55	flocks are separated slowly, serum remains white.
0.6	liquid.
1.0	liquid.

about 0.11 N, i.e. at the upper-limit of the liquid range (see fig. 2), the number of small lumps increases and the Brownian movement decreases, and at 0.13 N hardly any particles move, and only very few show Brownian movement. At 0.14 N the decreaming begins, which, at 0.15 N, leads into the range of coagulation. From 0.10 to 0.15 N therefore, there is a gradual transition from "free particles with Brownian movement" into lumps, particles yet free but not moving, and decreaming. Whether perhaps the few little lumps, which are found in the second liquid range, were formed by a local excess of acid during addition, was not examined.

If we keep a liquid from the middle part of the second liquid range, e.g. 2% latex with 0.06 N. hydrochloric acid, in a high cylindric glass, no decreaming takes place within the first few weeks, but the Brownian movement gradually decreases. After two months most of the particles have joined into small lumps, a few consisting of two or three, but most of them consisting of a great number of globules, so that, after that time, only a fairly small number of free particles remain in Brownian movement; yet only part of the rubber is decreamed, and superficially the liquid is still equally white.

We regret we were unable to examine, whether in the second liquid range the negative charge, which the rubber-globules show in the original latex, had given place to a positive one, as required by the theory of „change of charge" [1]). Some experiments concerning

[1]) Cf. F. Powis, Z. Phys. Chem. 89 (1915), 105.
H. R. Kruyt, these Proceedings 17 (1914), 615, and 19 (1917), 1021.

the coagulation with different salts, will be described in a following communication.

We shall discuss in § 8 a few examples of the influence of the original acidity of the latex on the position of the limits of the ranges.

§ 4. *Nitric acid.*

We likewise made serial determinations with nitric acid and sulphuric acid, but less detailed, so that the limits of the different ranges were only roughly determined. For these experiments latex was used from a different group of trees, containing 28 % rubber. Fig. 3 gives the determinations for nitric acid. The general type is

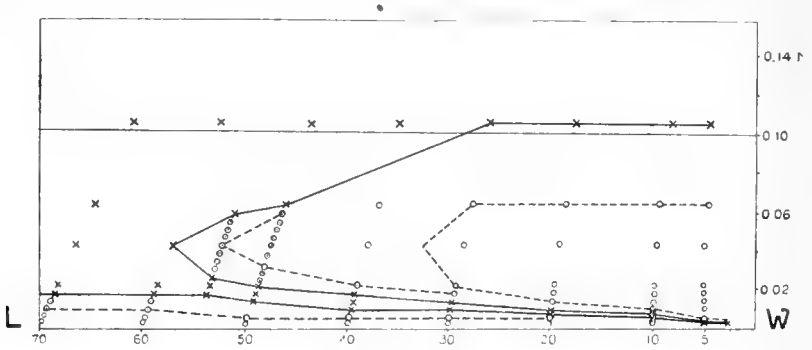

Fig. 3.

exactly the same as with hydrochloric acid, both the liquid and the pappy ranges are smaller. Fig. 3 only goes as far as mixtures with 70 % latex; the top of the pappy range, being with hydrochloric acid at 75 % latex and about 0.07 N, is found here at a little less than 60 % latex and about 0.04 N. The top of the totally liquid range is comparatively still more displaced towards the right, so that, between both these tops, a very wide „pappy" range is found, in which we separated, by a dotted line, that part where, after two hours a thick or fairly thick pap is formed, from the part still showing fairly liquid mixtures with streaks or a beginning curdling. With nitric acid the upper-limits lie at about half the normality of that with hydrochloric acid.

In § 7 we intend to compare more closely the figures for the four acids, and also discuss more detailed the data for mixtures with 5 and 2 % latex.

§ 5. *Sulphuric acid.*

The data, which we gathered for coagulation with sulphuric

acid, have been put together in fig. 4. The large range of coagulation at acid-concentrations above 0.1 N (normal = 49 Gr. H_2SO_4 per Liter), has again been quite left out, and also the mixtures with over 70 % latex, where coagulation constantly takes place as soon as more than 0.01 N acid is added. On account of the smaller number of observations, the course of the limits in fig. 4 seems to be somewhat irregular, yet the data are sufficient to

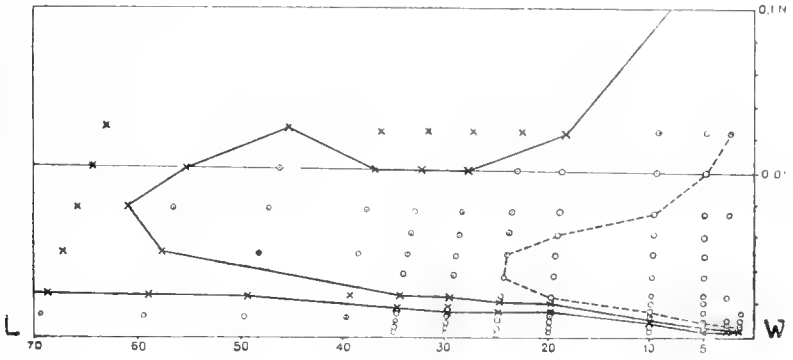

Fig. 4.

conclude, that the pappy and the liquid range, compared with hydrochloric acid and nitric acid, have shrunk still more. Figures of comparison are again found in § 7.

We may still mention, that, starting from a mixture with 70 % latex, we get a distinct indication regarding the existence of the „irregular series", though all the mixtures coagulate; the mixture with 0.04 N. acid gives a perceptibly softer coagulum than that with 0.025 or 0.05 N. acid.

§ 6. *Acetic acid.*

For acetic acid — the general and usual means of coagulation at rubber plantations — the course of the phenomena generally speaking appears to be the same as in the previous cases, but the proportions of the various ranges are quite different ones. Whilst with the three previous acids the whole range of the irregular series lies in a narrow strip along the latex-water side, which in a representation like fig. 1 is hardly discernible, the irregular series with acetic acid are extended to far higher acid concentrations, and a triangle-figure like fig. 5 opens the best general aspect. Here likewise the range of coagulation occupies by far the greater part, viz. almost ¾ of the triangle; but in the neighbourhood of the

angularpoint for water we find that over $^1/_4$ of the triangle is occupied by the liquid and pappy range, while naturally in this case also, close along the latex-water-side a first liquid range is found, not showing any coagulation on addition of a very small quantity of acid, but, after keeping, showing spontaneous coagulation by the action of bacteria.

The proper liquid range in fig. 5 is again limited by a dotted line; the pappy range is divided by a somewhat thicker dotted line into two parts, a fairly liquid and a more pappy one. Formation of a coherent coagulum takes place in the narrow strip parallel to

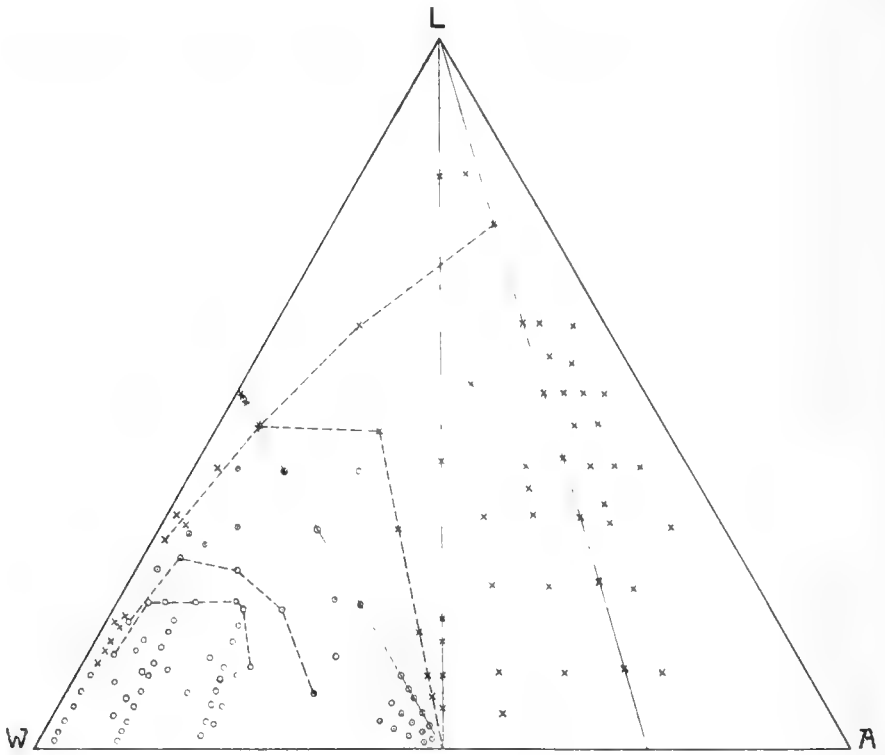

Fig. 5.

the latex - water - side and towards the side of the angulair-point Latex; the total range towards the side of the angulair-point acetic acid gives a perfect coagulation, but in the shape of flocks or as a pap, and not as a coherent lump. Both these ranges of coagulation are roughly separated in fig. 5 by a dotted line. Therefore in this respect too, there is an important difference between acetic acid and the three other acids, with which the whole range of coagulation gives a coherent coagulum.

We traced the coagulation with acetic acid once more by a considerable number of determinations, viz. in the latex of both the above-mentioned groups of trees; in fig. 5 we have represented the results, obtained with the 28 % latex of the second group (see § 4). The normality of acetic acid added is given in table 2 for the limits of the various ranges.

TABLE 2.

	Quantity of latex in the mixture in %.									
	100	80	60	50	40	30	20	10	5	2½
Limit lower liquid range.	0.004	0.003	0.003	0.003	0.003	—	0.003	0 0015		—
Beginning lower creamy range.	0.008	0.008	0.008	0.009	0.009	—	0.009	0.003	—	—
Lower limit range of coagulation.	0.017	0.024	0.031	0.030	0.028	—	0.016	0.006	0.0015	0.0012
Upper limit first range of coagulation.	—	—	—	0.52	0.35	0.21	0.13	0.08	0.05	0.026
Lower limit second liquid range.	—	—	—	—	—		0.8	0.24	0.16	0.10

On comparing these figures and fig. 5 with the results described in §§ 3—5, we distinctly see the great difference in the distance between the limits. A comparative review is given in § 7.

In judging the above figures one has to bear in mind, that the phenomena, in the sense in which we consider them here, are not exactly the same as in plantation practice. So here we take as lower limit of the range of coagulation those mixtures, where a coherent coagulum is formed after two hours, whilst with regard to the coagulation at the plantations it is moreover required, that the serum is clear or almost clear, and the coagulum sufficiently stiff to be mangled. With undiluted latex the lower limit of the range of coagulation, as it is described here (0.017 N or about 1 Gram acetic acid per Liter latex), will be lower than the amount used in practice, if we wish to mangle a few hours after the coagulation. With 50 % latex (i. e. 1 : 1 diluted) the dose (0.030 N = 1.8 Gr. acetic acid per Liter is higher, because with diluted latex one

mangles the next day, when with a much smaller quantity of acetic acid, a coagulum fit for use has formed itself.

To this we add the results of a less complete series of observations, made in November 1922 with latex from the Botanical garden at Buitenzorg, where a few groups of trees were tapped with a cut over $^1/_4$ of the circumference.

This latex contained 37 % rubber, and had an acidity of about 0.025 N. We see that the general type is the same, that the lower limits fairly well coincide, but that, with regard to other limits, rather important differences appear, that may be attributed partly to the difference in composition and acidity of the latex, partly however, to the difference of appreciation between the observers. This example illustrates, together with the cases to be discussed in § 8, the restriction we made already in § 1, regarding the quantitative value of the results.

TABLE 3.

	Quantity of latex in the mixture in %.							
	80	60	40	20	10	5	2	1
Beginning lower creamy or pappy range.	0.009	0.010	0.009	0.006	0.0026	—	—	—
Lower limit first range of coagulation.	0.018	0.022	0.017	0 009	0.0053	0.0026	0.0020	0.0016
Upper limit first range of coagulation.	—	—	0.40	0.20	0.083	0.04	0.033	0.023
Lower limit second liquid range.	—	—	—	0.5	0.17	0.066	0.059	0.040

§ 7. *Comparison of the four acids.*

We now intend to compare amongst each other the results, obtained with the four acids. Whilst, roughly speaking the general course is exactly the same, we may notify interesting differences and conformities.

Considering first of all the top and the upper limit of the liquid range, we can use for that purpose the data mentioned in § 3—6, although they refer to two different latices, and the principal observations covered a period of over half a year, because these limits, can only be roughly determined. So we get:

TABLE 4.

	HCl	HNO_3	H_2SO_4	$C_2H_4O_2$
Top liquid range, with mixtures with latex	70%	35%	25%	25%
Top pappy range, with mixtures with latex	77%	57%	65%	57%
Upper limit liquid range for 20% latex, at acidity	0.12 N	0.06 N	0.03 N	3—4 N
Upper limit pappy range (lower limit second range of coagulation) for 20% latex, at acidity	0.155 N	0.10 N	0.06 N	7—8 N

The limit, at which irregular series do not appear any more — the top of the pappy range — is found for nitric acid, sulphuric acid and acetic acid at almost the same percentage of latex, but for hydrochloric acid it is somewhat higher. With all this we have to bear in mind that with nitric acid in a mixture with 60 %, with sulphuric acid in one with 70 %, a distinct interruption in the series can still be observed, owing to the coagulum, at a level of the above-mentioned top, being softer than at higher or lower concentrations of acid. A striking difference in the position of this top cannot therefore be stated with the four acids.

On the other hand there is an undeniable difference with regard to the top of the really liquid range, which, with hydrochloric acid extends to much higher latex-concentrations, than with the three remaining acids.

In the upper limit of the liquid range, i.e. the beginning of the upper curdling range, and likewise in the upper limit of this range, i.e. the lower limit of the second range of coagulation, the difference is very striking too. With acetic acid these limits are by far the highest; then follows hydrochloric acid, about halfway lower nitric acid, and half way lower again sulphuric acid. If we assume, that in the second liquid range the colloid rubber particles have changed their charge from negative into positive, the stronger coagulating action of the bivalent sulphate-ion would be fully explained; monovalent ions then would show a decided difference in the series nitrate-, chlorine-, acetate-ion.

A comparison of the action of the four acids in the first range of coagulation seemed of particular interest to us, viz. with small latex-concentrations, where, with a small increase of the acid-

concentration we so sharply get with the three inorganic acids the phenomenon liquid-rapid coagulation-liquid, described in § 3. Therefore, for the same mixture of latex, we once more determined these limits for all four acids separately, in order to get absolutely comparable figures (which figures therefore do not fully correspond with those in §§ 3—6, as we explained in § 1 and intend to discuss more in detail still in § 8).

The figures were for the acid-concentration in normality:

TABLE 5.

	HCl	HNO_3	H_2SO_4	$C_2H_4O_2$
5 % latex, lower limit	0.0011	0.0011	0.0011	0.0015
5 % latex, upper limit	0.00265	0.00265	0.0029	—
2 % latex, lower limit	0.0007	0.0007	0.0007	0.0010
2 % latex, upper limit	0.0013	0.0013	0.0014	—

The lower limit of the range of coagulation is exactly the same with the three strong inorganic acids, and here it is quite clearly demonstrated, that, at least in this range of strongly diluted latices, the phenomenon is ruled by the positive H ions; the action of acetic acid is somewhat weaker.

With hydrochloric acid and nitric acid the upper limit again is exactly the same; also the strips of transition (which are very narrow with these strongly diluted latices) show exactly the same phenomena if the same quantity of acid is added; so the action of hydrochloric acid and nitric acid in the lower range of coagulation is exactly the same, whilst the limit of the upper range of coagulation, as we have seen just now, is considerably lower with nitric acid. With sulphuric acid the upper limit of the first range of coagulation is a little higher; the difference is not important, but for all that, with this exclusively comparative experiment, it could be stated clearly, also because corresponding differences were noticed in the strip of transition lying above the range of coagulation. With acetic acid the upper limit is much higher (at about 0.05 and 0.026 N, see table 2) and has not been determined again in this experiment.

A determination of the hydrogen-ions concentration in these various liquids, which would be necessary for a correct interpretation of the phenomena, could not as yet take place; we only wish to draw the attention to the fact, that the subsequency of the four acids at

the upper limit of the first range of coagulation (hydrochloric acid and nitric acid — sulphuric acid — acetic acid) is not the same as at the lower limit of the second range of coagulation (sulphuric acid — nitric acid — hydrochloric acid — acetic acid).

§ 8. *Influence of the acidity of the latex itself.*

As already stated in § 1 latex is feebly acid, and on being left to itself gradually increases in acidity. The acidity of the latex, which is used for the researches, is of course not without influence on the figures obtained, though the relation need not be purely additive, as the acidity in latex is caused by carbonic acid and organic acids amongst which, after the action of bacteria, lactic acid, acetic acid and butyric acid.

First of all we made a few observations in ordinary latex and in the same latex after neutralisation with hydroxide of potassium, i.e. again for the limits, to be fixed sharply, of the first range of coagulation in mixtures with little latex. A mixture with 5% latex (percentage of rubber 1.43% needed) for the neutralisation (phenolphthalein as indicator) 16.6 cc. $^1/_{10}$ N hydroxide of potassium per Liter, and therefore was 0.00166 normal; for the original latex we calculate from these data an acidity of 0.033 N. A mixture with 2% latex (percentage of rubber 0.54% required 6.6 cc. hydroxide of potassium and therefore was 0.00066 normal (i.e. also 0.033 N calculated for original latex).

The limits of the first range of coagulation appeared to be with hydrochloric acid:

TABLE 6.

	Own acidity	Addition hydrochloric acid in normality	
		Lower limit	Upper limit
5 % latex, original	0.00166	0.0015	0.0032
id. , neutralized	—	0.0030	0.0048
2 % latex, original	0.0066	0.0013	0.0020
id. , neutralized	—	0.00195	0.0027

We see, that the neutralization has increased the necessary addition of acid with about the amount of the own acidity of the latex. In judging the figures we should bear in mind that the neutralized

latex contains by the neutralization a small quantity of potassium salts, that may somewhat displace the limit of the ranges.

A second experiment related to the increase of the own acidity of the latex, when left to itself. The latex used for this purpose titrated, when left to itself undiluted, at 10 o'clock 0.026, at noon 0.030 and at 1.45 p.m. 0.032 N. From the observations resulted:

44½ cc. 70% latex, diluted at 10 o'clock with 5½ cc. 10% HCl, (i.e. mixture 0.1 normal, belonging in the upper pappy range of transition, see Fig. 2): after one hour still liquid, but containing a fair-sized lump of streaks, and after three hours a thick pap, fairly well coagulated, with quite white serum;

the same mixture, but prepared only at 12.30 p.m. from the undiluted latex, was already coagulated, after being left to itself for one hour, though the coagulum was still very soft. So the influence of the higher own acidity of this latex was quite noticeable.

43 cc. 40% latex, prepared at 10 o'clock with 7 cc. 10% HCl (i.e. about 0.13 N, again in the middle of the upper pappy range of transition, see Fig. 2) caused after one hour a small lump of little skins, and was still liquid after three hours with a fairly strong skin;

the same mixture, prepared at 12.30 was still liquid after one hour with a small lump of skins, which was somewhat larger than in the above-mentioned mixture after one hour. So in this case the difference was noticeable, though not important.

It appears from these experiments, as might be expected, that,

TABLE 7.

	May 1920	Oct 8th 1920	Oct. 9th and 12th 1920	Oct. 14th 1920	May 1922
Own acidity undiluted latex	0.026–0.030	–	0.041–0.044	0.033	0.022
Upper limit 5 % latex	0.0027	0.0025	0.00265	0.0032	0.0044
Lower limit ib.	0.0018	0.0012	0.0011	0.0015	0.0020
Upper limit 2½ % latex	0.0018	—	—	—	—
Lower limit ib.	0.0009	—	—	—	—
Upper limit 2 % latex	—	0.0014	0.0013	0.0020	0.0026
Lower limit ib.	—	0.0007	0.0007	0.0013	0.0014
Upper limit 1 % latex	0.0011	—	—	—	0.0020
Lower limit ib.	0.0008	—	—	—	0.0014

by operating with the latex later, the quantity of acid that has to be added in order to reach a certain stage, is found to be a little smaller.

We will still give a few examples, how much the percentages of acid found may vary when latex from different origin is used, viz. for hydrochloric acid and for the limits of the first range of coagulation with mixtures with 5 and 2% latex.

If we calculate the differences in own acidity of the diluted latices, we see that the differences in acidity for the limits differ fairly strongly from them, though a general relation can be clearly noticed. In fact a strictly quantitative correspondence could not be expected as the latices differed not only in acidity but also in percentage of rubber and in secundary substances.

§ 9. *Investigations of others.*

As mentioned in the introduction, we find in literature a good many investigations, pointing to the existence of irregular series with Hevea latex.

J. PARKIN, one of the first investigators who was engaged with acid-coagulation of Hevea latex [1]), used for his experiments ten times diluted latex and stated therewith the transition liquid — coagulated — liquid. PARKIN, whose experiments were limited to small additions of acid, did not notice the second range of coagulation. As an explanation PARKIN assumed, that the protein, present in latex, is insoluble in a neutral liquid, but dissolves in alkali or acids. PARKIN was of opinion that Hevea latex is alkaline; therefore addition of acid would first cause neutralization, with precipitation of the protein and, as a result, of the rubber as well, whilst, at a higher acidity the protein would dissolve again. PARKIN further stated that with acetic acid the range of coagulation is wider than with other acids, and thought this a decided advantage for practice, because by addition of too much acid the coagulation would not fail so soon.

Because in the practice of plantations one never causes the percentage of rubber of the latex to sink below 15 or 12 % (i. e. in our terminology, one never uses mixtures with less than 50 to 40 % latex), where with acetic acid no irregular series occur, there was for a long time no further interest for these phenomena. W. CROSSLEY [2]), again gave a few figures for upper — and lower limit of the

[1]) Circulars Royal Botanic Gardens Peradeniya Vol. I (1899), 149.

[2]) India Rubber Journal 41 (1911), 1206.

first range of coagulation with a mixture with 7 % rubber (i. e. about 25 % latex) which had been preserved with formaline. We found the lower limit at 0.014 N. acetic acid, the upper limit at 0.29 N, whilst the own acidity of the diluted latex was 0.015 N. These figures correspond fairly well with ours (tables 2 and 3). CROSSLEY's lower-limit is somewhat lower and his upper-limit somewhat higher, whereby the unknown action of formaline, may have been of influence. Moreover CROSSLEY determined the lower limit of the first range of coagulation for dilutions of the above-mentioned latex with 7 % rubber, and found that, as far as a hundredfold dilution, the total acidity (acetic acid added plus calculated own acidity) decreased with great exactness proportional to the percentage of latex. For dialysed latex with a percentage of 12 % totally solid substance (i.e. a mixture with about 40 % latex) CROSSLEY [1]) found the following figures for the lower- and upper-limit of the first range of coagulation:

TABLE 8.

	Lower limit	Upper limit
Acetic acid	0.02 N	0.18 N
Trichloracetic acid	0.005	0.026
Formic acid	0.008	0.022
Hydrochloric acid	0.004	0.016
Sulfuric acid	0.005	0.018

The dialysed latex had an acidity of only 0.001 N; all the limits (except the upper-limit with sulphuric acid) are lower than those we found for normal latex, so that the dialysable serum substances in natural latex would have an anti-coagulating action.

As a criticism of these investigations B. J. EATON [2]) published a few series of observations with hydrochloric acid, nitric acid, sulphuric acid and acetic acid, which however are very incomplete and did not throw much light on the phenomena; EATON found mixtures which remained liquid, but this he attributes to a retardation of the coagulation on account of high dilution, or to an inclusion of the acid in the little lumps on partial coagulation. EATON denies the

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1) India Rubber Journal **42** (1911), 1345.

2) Bull. of the Dept. of Agric., Fed. Malay States No. 17 (1912), p. 10.

existence of a maximum-limit for the first range of coagulation, as fixed by CROSSLEY; from the above it is perfectly clear that this criticism is absolutely without ground, and that the maximum-limit, described by PARKIN and CROSSLEY does really exist; but only with mixtures with a percentage of latex below a certain limit.

G. S. WHITHBY[1]) was the first one who emphatically pointed out the existence of the second range of coagulation above the second liquid range and described a few complete series liquid coagulated — liquid — coagulated. WHITBY for these phenomena assumed the explanation that small quantities of acid have an activating influence on an enzym, which is found in latex, coagulase, which, at a small acidity, would cause the coagulation, but at a higher acidity would become inactive; the second range of coagulation then would be a direct precipitation of protein by larger quantities of acid.

We shall now compare the observations of the last two investigators with our own.

1. *Hydrochloric acid.* In Fig. 6 the limits have been taken from Fig. 2, and therein have been drawn the observations made by EATON and WHITBY.

Starting from undiluted latex EATON found with 10% acid (line 1 in fig. 6) a continual series of coagulations, but with 1 % acid

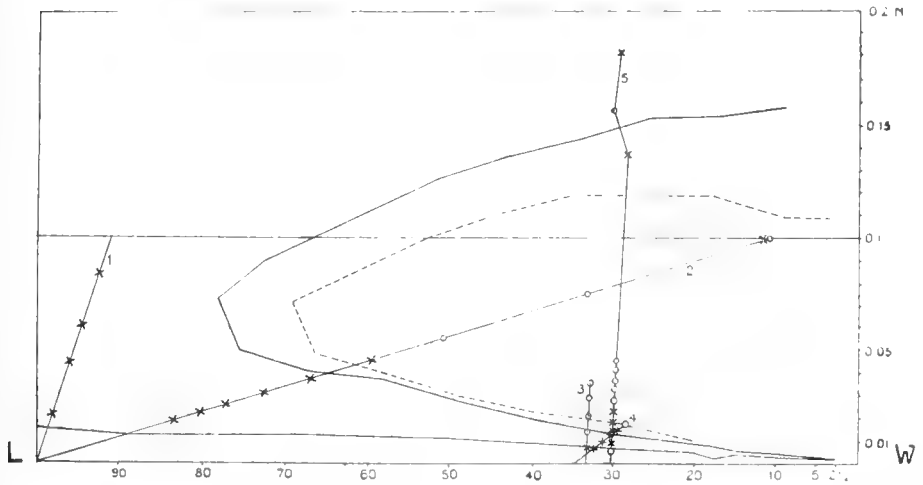

Fig. 6.

(line 2) he got into the liquid range. Two series with 1 : 2 diluted

[1]) Zeitschr. Koll. Chem. 12 (1913), 156, India Rubber Journal (London) 45 1913), 945; further Agric. Bull. of the Dept. of Agr. F.M.S. (Kuala Lumpur) 6 (1918), 381.

latex (our $33^1/_3$ %) showed him the transition from coagulated to pappy, but did not show distinctly, that he had got again into the second liquid range (lines 3 and 4). EATON did not observe the upper range of coagulation.

WHITBY made a complete series at about 30 % latex; his limits do not fully coincide with ours, which for the reasons already mentioned (own acidity latex etc.) is not astonishing, and also may be caused by wrong reproduction, as WHITBY does not mention the exact titre of his hydrochloric acid. So except small differences the observations of both investigators fit satisfactorily in the frame of our recapitulating-figure (see fig. 6 and 2).

2. *Nitric acid.* EATON made two series of observations, starting from undilated latex, and always found coagulation at increasing acidity, corresponding with Fig. 3. Moreover a series with 1% acid with 1 : 2 diluted latex, with which he passed from the range of coagulation into a pappy range ("incomplete coagulation"), which again he attributes to the above mentioned causes (inclusion of acid in the lumps).

WHITBY also described for nitric acid a complete series, viz. for a latex with 12 % rubber (corresponding with a mixture with 40% latex); he found at 0.016 N coagulation, at 0.021 a pap, at 0.032 and 0.052 liquid mixtures, at 0.063 a pap again, at 0.105 and 0.21 coagulation. These observations tally with ours (see Fig. 3), except both the liquid mixtures (WHITBY only says "coagulation failed to occur", which possibly may correspond with our mixtures with a little curdling).

3. *Sulphuric acid.* EATON made a series with undiluted latex, which (as might be expected) showed coagulation at all acidities; moreover one with latex diluted 1 : 3 where after the range of coagulation came a few mixtures with incomplete coagulation, and a series with latex diluted 1 : 10, where coagulated — incompletely coagulated — liquid was stated. The fact of remaining liquid is attributed again by EATON to a retardation of the coagulation with strongly diluted latex, but he does not explain in which way he accounts for the coagulated mixtures with less acid found in this series.

WHITBY only gives a short indication about a series liquid — coagulated — incompletely coagulated (pap) — coagulated, without mentioning the percentages of acid and the percentage of rubber. Probably this has been the same diluted latex with 10 % rubber (30 % latex) as in his experiments with hydrochloric acid, and therefore WHITBY probably remained at a concentration, up to which the liquid range does not reach. (cf. Fig. 4).

4. *Acetic acid.* EATON again mentions a few series with undiluted and diluted latices, in which for the diluted latices the pappy, skinny or liquid range was reached at acidities, corresponding fairly well with those found by us. For this acid WHITBY does not give any quantitative data, but only says that the first range of coagulation is much wider than with the previous acids, and that, after that, liquid mixtures are reached. With 30 % latex we did not find any liquid mixtures (top at 25 % latex), but probably WHITBY's mixture had come, by the addition of diluted acetic acid, to a lower percentage of rubber. WHITBY did not find an upper limit of the liquid range, as could not be the case (see Fig. 5) on dilution of 30 % latex with acetic acid of less than 50 %.

As we see, the data of both these investigations fit in a satisfactory way in the frame of our recapitulating-figures and their observations, partly seeming confused, are explained by the system of ranges, as they have become known to us at present.

SUMMARY.

Mixtures of Hevea Latex and water show, on addition of acids, the phenomenon of the irregular series. For hydrochloric acid, nitric acid, sulphuric acid and acetic acid the limits of the ranges (first and second liquid range, first and second range of coagulation, strips of transition) were completely fixed for all mixing-proportions of latex, water and acid (see fig. 1—5), and a comparison was made between the position of the limits for these four acids.

Buitenzorg, December 1922.

Histology. — "*On the Determination of Polarity in the Epidermal Ciliated cell. (After experiments on Amphibian Larvae)*". By Dr. M. W. WOERDEMAN. (Communicated by Prof. L. BOLK).

(Communicated at the meeting of September 29, 1923).

It is a well-known fact that in the early stages of their life the larvae of amphibians have an epidermis, provided with ciliated cells. This cannot be observed distinctly in all species, for they differ largely as to the number of ciliated cells. Nor are these cells evenly distributed over the epidermis of one and the same larva; there are spots where they are scattered thickly, while they occur more sparsely in other spots.

The ciliary movement causes a slow rotation of the larvae while the latter are still inclosed in their jelly-like envelope. When this envelope is removed, the exposed larvae will be seen to keep up their rotatory motion owing to the ciliary movement, just as the larvae that have already left their envelope. At the same time a rather violent current may be observed in the water encircling the larva. It is self-evident that this current is strongest where most ciliated cells are collected. Strong currents are, therefore, distinguishable along certain parts of the larval body, weaker streams along other parts, which e.g. have been minutely examined by ASSHETON [1]) for Rana temporaria and Triton cristatus and have been represented in plates for larvae of various age-periods.

It appears that in these animals the first action of cilia is noticeable in larvae where the neural folds are still open, shortly before their closure. There is a strong current in the water round about the larva from head to tail along the neural walls. My own researches were made on Rana esculenta and Triton alpestris larvae. I found that in these amphibians the ciliary movement begins when the neural walls are in part united. The direction of the fluid-streams along the larval body I found to agree in the main with ASSHETON's schemata, although there were also some differences. This, however, is not to the purpose. The direction of the ciliary movement in normal larvae of Rana esculenta and Triton alpestris was,

[1]) R. ASSHETON. Quarterly Journ. of micr. Science New Series. Vol. 38. 1896, p. 465.

therefore, closely examined and represented in diagrams. It was further established that the fluid-streams flow invariably in the same direction. A reversed direction of the ciliary movement seems to have rarely been observed in metazoa. (ERHARD)[1]).

This implies such a structure of the ciliated cells that a ciliary movement is only possible in one direction, the cells present a certain asymmetry in their structure; besides their polarity (by which base and ciliated free surface are distinguished) there is an "accessory polarity" (vide ROUX[2]) for these ideas). The question has been considered whether this accessory polarity could be reversed artificially, in other words, whether the ciliated cell could by some artificial method be made to move in the opposite direction. This question is connected with another, viz. in how far the ciliary movement depends on the position of the ciliated cells relative to the axis of the body.

Experiments performed by v. BRÜCKE[3]) and those made this very year by MERTON[4]) bear on this question. They did not succeed in bringing about a reversion of the polarity. Now it has been evidenced by numerous experiments that in the embryonic development there is a period in which the ectoderm, from which the larval epidermis is derived, is still indifferent. SPEMANN[5]) e.g. found that at the beginning of the gastrulation ectoderm, destined to build up the medullary plate (so-called presumptive medullary plate), could be replaced by presumptive epidermis. Larvae developed with normal medullary plate and normal epidermis. The fate of the ectoderm-cells in that stage of development has not been, or has not yet been determined. The ectoderm is still in a high degree liable to change („umbildungsfähig"). Whether in that phase it is still completely indifferent cannot be decided without a detailed inquiry. It occurred to me that an inquiry into the polarity of the cell might afford some indication, as the polarity of the cell may already be determined before its organogenetic function. SPEMANN's experiments regard the organ-determination. Now, how about the polarity of the cell? When is it determined? The experiments in which I tried to solve these questions, I performed on larvae of Rana esculenta and of Triton alpestris in the Zoological Institute of the Freiburg University (Director Geheimrat Prof. Dr. H. SPEMANN).

[1]) ERHARD in ABDERHALDEN's Handbuch der biologischen Arbeitsmethoden.

[2]) W. ROUX. Terminologie der Entwicklungsmechanik der Tiere und Pflanzen. Leipzig. Engelmann. 1912.

[3]) E. TH. v. BRÜCKE. Pflüger's Archiv. f. d. ges. Physiol. Bnd. 166. 1917.

[4]) H. MERTON. Pflüger's Archiv. f. d. ges. Physiol. Bnd. 198. 1923.

[5]) H. SPEMANN. Sitzungsber. d. Gesellsch. naturf. Freunde. Berlin. 1916. N⁰. 9.

I started by ascertaining whether there were developmental stages in which the polarity of the ciliated cell is reversible, that is stages in which the ciliated cells can be forced to move in a direction other than the normal.

After circumcision with fine glass-needles patches of ectoderm were detached from their sublayer and after a rotation of 180°—90° brought again to coalescence. After the wounds thus made were healed, which occurred in a marvellously short time, the direction of the ciliary movement was determined by examining the larvae in water in which granules of carmine had been suspended. A disadvantage of this procedure appeared to be that the borders of the wound are soon altogether invisible, so that the extent of the reversed regions cannot be traced out. For this reason I used the method adopted by W. Vogt [1]), who interchanged ectoderm patches of larvae stained vitally and those of nonstained larvae. After it had first been ascertained that vital staining with Nile-blue sulphate did not affect the ciliary action, I stained one of two larvae of the same age-period, and the other I did not. Of these two larvae fragments of ectoderm of a very well-defined shape and of the same size were excised and interchanged. In the transplantates the colour remains very well localized, it does not diffuse and enables us to recognize the contour of the implantate for many days still. Moreover, the shape of the implantate is indicative of its original position, consequently of the direction of the currents produced by the ciliary movement under normal circumstances. I shall not give an account of the various experiments, but I will describe briefly the final result of all of them.

It became evident that when a ciliated cell has once begun to vibrate it cannot be made to move in another direction. Patches of ectoderm being implanted in the wrong direction persisted to move in their original direction for days, nay, even till the ciliated cells had disappeared from the epidermis. Even before the ciliary movement has begun, its direction has already been established. When ectoderm fragments are reversed 180° before the ciliary movement begins, the ciliated cells will afterwards reveal a vibration opposite to that under normal circumstances. The youngest stages of development, however, are excepted in this respect, as it appeared that in blastulae and in incipient gastrula-stages the blastula-roof resp. ectoderm-patches can be reversed, without affecting the direction of the movement, when afterwards the larvae begin their ciliary action.

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[1]) W. Vogt. Verhandl. deutsch. zoolog. Gesellsch. Bnd. 27. Sept. 1922. p. 49.

It is evident, then, that in the young stages, just referred to, the polarity of the cell has not yet been determined. It also appeared from the experiments that the determination takes place during the gastrulation. If the blastopore is still like a straight or slightly crescent-shaped slit, the future ectodermal ciliated cell is still indifferent. But as soon as the blastopore has become horseshoe-shaped and still later circular, reversion of ectoderm without reversion of the future direction of the ciliary movement is not possible.

It follows, then, that the period of determination of the polarity of the epidermal ciliated cell falls in an early stage of gastrulation.

Now we had to ask if the determination of the polarity of the cell coincided with the organ-determination.

To ascertain this we interchanged patches of presumptive epidermis and presumptive medullary plate in very young larvae, and we watched the subsequently developing ciliary movement while giving due attention to the original position (vital staining). Stated briefly the results were to the following effect: When after the operation larvae appeared with a normally developed medullary plate (part of which was consequently generated by presumptive epidermis) and with a normally developed epidermis (part of which was consequently formed by presumptive medullary plate), the larvae exhibited normal direction of ciliary movement i.e. the ciliated cells have not developed as they would have done originally, but have adapted themselves to their new environment. If the organ-determination has not yet been effected, the direction of the ciliary movement can still be influenced by the environment. But if abnormal larvae developed with a deficient medullary plate or with pieces of the medullary plate in their epidermis, then the direction of the movement appeared to have developed on the implantates according to the origin of the implantates and appeared not to have been influenced by the new environment.

Our experiments, therefore, seem to imply that the determination of the polarity of the cells and of the organogenetic function either occur synchronously or at all events with a very brief interval of time.

It should be borne in mind, however, that the organ-determination in the ectoderm does not occur everywhere at the same time. SPEMANN's and Mrs. MANGOLD-PRÖSCHHOLD's [1]) experiments have shown that this determination starts from what they have termed an „organisation centre", which is located in the dorsal lip of the blastopore.

[1]) H. SPEMANN. Arch. f. Entw. mech. der Organismen. Bnd. 48. 1921.

Furthermore, experiments by O. MANGOLD[1]) tend to show that after the conclusion of the gastrulation, i.e. when the region of the medullary plate has already been determined, ectoderm of the ventral half of the larva can still form mesoderm or entoderm. From this we see that this ectoderm has not yet been determined.

My experiments to find an answer to the question if there is any relation between the determination of the polarity of the cell and of its organogenetic function, were carried out in the region of the future medullary plate. A more extensive investigation is required for the purpose of ascertaining whether the phenomenon that the determination of the polarity of the cell almost coincides with that of the organogenetic function of the cells holds generally or only for the region of the medullary plate.

In a subsequent communication I intend to discuss the histo-physiological data regarding the ciliary movement obtained in the experiments reported in this paper.

[1]) O. MANGOLD. Verhandl. deutsch. zoolog. Gesellsch. Bnd. 27. Sept. 1922. p. 51.

Histology. — "*A Contribution to the Histophysiology of the Ciliated Epithelium*". By Dr. M. W. WOERDEMAN. (Communicated by Prof. G. VAN RIJNBERK).

(Communicated at the meeting of September 29, 1923).

The sudden reversion of the direction of the ciliary movement which we know to be a property of a number of protozoa, is of very rare occurrence in metazoa (literature ERHARD [1]). As far as I know it has hitherto not been found in ciliated cells of amphibians.

v. BRÜCKE [2]) hit upon the idea of detaching small patches of the oral mucous membrane in frogs and allowing them to coalesce again after having turned them 180°. These experiments were hampered by all sorts of difficulties, such as inflammation, necrosis of the patches, suppuration etc. Macroscopically it could be observed in two animals that the epithelium of the patch was not destroyed, so that v. BRÜCKE was able to study the direction of their ciliary movement for 40 days. The cells continued acting in the original direction.

In three other animals the epithelium of the patch was most likely (v. BRÜCKE did not examine it microscopically) displaced by epithelium that arose from the borders of the wound. This regenerated epithelium exhibited a normal direction of the ciliary movement. Experiments made by MERTON [3]) in the past year substantiate v. BRÜCKE's data, so that it seems quite certain that in adult frogs it is not possible to reverse the direction of the ciliary movement, i.e. to alter the polarity of the ciliated cell.

Indeed, the negative results of SCHÖNE's [4]) and of WEIGEL's [5]) experiments with other epithelia had already made us suspect this; but, then, it was exactly in ciliated epithelium that the direction

[1]) ERHARD in Abderhalden's Handb. d. biol. Arbeitsmethoden.
[2]) E. TH v. BRÜCKE. Pflüger's Arch. f. d. ges. Phys. Bnd. 166. 1917.
[3]) H. MERTON. Pflüger's Arch. f. d. ges. Phys. Bnd. 198. 1923.
[4]) SCHÖNE Die heteroplastische und homoioplastische Transplantation. Berlin 1912.
[5]) WEIGEL. Arch. f. Entw. mechan. der Organismen. Bnd. 36. 1913.

of the ciliary movement of the environment could readily be imagined to influence the movement of the cells of the turned implantate, considering our view of the conduction of the stimulus in the ciliated epithelium.

Classic experiments have in this field been carried out by VERWORN [1]). They tended to show that every ciliated cell, nay, every separate cilium has a movement of its own. However, for the regular action of the entire epithelium, in which not a single ciliated cell begins to move before its predecessor ("metachronic" ciliary movement after VERWORN), an interconnection of all those cells is indispensable. If one of the anterior vibrating elements (ciliated plates on the ribs of Beroë. Inquiry by VERWORN) is checked in its movement, all the rest will stop vibrating. If an incision is made, the part distad of the incision will not vibrate any longer with the same rhythm as the part proximad of it. The first element posterior to the incision now marks the rhythm, which is taken over by the succeeding vibrating elements.

We cannot but assume that a conduction of the stimulus must take place in the ciliated epithelium (in the free border of the cell), and that all the ciliated cells are interconnected (literature ERHARD). If this is the case, we might imagine the direction of the ciliary movement to reverse in the rotated patches of ciliated epithelium that have coalesced with the environment, since the conduction of the stimulus in these patches will now be just the reverse of the normal conduction.

But the fact that the healing of the patches of the oral mucous membrane was rather tardy and was attended with inflammation of the borders of the wound, justifies our doubt as to the existence of any normal organic connection between implantate and surroundings.

With a different object in view I have been working on larvae of Rana esculenta and of Triton alpestris, in the Zoological Institute of the Freiburg University (Director Prof. Dr. H. SPEMANN). Ectoderm patches were detached and after a rotation of 90° or 180° they were allowed to coalesce again. As the larval epidermis contains ciliated cells and exhibits a very regular ciliary movement (vide ASSHETON [2])), I was now in a position to study the effect of these rotations on the ciliary movement.

Beforehand it should be stated that the rotated patches of ectoderm in young amphibian larvae coalesce in a wonderfully short time

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[1]) M. VERWORN. Pflüger's Arch. f. d. ges. Phys. Bnd. 48. 1890.

[2]) R. ASSHETON. Quarterly Journ. of microsc. Science. New Series. Vol. 38. 1896.

without reaction, so that a few hours after the operation no traces are distinguishable of the borders of the wound, even under the microscope. Now in order to verify the extent of the rotated region we had recourse to a special technique, which enabled us to recognize the contour of the rotated patch for many days together (Transplantation and vital staining after W. Vogt [1]).

The commencement of the ciliary movement in amphibian larvae nearly coincides with the closure of the neural canal. When an ectoderm region is rotated in a stage, in which the ciliary movement has just commenced or has been proceeding for some time, the ciliary movement will keep up its original direction. This lasts for days until the ciliary cells disappear from the epidermis. An influence on the rotated region by its environment cannot be made out.

If the experiment is made after the conclusion of the gastrulation, that is hours before the commencement of the ciliary movement, the result is the same. So before the movement commences its direction has already been determined.

Only in blastulae and the youngest gastrula stages can the future ciliary movement be influenced successfully.

From these experiments it may, therefore, be concluded that after the conclusion of the gastrulation the polarity of the ciliated cell has been determined. The following experiments were now made with stages immediately succeeding the conclusion of the gastrulation. I have extended the experiments to various spots that might be considered as a source of the ciliary movement. They were turned long before the movement began. Nevertheless the process of the ciliary movement in the non-rotated regions progressed quite normally.

In another set of experiments vibrating patches of ectoderm were implanted into young stages that did not yet possess ciliary movement. Now it might be supposed that on the appearance of the movement, its direction would be dictated by that of the implantate. In every experiment this influence failed to appear.

Furthermore, non-vibrating patches of ectoderm (of very young stages) were implanted into older larvae with vibrating epidermis.

Now also it might be supposed that, when the ciliary movement of the implantate commences, its direction would be determined by the epidermis of the host.

It appeared, however, that the ciliary movement of the implantate commenced simultaneously with the movement of the larva

[1]) W. Vogt. Verhandl. deutsch. zool. Gesellsch. Bnd. 27. Sept. 1922. p. 49.

from which the implantate had been derived and that the direction of the movement was determined by the origin of the implantate, *not* by the new surroundings (a true case of "Selbstdifferenzierung" after ROUX).

Now it may justly be assumed that in the younger stages that I operated upon, the implantates are readily taken up into organic connection with their surroundings. In experimental embryology numerous cases are known in which such an implantate behaves in every respect like the region it has displaced. Nay, the fact that the implantate is competent to incite remote cells to display their organogenetic function, points indeed to conduction of a stimulus from the implantate to its environment, which also implies that the implantate has an organic relation with its environment.

In order to account for the beautiful metachronism in the ciliary movement it is generally supposed that there is a conduction of stimuli from one ciliated cell to the other. Recent experiments by WINTREBERT [1]) have proved, moreover, that this conduction exists and takes place in young stages without the help of the nervous system, i.e. in the epithelium alone.

The experiments on blastulae and young gastrulae go to show that the turned patches vibrate co-ordinately with their environment. This implies that not only the direction of the movement of every cell is opposite to that in which the cell would originally have moved, but also that the regulation of the ciliary movement is reversed and agrees with the sequence of vibrations in the environment of the rotated patch.

This co-ordinate movement simultaneous with the environment proves: 1° that the patch is apparently stimulated by the environment (so that the conduction of stimuli has not been interrupted); 2° that the polarity of the cell is reversed; 3° that the direction of the stimulus-conduction is reversed. If in an older larva a patch of epidermis is turned, then the cilia on this patch persist in moving co-ordinately, but not in co-ordination with the environment. There is not a single reason why the patch should not receive stimuli from its environment now. Various experimental embryological data point to the fact that also in these stages such a relation arises again after the wounds have been healed. If this is the case, the results of the experiments with older stages would imply 1° that then the polarity of the cell is not reversed; 2° that the conduction of the stimuli still takes place in the original direction.

[1]) P. WINTREBERT. Comptes rendus de l'Acad. des Sciences. Paris T. 172. 1921, p. 934.

We are, therefore, impressed with the idea that the direction of the conduction and the polarity of the ciliated cells are determined simultaneously, and that conduction of the stimulus is possible only in a special direction. We are justified in assuming that this phenomenon depends on the nature of the connection between the ciliated cells. However, microscopical researches have not yet produced positive evidence of this nature.

Meteorology. — "*A non-tangent infralateral arc*". By Dr. S. W. VISSER. (Communicated by Prof. E. VAN EVERDINGEN Jr.).

(Communicated at the meeting of October 27, 1923).

On 24th June 1923 I saw at the Astronomical Observatory at Lembang a beautiful halo, which I will describe in the following pages.

Already early in the morning a mock-sun was visible on the right of the sun. Direct measurements of its distance were impossible, as the sun itself was hidden by thick clouds. About twelve o'clock a very bright lower tangent arc appeared, which after a few minutes became so intensely luminous as to be visible from time to time through the lower clouds. Soon this arc spread and developed into a complete circumscribed halo within which a weak ordinary ring became also visible. I succeeded between $12^h\,17^m$ and $12^h\,49^m$ in taking some 26 measurements of both rings by means of the cloud theodolite, mounted at the Observatory expressly for observations of halo's. To these measurements I will refer afterwards. In the mean time I kept a keen lookout for other halo's. Not before $12^h\,49^m$ my effort was rewarded by the apparition of a spot of light on the left below the sun, near the place where the mayor ring (46°) was to be expected. This spot soon grew more intense and developed into a short, oblique arc. Colours (red and green) were visible. On the other side of the sun nothing could be observed, because there the Cu-cloud around the Tangkoeban Prahoe shielded the Cirrus layer from our vision. I now concentrated my full attention on this arc and obtained 12 measurements until $1^h\,4^m$. Sometimes clouds prevented the observation. Moreover between $12^h\,16^m$ and $1^u\,2^m$ fourteen control-observations of the sun were made. At $1^h\,4^m$ the lower cloud had so much increased, that the measurements had to be finished. At half past seven in the evening the Ci-St proved to be still present, there was a bright lunar halo, but without any particularities.

The same halo's were seen by M.M. VOÛTE and RIJKEN RAPP

during their railway journey between Tjimahi and Bandoeng. However they saw the small arc not on the left (west) of the sun, but on the right (east). Though on the left hand side the sky presented an equally smooth Cirrus-veil as on the right, nothing was to be seen there. According to RIJKEN RAPP the arc was intensely coloured and bent like a portion of the greater ring. I have not been able to note any curvature at Lembang.

Before discussing my measurements I give here a short review of the theory of the infralateral arc.

BRAVAIS explains the arc by the refraction of light in ice-crystals with a horizontal principal axis, the light entering at a vertical basal plane (the hexagonal terminal plane of the crystal) and leaving at a sideplane of the prism. The refracting angle is 90° then. For a definite position of the principal axis (defined f. i. by its azimuth) we get a circular arc perpendicular to this axis and at a distance from the sun, depending on the sun's height. In a simple way we may imagine this circle by drawing the case of the circumzenithal arc and rotating the drawing then over 90°, so that the axis which at first was vertical, now gets a horizontal position. To each azimuth of the axis such a circle belongs. The envelope of all these minor circles is the infralateral arc. One among these circles is tangent to the greater ring. For the rest, this arc does no more then the circumzenithal arc fulfil the conditions for minimum deviation of the refracted rays of light.

PERNTER (Meteorologische Optik, 1st Edition) sticks to these conditions. He considers the arc as a "Lowitz arc of the greater ring" and deduces the form and position of the lateral arcs to the smaller and greater rings in an exactly analogous way. Without going into the details of the calculations, we may state, that the arc according to PERNTER in consequence of the conditions for minimum deviation which he imposes, generally will be less distant from the ring than BRAVAIS's arc.

BESSON (Sur la Théorie des Halos, Paris 1909, p. 51, p. 70) has shown, that PERNTER's theory is not very satisfactory. EXNER (PERNTER-EXNER, Meteor. Optik, 2nd Edition 1922 p. 405) concurs in this opinion and develops a new theory. During the normal fall of an ice-prism the principal axis and one of the bigger diagonals of the hexagon are placed horizontally. An infralateral arc may than be formed by light, entering the basal plane and emerging from one of the oblique prism-planes. The plane perpendicular to the refracting edge is inclined to the horizon at an angle of 30°. For one definite

height of the sun (27° 45′) the lateral arc is tangent to the ring. For all other suns-heights the arc deviates towards the outer side. If we allow rotations about the principal axis, minimum deviations are possible up to a suns-height of 80° 50′. According te EXNER (f.i. pag. 402) measurements are lacking. However there exists one by BESSON (l.c. pag. 71). 23rd April 1908 with a suns-height of 53° he saw an infralateral arc on the left below the sun at a height of 19°, whereas from BRAVAIS's theory a height of 18° 57′ would follow.

This case bears some resemblance to that of Lembang. "Three minutes afterwards" BESSON writes "the ring of 22° and the circumscribed halo appeared, complete but scarcely visible". In both observations the same forms of halo's appear. [1])

For the measurements at Lembang as a rule the red of the arc was vised at. Once green was measured. Two times the left- and righthand ends of the red were determined.

The readings and some distances and angles calculated from these have been entered in the following table.

Nr.	M. J. T.	⊙ az.	⊙ h′	Az_w	h_w	Δ_w	A_w	h_b	Δ_b	A_b	h_{w-b}	Δ_{w-b}	A_{w-b}
1	12u 50m	N 17.2 W	58.2	N 60.1 W	22.8	46° 44′							
2	„	„	„	58.3	22.4	46 17	58° 19′	20.6	45.7	48.0	+1.9°	+0.9	10.1°
3	12 51	17.6	„	60.3	22.4	47 0							
4	„	„	„	58.9	22.5	46 17							
5	12 52	18.0	58.1	60.4	22.9	46 23	56 10	20.5	45.5	48.4	+1.4	+0.9	7.8
6	„	„	„	56.4	20.9	46 24							
7	12 54	18.8	57.9	58.2	21.5	46 10	54 57	20.5	45.3	48.8	+1.0	+0.9	6.2
8	12 55	19.3	57.8	56.4	20.3	46 11	54 10	20.5	45.2	49.1	+0.5	+1.0	5.1
9	12 56	19.7	57.7	60.3	21.8	46 19							
10	12 57	20.1	57.6	60.2	20.3	47 22	55 13	19.4	46.2	50.5	+0.4	+1.2	4.8
11	12 58	20.5	57.5	59.6	21.1	46 10	54 39	20.4	45.3	49.6	+0.7	+0.9	5.1
12	1 4	23.0	57.1	59.5	20.0	45 46	51 17	20.3	45.2	50.3	−0.3	+0.6	1.0

The observations 5 and 9 refer to the lefthand end, 6 and 8 to

[1]) See also: E. VAN EVERDINGEN. Halo's in April, Hemel en Dampkring 21, 1923, p. 216, 217.

the righthand end, 10 to the green. The time is Middle-Java time. The suns-height and azimuth were calculated and with these the readings of the theodolite were reduced. Az_w and h_w stand for the observed azimuth and height of the arc; Δ_w is the distance of the observed points from the sun calculated from the 4 foregoing columns; A_w is the angle between the suns vertical and the radiusvector from the sun to the arc, deduced from the observations.

The column under Δ_w shows, that the points measured deviate sensibly from the ring, for the red of the ring is formed at a distance of 45°6′ from the sun. The mean deviation is 1.1°. Gradually the distance decreases, but for Nr. 12 it is still 0.6° larger than that of the ring. This deviation is so big and so systematic, that it is impossible to think of observational errors. Indeed there is question here of a *non tangent* arc. The position of the tangent-point of the arc was calculated according to BRAVAIS's theory. The results have been entered under h_b, Δ_b and A_b. The calculation was carried through for the 10th observation for green ($n = 1.3115$) for the rest for red ($n = 1.307$). In taking the differences between observation and calculation the first four points, which in consequence of the initial weakness of the arc happened to be less accurate than the others, were combined to a mean value. The observations 5 and 6, 8 and 9, which refer to the ends of the arc, were substituted by their mean values.

Almost all the observed points are too high (column h_{w-b} gives the difference observation and calculation), but they approach the height calculated from theory. The angle A, which according to theory should increase for a sinking sun, in reality rapidly decreases. In consequence the difference between observation and calculation decreases from 10° tot 1°. Finally, the distance from the sun remains almost constantly 0.9° too big, hardly showing any tendency to decrease.

During the whole time of observation the arc remains outside of Bravais's arc; the position with respect to the sun approaches more and more that of the theoretical tangent point.

This arc deviates from that of BRAVAIS and hence still more from that of PERNTER. No more is it in harmony with EXNER's theory. For in this case we have to assume a normal plane inclined at an angle of 30°. In our case the rays of the sun are in their turn inclined to this plane at an angle of at least 57.1°—30° = 27.1°. The smallest distance from the arc to the sun is then 57.6°, which is quite out of question for the observed arc.

As was explained above, crystals showing various orientations of the principal axis in the horizontal plane contribute to the formation of the infralateral arc.

That is why I calculated what position in space the axis ought to present in order to give rise to the phenomenon as it was observed. I supposed, that the refraction took place in the normal plane — for in this case the deviation is a minimum and the intensity of light a maximum.

We consider the spherical triangle ZSN, formed by the zenith Z, the sun S and the vanishing point of the crystal-axis N. We know ZS, the complement of the suns-height, $\angle$S, the supplement of the angle A we already determined, and arc SN. The latter is the angle of incidence i of the rays of light and is to be deduced from the observed Δ. Arc ZN and $\angle$Z may then be calculated, ZN gives the height of the vanishing point, $\angle$Z is the difference in azimuth with the sun. From this follows the azimuth of the axis, as the sun's azimuth is known.

The results are as follows:

Nr	i	ZN	Z	az.ax.
1—4	74.6°	92.4°	55.2°	N 72.7°W
5—6	74.2	93.0	53.2	71.2
7	73.5	93.0	51.8	70.6
8—9	73.2	93.7	51.3	70.8
10	74.4	93.6	52.4	72.5
11	73.5	93.4	51.6	72.1
12	72.2	93.9	48.1	72.1

Hence in the mean the crystal-axis is inclined at an angle of 3°.3 to the horizon and its azimuth is N 71.8 W.

The position of the axis appears to be stationary. The differences with the mean value are as a rule below 1°. The conclusion is the more remarkable for the azimuth, as the difference in azimuth with the sun decreases more than 7° during the observations.

In trying to find an explanation of such a position by taking into account the influence of gravitation, wind [1]) and atmospheric

[1]) M. Pinkhof. Bijdrage tot de theorie der halo-verschijnselen. Verhandelingen Kon. Akademie van Wetenschappen 1e Sectie, Dl. 13, N°. 1, p. 21, 1919.

electricity on the position of the ice-crystals, I met among others with the difficulty, that the complete development of the circumscribed halo seemed at variance with the explanation proposed. Therefore I hope to come back to this point afterwards. For each explanation however the observations on the ring and its envelope may be wanted. They follow therefore as the concluding part of my remarks.

BRB = lower tangent arc; O.H = circumscribed halo; K = ring of 22°; l = left; r = right. The remaining symbols have the same meaning as in the other tables.

The mean of the 6 observations on the red of the ring is 21° 54′, only 2′ more than that found from the measurements on the top.

The calculated Δ_b is meant for white light, the observed Δ_w for red. Leaving apart the 4 very discordant differences for the first 4 observations, the mean difference observation minus calculation is —0.3°, that means exactly the difference in distance for red and white. Hence these observations of the circumscribed halo are in harmony with the calculation for red. In the 15 measurements on the ordinary ring however, on the contrary a very distinct difference of +0.3° remains.

A. Measurements of the upper and lower top (red).

M. J. T.	⊙ h	height of the top		Δ	
		lower	upper	lower	upper
12u 18m	59.7°	37.5°	—	22.2°	—
21	59.6	—	81.3°	—	21.7°
22	59.6	—	81.7	—	22.1
24	59.5	38.0	—	21.5	—
31	59.4	—	81.1	—	21.7
32	59.3	37.7	—	21.6	—
35	59.2	37.1	—	22.1	—
36	59.1	—	80.9	—	21.8
48	58.4	36.3	—	22.1	—
			mean	21.9	21.8

Mean of all measurements 21° 52′, for red according to PERNTER 21°34′

B. Measurements of the ring and the circumscribed halo.

Nr.	Time	⊙ h	⊙ az	h_w	az_w		Δ_w	A_w	Δ_b	Δ_{w-b}
1	12u 17m	59.7°	N2.8°W	38.1°	No°. oW	BRB r	22°37′	5°49′	21.9°	+0.7°
2	17	59.7	2.8	38.1	7.6	„ 1	22 48	9 55	22.0	+0.8
3	19	59.7	3 5	39 3	—11.7	„ r	22 32	31 58	22.8	—0 3
4	20	59.7	4.2	39.3	17.2	„ 1	21 59	27 42	23.0	—1.0
5	26	59.5	7.0	58.5	55.0	O.H 1	24 11	108 34	24.4	—0.2
6	27	59.5	7.4	58.5	50.6	K 1	21 53	106 20	—	—
7	29	59.4	8.3	59.4	56.0	O.H 1	24 11	110 40	24.4	—0.2
8	29	59.4	8.3	59.4	52.3	K 1	21 58	109 1	—	—
9	37	59.0	11.8	59.0	60.0	O.H 1	24 15	110 47	24.5	—0.2
10	38	59.0	12.2	59.0	—36.7	O H r	24 35	111 5	24.5	+0.1
11	41	58.9	13.5	58.9	56.8	K 1	21 58	108 45	—	—
12	43	58.8	14.3	58.8	—28.8	K r	21 54	108 22	—	—
13	43	58.8	14.3	58.8	—33.0	O.H r	23 58	110 23	24.5	—0.5
14	46	58.5	15.6	43.7	—16.4	O.H r	24 34	69 11	25 0	—0.4
15	47	58.5	16.0	51.6	—20.8	K r	21 52	87 15	—	—
16	47	58.5	16.0	50.9	—25.7	O.H r	24 31	90 0	25.1	—0.6
17	49	58.3	16.8	41.1	44.8	K r	21 46	72 32	—	—

Weltevreden, July 1923.

Chemistry. — "*In-, mono- and divariant equilibria*". **XXIV**. By Prof. F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of October 27, 1923).

Components and composants.

In our considerations we have represented the composition, the thermodynamical potential etc. of the different phases with the aid of the quantities of the components; we may, however, also represent them in another way.

For example we take a quaternary system with the components X Y Z and U. The composition of an arbitrary phase may be represented by:

$$F = x\,X + y\,Y + z\,Z + (1 \quad x - y - z)\,U \quad . \quad . \quad . \quad (1)$$

wherein xX, yY etc. represent x quantities of X, y quantities of Y, etc. In a system of coördinates with the axes x y z the component U is situated, therefore, in the origin of the coordinates; we call U the fundamental-component.

We now take in the quaternary system under consideration, four arbitrary phases M N P and Q; we may represent the composition of the phase F by:

$$F = m\,M + n\,N + p\,P + (1 - m - n - p)\,Q \quad . \quad . \quad . \quad (2)$$

As definite values of m n and p belong to each composition of F, we may, therefore, also consider the composition of F as a function of m n and p.

We call the phases M, N, P and Q, in which we express the composition of a phase F, the *composants* of the system; we shall call Q the *fundamental composant*.

When we represent the composition of a phase F by (1), consequently expressed in its components, then its thermodynamical potential, its free energy etc. a function of x y and z; when we represent the composition by (2), consequently expressed in composants, then we may represent its thermodynamical potential, its free energy etc. also as functions of m n and p. Of course there exist relations between those two way of representations; we shall deduce them further.

We now consider the equilibrium between a variable (f.i. liquid)

phase L and a constant (f.i. solid) phase F. The composition of L may be x, y, z and $1-x-y-z$ expressed in the components, the composition of F: a, b, c and $1-a-b-c$.

When we deduce in some way the condition of equilibrium for this system $F+L$, then we find:

$$\zeta-(x-a)\frac{\partial\zeta}{\partial x}-(y-b)\frac{\partial\zeta}{\partial y}-(z-c)\frac{\partial\zeta}{dz}=\zeta_1 \quad . \quad . \quad . \quad (3)$$

wherein ζ represents the thermodynamical potential of L and ζ_1 that of F.

We now express the composition of L and F in the composants M, N, P and Q. Let be the composition of L: m, n, p and $1-m-n-p$; that of F: α, β, γ and $1-\alpha-\beta-\gamma$. In a similar way as we may deduce (3) we then find:

$$\zeta-(m-\alpha)\frac{\partial\zeta}{\partial m}-(n-\beta)\frac{\partial\zeta}{dn}-(p-\gamma)\frac{\partial\zeta}{\partial p}=\zeta_1 \quad . \quad . \quad . \quad (4)$$

Let us take two variable phases L and L_1 (f.i. two liquids or vapour + liquid or mixed crystals + liquid etc.). We express the composition of those phases with the aid of the components viz. $x\ y\ z$ and $x_1\ y_1\ z_1$, with the aid of the composants viz. $m\ n\ p$ and $m_1\ n_1\ p_1$. In the first case we find as conditions for equilibrium:

$$\left.\begin{aligned}&\zeta-x\frac{\partial\zeta}{dx}-y\frac{\partial\zeta}{\partial y}-z\frac{\partial\zeta}{\partial z}=\zeta_1-x_1\frac{\partial\zeta_1}{\partial x_1}-y_1\frac{\partial\zeta_1}{\partial y_1}-z_1\frac{\partial\zeta_1}{\partial z_1}\\&\frac{\partial\zeta}{\partial x}=\frac{\partial\zeta_1}{dx_1}\qquad\frac{\partial\zeta}{\partial y}=\frac{\partial\zeta_1}{\partial y_1}\qquad\frac{\partial\zeta}{\partial z}=\frac{\partial\zeta_1}{\partial z_1}\end{aligned}\right\} \quad . \quad (5)$$

When expressed in the composants, we find:

$$\left.\begin{aligned}&\zeta-m\frac{\partial\zeta}{\partial m}-n\frac{\partial\xi}{\partial n}-p\frac{\partial\zeta}{\partial p}=\zeta_1-m_1\frac{\partial\zeta_1}{\partial m_1}-n_1\frac{\partial\zeta^1}{\partial n_1}-p_1\frac{\partial\zeta_1}{\partial p_1}\\&\frac{\partial\zeta}{\partial m}=\frac{\partial\zeta_1}{\partial m_1}\qquad\frac{\partial\zeta}{\partial n}=\frac{\partial\zeta_1}{\partial n_1}\qquad\frac{\partial\zeta}{\partial p}=\frac{\partial\zeta_1}{\partial p_1}\end{aligned}\right\} \quad . \quad (6)$$

Generally we may say that the equations for equilibrium have a same form, independent on the fact whether they are expressed in components or in composants.

We now shall consider more in detail the relations between components and composants. For this we take again the composants M N P and Q. We represent, expressed in components, the composition:

of M by α_1 β_1 γ_1 and $1-\alpha_1-\beta_1-\gamma_1$
,, N ,, α_2 β_2 γ_2 ,, $1-\alpha_2-\beta_2-\gamma_2$
,, P ,, α_3 β_3 γ_3 ,, $1-\alpha_3-\beta_3-\gamma_3$
,, Q ,, α_4 β_4 γ_4 ,, $1-\alpha_4-\beta_4-\gamma_4$

In order to express the composition of a phase

$$F = x\,X + y\,Y + z\,Z + (1 - x - y - z)\,U \quad . \quad . \quad . \quad . \quad (7)$$

in the four composants, we put:

$$F = m\,M + n\,N + p\,P + (1 - m - n - p)\,Q \quad . \quad . \quad . \quad (8)$$

so that Q is the fundamental composant. As (7) and (8) represent the same phase F, it follows:

$$\left.\begin{aligned} m(\alpha_1 - \alpha_4) + n(\alpha_2 - \alpha_4) + p(\alpha_3 - \alpha_4) &= x - \alpha_4 \\ m(\beta_1 - \beta_4) + n(\beta_2 - \beta_4) + p(\beta_3 - \beta_4) &= y - \beta_4 \\ m(\gamma_1 - \gamma_4) + n(\gamma_2 - \gamma_4) + p(\gamma_3 - \gamma_4) &= z - \gamma_4 \end{aligned}\right\} \quad . \quad (9)$$

so that m n and p are defined.

In order to define, however, m n and p from (9) the determinant, formed by the coefficients of m n and p may not be zero. Consequently in general we have the following:

in a system of n components we may choose n arbitrary phases like composants, notwithstanding their determinant is not zero.

For a ternary system this means: we may choose three arbitrary phases as composants notwithstanding those are not situated on a straight line. In a quaternary system we may take 4 arbitrary phases as composants notwithstanding those are not situated in a flat plane.

When we represent the composition of a phase F as in (8) with the aid of composants, then we may consider the thermodynamical potential ζ of this phase also as a function of m n and p. Hence it follows:

$$\frac{\partial \zeta}{\partial m} = \frac{\partial \zeta}{\partial x}\cdot\frac{dx}{dm} + \frac{\partial \zeta}{\partial y}\cdot\frac{dy}{dm} + \frac{\partial \zeta}{\partial z}\cdot\frac{dz}{dm} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and still 2 similar relations, which we obtain by substituting in (10) m by n and p. With the aid of (9) we now find:

$$\left.\begin{aligned} \frac{\partial \zeta}{\partial m} &= (\alpha_1 - \alpha_4)\frac{\partial \zeta}{\partial x} + (\beta_1 - \beta_4)\frac{\partial \zeta}{\partial y} + (\gamma_1 - \gamma_4)\frac{\partial \zeta}{\partial z} \\ \frac{\partial \zeta}{\partial n} &= (\alpha_2 - \alpha_4)\frac{\partial \zeta}{\partial x} + (\beta_2 - \beta_4)\frac{\partial \zeta}{\partial y} + (\gamma_2 - \gamma_4)\frac{\partial \zeta}{\partial z} \\ \frac{\partial \zeta}{\partial p} &= (\alpha_3 - \alpha_4)\frac{\partial \zeta}{\partial x} + (\beta_3 - \beta_4)\frac{\partial \zeta}{\partial y} + (\gamma_3 - \gamma_4)\frac{\partial \zeta}{\partial x} \end{aligned}\right\} \quad . \quad . \quad (11)$$

From those equations it follows also, with the aid of (9)

$$m\frac{\partial \zeta}{\partial m} + n\frac{\partial \zeta}{\partial n} + p\frac{\partial \zeta}{\partial p} = (x - \alpha_4)\frac{\partial \zeta}{\partial x} + (y - \beta_4)\frac{\partial \zeta}{\partial y} + (z - \gamma_4)\frac{\partial \zeta}{\partial z} \quad . \quad (12)$$

Above we have seen that for an equilibrium $F + L$ as well equation (3) as (4) is valid; we are able also to prove this by converting equation (3) into (4) with the aid of the above relations. We write (3) in the form:

$$\zeta - x\frac{\partial\zeta}{\partial x} - y\frac{\partial\zeta}{\partial y} - z\frac{\partial\zeta}{\partial z} = \zeta_1 - a\frac{\partial\zeta}{\partial x} - b\frac{\partial\zeta}{\partial y} - c\frac{\partial\zeta}{\partial z}$$

With the aid of (12) we may write:

$$\zeta - m\frac{\partial\zeta}{\partial m} - n\frac{\partial\zeta}{\partial n} - p\frac{\partial\zeta}{\partial p} = \zeta_1 - (a-\alpha_4)\frac{\partial\zeta}{\partial x} - (b-\beta_4)\frac{\partial\zeta}{\partial y} - (c-\gamma_4)\frac{\partial\zeta}{\partial x} \quad . \quad (13)$$

The composition of the phase in components is represented by a, b and c; α β and γ represent the composition of this same phase in composants. In accordance with (9) the following relations are valid:

$$\begin{aligned}
\alpha(\alpha_1-\alpha_4) + \beta(\alpha_2-\alpha_4) + \gamma(\alpha_3-\alpha_4) &= a-\alpha_4 \\
\alpha(\beta_1-\beta_4) + \beta(\beta_2-\beta_4) + \gamma(\beta_3-\beta_4) &= b-\beta_4 \\
\alpha(\gamma_1-\gamma_4) + \beta(\gamma_2-\gamma_4) + \gamma(\gamma_3-\gamma_4) &= c-\gamma_4
\end{aligned}$$

When we add those three equations to one another, after having multiplied the first one with $\frac{\partial\zeta}{\partial x}$, the second one with $\frac{\partial\zeta}{\partial y}$ and the third one with $\frac{\partial\zeta}{\partial z}$, then we find, with the aid of (11)

$$\alpha\frac{\partial\zeta}{\partial m} + \beta\frac{\partial\zeta}{\partial n} + \gamma\frac{\partial\zeta}{\partial p} = (a-\alpha_4)\frac{\partial\zeta}{\partial x} + (b-\beta_4)\frac{\partial\zeta}{\partial y} + (c-\gamma_4)\frac{\partial\zeta}{\partial z}$$

With the aid of this (13) now passes into:

$$\zeta - m\frac{\partial\zeta}{\partial m} - n\frac{\partial\zeta}{\partial n} - p\frac{\partial\zeta}{\partial p} = \zeta_1 - \alpha\frac{\partial\zeta}{\partial m} - \beta\frac{\partial\zeta}{\partial n} - \gamma\frac{\partial\zeta}{\partial p}$$

which is in accordance with (4).

We may also write the four equations (5) in the form (6). For the first one of the equations (5) we may viz. write:

$$\left.\begin{aligned}
&\zeta - (x-\alpha_4)\frac{\partial\zeta}{\partial x} - (y-\beta_4)\frac{\partial\zeta}{\partial y} - (x-\gamma_4)\frac{\partial\zeta}{\partial z} \\
&= \zeta_1 - (x_1-\alpha_4)\frac{\partial\zeta_1}{\partial x_1} - (y_1-\beta_4)\frac{\partial\zeta_1}{\partial y_1} - (z_1-\gamma_4)\frac{\partial\zeta_1}{\partial z_1}
\end{aligned}\right| \quad . \quad . \quad (14)$$

With the aid of (12) (14) passes into the first one of the equations (6).

The three equations (11) excepted, which are valid for the phase without index, we have still also three similar equations, which we obtain from (11) by giving to all variables and to ζ also, the index 1.

We call those three equations 11. As, however, in accordance with (5) $\frac{\partial \xi}{\partial x} = \frac{\partial \zeta_1}{\partial x_1}$ etc., it follows from (11) and (11^a) also $\frac{\partial \zeta}{\partial m} = \frac{\partial \zeta_1}{\partial m_1}$ etc.

For a ternary system with the composants F_1 F_2 and F_0 we have, when we choose F_0 as fundamental component:

$$F = m F_1 + n F_2 + (1 - m - n) F_0 \quad . \quad . \quad . \quad . \quad (15)$$

When we represent the compositions of the components by α and β with the corresponding index, then the equations (9) pass into:

$$\left.\begin{aligned} m(\alpha_1 - \alpha_0) + n(\alpha_2 - \alpha_0) &= x - \alpha_0 \\ m(\beta_1 - \beta_0) + n(\beta_2 - \beta_0) &= y - \beta_0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (16)$$

We now shall deduce those equations also in another way, by which at the same time the meaning of m and n in the graphical representation becomes clear.

We take a system of coordinates with the axes OX and OY (Fig. 1) in which we represent the composition of the phases,

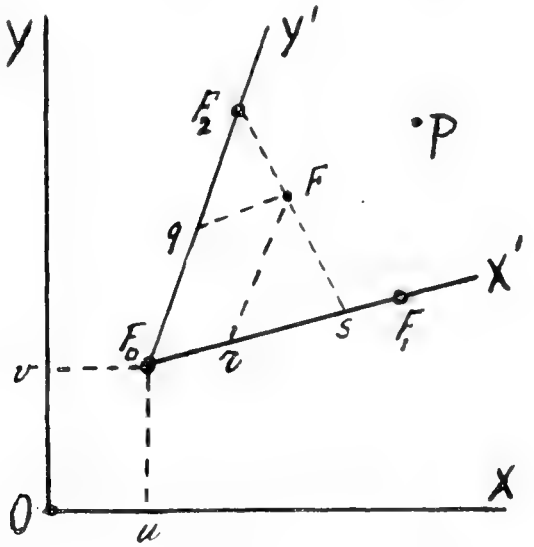

Fig. 1.

expressed in the components. We imagine the three composants F_0 F_1 and F_2 and the arbitrary phase F to be represented by the points F_0 F_1 F_2 and F. Consequently in the figure is $F_0 v = \alpha_0$, $F_0 u = \beta_0$ etc. Now we take F_0F_1 as new X-axis and F_0F_2 as new Y-axis; then the new coordinates of the point F are Fq and Fr; we put $Fq = x'$ and $Fr = y'$. When we call the angles, which the new axes F_0F_1 and F_0F_2 are making with the original X-axis φ_1 and φ_2 then it follows from the figure:

$$\left.\begin{aligned} x &= \alpha_0 + x' \cos\varphi_1 + y' \cos\varphi_2 \\ y &= \beta_0 + x' \sin\varphi_1 + y' \sin\varphi_2 \end{aligned}\right\} \quad \ldots\ldots (17)$$

When we represent the length of F_0F_1 and F_0F_2 by l_1 and l_2, then we may write for (17)

$$\left.\begin{aligned} x-\alpha_0 &= \frac{x'(\alpha_1-\alpha_0)}{l_1} + \frac{y'(\alpha_2-\alpha_0)}{l_2} \\ y-\beta_0 &= \frac{x'(\beta_1-\beta_0)}{l_1} + \frac{y'(\beta_2-\beta_0)}{l_2} \end{aligned}\right| \quad \ldots\ldots (18)$$

Now we shall express the composition of the phase F in that of the three composants: F_0, F_1 and F_2. We find:

quantity of F_2: quantity of $(F_0 + F_1) = Fs : FF_2$

or: quantity of F_2: quantity of $(F_0 + F_1 + F_2) = Fs : F_2s$

When we put the total quantity of $F = F_0 + F_1 + F_2$ equal to zero, and when we bear in mind that:

$$Fs : F_2s = Fr : F_2F_0 = y' : l_2$$

then follows: quantity of $F_2 = \frac{y'}{l_2}$.

In a similar way we find: quantity of $F_1 = \frac{x'}{l_1}$.

Consequently there are wanted for forming the unit of quantity of the phase F: $\frac{x'}{l_1}$ quant. of F_1 and $\frac{y'}{l_2}$ quant. of F_2 and consequently also $1 - \frac{x'}{l_1} - \frac{y'}{l_2}$ quantities of F_0. We may write, therefore;

$$F = \frac{x'}{l_1}F_1 + \frac{y'}{l_2}F_2 + \left(1 - \frac{x'}{l_1} - \frac{y'}{l_2}\right)F_0 \quad \ldots\ldots (19)$$

When we put $\frac{x'}{l_1} = m$ and $\frac{y'}{l_2} = n$ then (18) and (19) pass into (15) en (16).

Hence it appears a.o. that m and n do not represent the coordinates x' and y' of the phase F, but they are functions of them; when m and n are known, then also x' and y' are known and reversally. For this reason we may call m and n yet also coordinates.

The coordinates of the composant

F_0	are	$x' = 0$	$y' = 0$	consequently	$m = 0$ and $n = 0$
F_1	,,	$x' = l_1$	$y' = 0$	,,	$m = 1$,, $n = 0$
F_2	,,	$x' = 0$	$y' = l_2$	,,	$m = 0$,, $n = 1$

Of course this is also in accordance with (15); when herein we put f.i. $m=1$ and $n=0$ then phase F represents the composant F_1.

When we express the composition of a phase in its components, consequently in x and y, then x and y are positive and $x+y \leqq 1$. When, however, we express its composition in composants, then m and n may also be negative and also $m+n>1$. The latter is the case f.i. for a phase, represented by the point P. In (15) m and n are then positive and $1-m-n$ is negative.

When we have a quaternary system then similar relations exist between the coordinates viz.

$$x' = m\,l_1 \qquad y' = n\,l_1 \qquad z' = p\,l_1$$

Till now we have assumed that each of the n composants of a system of n components contains also those n components. It is apparent, however, that we may choose the composants also in such a way that one or more or even all composants contain less than n components. Of course the n composants together must contain the n components. We may consider the representation with the aid of components as a special case of the representation with the aid of composants; each of the composants then contains a single component only. We shall, however, continue by calling this a representation with the aid of components. When, however, there is at least one composant, which contains more than one component, then we shall speak of a representation with the aid of composants.

As it is known, the deduced functions of the thermodynamical potential become infinitely large when the quantities of one or more of the components approach to zero. In a quaternary system f.i. $\frac{\partial \zeta}{\partial x}$ becomes infinitely large when x or $1-x-y-z$ approaches to zero; $\frac{\partial \zeta}{\partial y}$ when y or $1-x-y-z$ and $\frac{\partial \zeta}{\partial z}$ when z or $1-x-y-z$ approaches to zero.

Using composants this is otherwise, however. It follows viz. from (11) that $\frac{\partial \zeta}{\partial m}$, $\frac{\partial \zeta}{\partial n}$ and $\frac{\partial \zeta}{\partial p}$ become infinitely large, only then when one or more of the functions $\frac{\partial \zeta}{\partial x}$, $\frac{\partial \zeta}{\partial y}$ and $\frac{\partial \zeta}{\partial z}$ are infinitely large and this may take place, as we have seen above, only when one or more of the conditions:

$$x=0 \qquad y=0 \qquad z=0 \qquad 1-x-y-z=0 \quad . \quad (20)$$

is satisfied. In general $\frac{\partial \zeta}{\partial m}$, $\frac{\partial \zeta}{\partial n}$ or $\frac{\partial \zeta}{\partial p}$ become, therefore, infinitely

large when we give such values to m, n and p, that one or more of the conditions (20) are satisfied. It is apparent that this may be casually only for $m = 0$ or $n = 0$ or $p = 0$ or $1 - m - n - p = 0$.

At the same time the following is apparent. When we give to m, n and p such values that f.i. x becomes $= 0$, then in (11) $\frac{\partial \zeta}{\partial x}$ becomes infinitely large, so that $\frac{\partial \zeta}{\partial m}$, $\frac{\partial \zeta}{\partial n}$ and $\frac{\partial \zeta}{\partial p}$ become infinitely large at the same time. When, however, we have chosen the composants in such a way that $\alpha_1 = \alpha_4$ then only $\frac{\partial \zeta}{\partial n}$ and $\frac{\partial \zeta}{\partial p}$ become infinitely large, while $\frac{\partial \zeta}{\partial m}$ remains finite.

When a liquid has the composition:

$$L = xX + yY + zZ + \ldots$$

wherein X, Y etc. represent components, then the stability requires that for all values of dx, dy etc.

$$\left(\frac{\partial \zeta}{\partial x} dx + \frac{\partial \zeta}{\partial y} dy + \ldots\right)^{(2)} > 0 \quad \ldots\ldots \quad (21)$$

When we imagine L to be divided into

$$L = \varkappa L_1 + (1 - \varkappa) L_2$$

wherein:

$$L_1 = (x + dx_1) X + (y + dy_1) Y + \ldots.$$
$$L_2 = (x + dx_2) X + (y + dy_2) Y + \ldots$$

then must

$$\zeta < \varkappa \zeta_1 + (1 - \varkappa) \zeta_2$$

from which (21) is following. When we now express the composition of L in composants viz.:

$$L = m M + n N + p P + \ldots.$$

then it follows in the same way that

$$\left(\frac{\partial \zeta}{\partial m} dm + \frac{\partial \zeta}{\partial n} dn + \ldots\right)^{(2)} > 0$$

must be true for all values of dm, dn etc.

(To be continued)

Leiden. Inorg. Chem. Lab.

Anatomy. — "*Thymus, spiracular sense organ and fenestra vestibuli (ovalis) in a* 63 *m.m. long embryo of Heptanchus cinereus*". By Prof. J. W. VAN WIJHE.

(Communicated at the meeting of September 29, 1923).

Many years ago I received this embryo from the Zoological Station at Naples. It was fixed in sublimate and preserved in alcohol. Just as another specimen it was treated with methylene blue, in order to make a skelet preparation of it.

This having proved quite successful with the one embryo, I decided to preserve the other, so as to make a series of cross sections later, in order also to be able to examine the remaining organs. I intended to wait with this until I had more of this rare material in different stages. I however received only one more embryo, 255 m.m. long, which was simply treated with alcohol and was much too large for making a series of sections. Here one would have to restrict oneself to only a few parts. For this specimen I am again indebted to the direction of the Station at Naples.

When in the autumn of 1922 I had finished with the development of the skeleton of Acanthias vulgaris [1]) I decided not to wait any longer, and a series of cross sections of the 63 mm. long embryo was made. The preservation proved to be excellent, notwithstanding the previous long treatment of HCl alcohol necessary for the elimination of the methylene blue from the remaining tissues, in order to restrict the colour to the cartilage. The staining of the sections with ammonia-carmine was also successful; but the light blue tint of the cartilage could not be intensified by the after-treatment with methylene blue or victoria blue. The reason for this remained unknown to me.

In the 255 m.m. long embryo, which had been in alcohol for many years, the cartilage suffered itself to be stained deep blue.

[1]) VAN WIJHE, J. W. Frühe Entwicklungsstadien des Kopf- und Rumpfskeletts von Acanthias vulgaris. Bijdragen tot de Dierkunde, publ. by the Kon. Zool. Genootsch. Natura Artis Magistra at Amsterdam. Afl. 22, Feestnummer voor MAX WEBER, 1922.

1. *Thymus.*

The development of the thymus in the Selachians was first described by Dohrn (1884). The facts then found by him were principally confirmed by later investigators. Hammar, who had given many years to the study of the structure, development and function of this organ in nearly all the principal groups of vertebrates, described the development in the Selachians in 1912, and gave a detailed account of the results of his predecessors.

He found, that in all vertebrates from fish up to man, the thymus continues to grow till the time of puberty. Then an involution period begins, wherein it as a rule atrophies, without totally disappearing.

The thymus, in all vertebrates, begins to form as a local proliferation of the epithelium of the gill clefts.

In man it appears principally, if not exclusively, on the third gill cleft, but in the Selachians, which generally have six gill clefts, a beginning of the thymus is described on each gill cleft. These however speedily disappear on the first and last, sometimes even on the last two gill clefts.

Not all investigators are of opinion that the thickening of epithelium cells of the first gill cleft (spiracle) may be considered as a thymus, and it is possible that here an interchange may have taken place with the place of origin of the spiracular sense organ.

Soon after its appearance, one can distinguish in the thymus two different kinds of cells, viz. a network of flat epithelial cells, which encloses groups of round cells in its meshes.

These round cells multiply themselves so quickly, that the network can no longer be discerned unless in very thin sections.

The whole organ, which formerly was pear-shaped and afterwards has the shape of a grape bunch, appears to be wholly constituted of round cells, which form a solid mass without lumen. These cells hardly have any protoplasm, and therefore give the appearance as if one only has to do with an accumulation of nuclei.

There are two opinions concerning the derivation of these round cells, which strongly resemble the lymphocytes of the blood. Many hold them for epithelium cells, which have rounded themselves off; others again take them to be true lymphocytes, which have penetrated the organ from the bloodvessels and the neighbouring mesenchym. The latter opinion is emphatically upheld by Hammar for all classes of vertebrates.

The question as to which of the two opinions is correct, cannot

be settled by the study of the 63 m.m. long embryo of Heptanchus, but a further question can be explained thereby, viz. whether the thymus has to be considered as a gland which has lost its original excretory duct and thus only has internal secretion left. It would then find itself in a similar condition as the anterior lobe of the hypophysis and the thyroid gland, which, however, in the embryo of vertebrates, always have an excretory duct which is only lost during the further course of development.

The thymus does not sever itself from the epithelium of the branchial gut in Cyclostomes and most of the bony fishes. This is however the case with the remaining vertebrates. But a true excretory duct, as a rule, does not appear. This would be expected in sharks, but FRITSCHE (1910) says: "Ein Lumen und einen Ausführungsgang habe ich bei Spinax ebensowenig auffinden können wie DOHRN bei seinen Haifischen."

In a very early stage of rays (Torpedo), they however noticed something which resembled an excretory duct.

In some of the sharks examined up to now, the body of the thymus separates itself directly, without a pedicle, from the epithelium of the branchial gut; while in others it still remains connected for some time by a stalk to the epithelium.

This stalk lacks the characteristics of an excretory duct, because it not only has no lumen, but also shows the same structure as the body of the thymus and consists almost exclusively of the rounded cells, which resemble lymphocytes.

In our embryo of Heptanchus we on the contrary find an excretory duct *in optima forma* for each of the thymus divisions (thymomeres) which are found on both sides of the body, one for each side from the second to the seventh branchial cleft. There are 8 gill clefts, but in the first (spiracle) and last the thymus is absent.

It is the largest in the second and third cleft and has the form of a bunch of grapes. The bunch is smaller in the 4th cleft, in the 5th still smaller, and in the 6th the thymus no longer has the bunch form, but is composed of a single acinus, into which the excretory duct opens.

In the 7th cleft every acinus is found missing from the short excretory duct.

In the figure of the section we see the large thymus of the 2nd branchial cleft. It runs over the top of the 1st epibranchial and then continues as the fairly long excretory duct. This has an obvious lumen, which with its one end opens at the top of the branchial

cleft, with the other reaches to the body of the thymus without entering it.

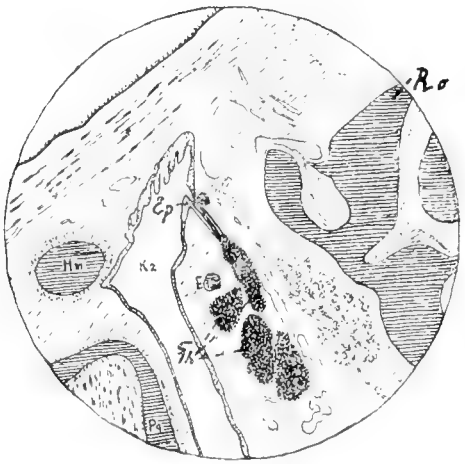

Fig. 1. Cross section through 2nd branchial cleft of a 63 m.m. long embryo of Heptanchus cinereus. In this and in the following figs. the cartilage (stained blue in section) is striated horizontally.

The wall of the duct is 2 cells thick, and is constituted of a double layer of fairly flat epithelium cells, amongst which not a single round cell is to be found.

The excretory duct of each of the remaining thymomeres shows a similar structure, viz. a double layered epithelial wall, encircling a lumen, which opens into its respective gill cleft.

These ducts from the 2nd caudally, gradually shorten; the last (6th) forming a rather unimportant, yet distinct attachment to the 7th branchial cleft.

The excretory ducts are not permanent. They later on lose their epithelial structure and lumen. This e. g. happened in the 225 m.m. long embryo. Here, in the place of the excretory duct of the first (anterior) thymomere, one finds a long pedicle, which appears as an outgrowth of the thymus. The pedicle runs over the top of the 1st epibranchial and reaches the wall of the branchial cleft.

It shows itself as a chord, which appears entirely to consist of lymphocyt-like round cells. No traces are left of the original epithelial structure and lumen. I however do not wish to deny the presence of a reticulum. It would also be possible to make it clear in the pedicle by appropriate methods.

For completeness the so called epithelial bodies and the suprapericardial organ should also be mentioned. In the 63 m. m. embryo an epithelial body is found, immediately above the opening of the 1st and 2nd thymomere. Each little body is a round isolated cellmass, which resembles an acinus of the thymus in form and

size, but is more compact; owing to the fact that it has finer lymph-spaces than the thymus. No trace of such a body was to be found at the 3rd, 4th, 5th and 6th thymomere.

The suprapericardial body was discovered by VAN BEMMELEN [1]) (1885) at the end of the branchial gut. Later it was found in all classes of vertebrates. It is generally taken as the last indication of an abortive branchial pouch, and mostly appears on only one side of the body.

In 1906 BRAUS found it in the 67 m.m. long embryo of Heptanchus, which very likely originates from the same mother-animal as mine, and I can corroborate his statement. It is only well developed in the left half of the body, and shows itself as a little bladder, the lumen of which is encircled by a single layer of fairly columnar epithelium cells. It is to be seen on 35 sections, and is situated as BRAUS stated, behind the last visceral arch, in the angle which this makes with the ceratobranchial. Just as BRAUS, I found it near its posterior margin connected with the epithelium at the base of the branchial gut by a short pedicle.

On the right side the organ is rudimentary.

I found it represented by a flattened little group of epithelial cells without a lumen, and totally severed from the gut epithelium. This is visible in the sections passing through the posterior half of the vesicle on the left. BRAUS does not mention this little group.

His specimen was probably somewhat further developed than mine,

[1]) Owing to the presence of a suprapericardial body in the embryos of Heptanchus (VAN BEMMELEN in vain sought for it in the adult animal) one cannot assume that, in higher animals, this little body is the remains of a branchial cleft, which is present in the Notidanides as such. The morphological significance of this organ is a problem. One may of course believe that it is the remains of a branchial cleft, which still lies further caudally than the last (8th) of Heptanchus. BRAUS e. g. takes it to be the rest of a 10th branchial pouch.

He professes to find the remains of a (9th) branchial pouch in a slight protrusion of the intestinal wall behind the last branchial arch, in the angle between the last (7th) ceratobranchial, and a caudalwards directed protuberance on its ventral side.

Although this protuberance chondrifies continuous with the 7th ceratobranchial, he considers it to be the remains of an 8th branchial arch.

I cannot agree with these conceptions. In my specimen the rather long protuberance is still quite prochondral, and just like the prochondral cardiobranchial end, lies in the beginning of the oesophagus. In the protuberance I can only discern a processus muscularis of the 7th ceratobranchial, morphologically insignificant. An intestinal protrusion which could also be considered as a 9th branchial pouch, is not present, and I must consider it as an artificial product in the specimen of BRAUS.

and this little group more atrophied. He thought he saw an indication of an antimere of the left vesicle on the right side of the body, in the shape of a more caudally situated diverticulum of the branchial gut.

Let us however return to the thymus. The *genus* Heptanchus is indeed rightly regarded as the most primitive of the living Selachians. The number of visceral pouches (i. e. 8) surpasses that of all other fishes and higher animals. Only the anterior 5 are still formed in mammals.

Concerning the 63 m.m. long embryo of Heptanchus, we may now assume, that also its thymus appears in a more primitive form than in the development of higher animals.

The original function of the thymus could then not have been internal secretion only, but it must also have removed products through its excretory ducts.

Originally each thymomere was a true gland, according to the old notion, with an excretory duct even as was the case with the thyroid and the anterior lobe of the hypophysis.

The presence of excretory ducts is also of importance for the conception of the morphological significance of the gland. Since the researches of Dohrn, it is generally accepted that the thymus is a branchiomere organ, a division of which occurred on each branchial cleft.

Now Amphioxus has on each of its many branchial clefts a glandular body, which opens with its excretory duct into the top of the cleft. This branchionephros functions as an excretory organ, and for many years I have presumed, that it would prove homologous to the thymus of higher animals.

This presumption was strengthened, when in 1909 Goodrich found that the branchionephros does not develop from the coelomic epithelium, as one would rather be inclined to assume for an excretory organ in chordates.

But he does not state that it develops from the branchial epithelium. His drawings however give this impression. Might this impression prove to be correct by later investigations, then the branchionephros develops from the same tissue as the thymus of higher animals. Cells resembling lymphocytes are never found in it. Lymphocytes do not occur in the blood of Amphioxus, the blood of which only consists of plasma, without any red or white blood corpuscles, just as the blood in its earliest stage in craniates.[1])

[1]) A few investigators profess to have found cells in the blood of Amphioxus. I have never observed any in my numerous sections of larvae and adult animals.

The presumed homology of the thymus and branchionephros has also been supported from the side of the craniates, now that, in the development of such a primitive form as Heptanchus, the presence in the thymus of excretory ducts, which in Amphioxus analogously open into the branchial clefts, has been shown.

If the branchionephros develops from the branchial epithelium, the chief difficulty to homologize it with the thymus, I think then lies in the period of development of this gland. One should expect the thymus to become perceptable in a very early period of its development, but this only happens very late.

The reason for this is because the original function no longer comes to development. It is taken over by the pronephros and the mesonephros. The other function of the thymus i.e. its internal secretion, caused by the lymphocytlike cells, must phylogenetically have originated much later.

2. *Spiracular sense organ.*

In no vertebrates does a division of the thymus come to development in the first branchial cleft (spiracle). It appears not even to be formed there at all. On the other hand, we find on the wall of the spiracle in the embryos or larvae of the more primitive fishes: Selachians, Ganoids and Dipnoi a sense organ, which is not met with on any of the remaining branchial clefts. These adult fishes also possess one.

We find it even in those forms (Dipnoi and Holostei) in which the spiracle, which is developed in the manner of an intestinal pouch, no longer breaks through outwardly.

It was discovered by Ramsay Wright in 1885, who found it as a protrusion of the medial wall of the spiracular visceral pouch of the Holostei (Lepidosteus and Amia). This protrusion (diverticulum) is directed upwards and surrounded by the cartilaginous auditory capsule; in other words, it lies in a canal of the lateral cartilaginous wall of the otic region of the skull, but otherwise has no relation to the auditory organ.

A similar canal in the cranial cartilage, into which a diverticulum of the spiracular wall penetrates, was discovered by Bridge in Polyodon. The same was also observed by Wright in the sturgeon. The presence of a sense organ in these Chondrostei is, however, not mentioned.

Wright found, that in the Holostei this sense organ is innervated by a branch of the ram. oticus of the facial nerve, which in the

Ganoids (Chondrostei and Holostei) is likewise overgrown by the cartilaginous auditory capsule, and of which (ram. oticus) it was known that it sends out branches in this region to the sense organs, belonging to the lateral line system.

These sense organs, called neuromasts (Nervenhügel) by WRIGHT, lie either free on the surface, or protected in little sacs, grooves or canals; all are of ectodermal derivation. Now it was noteworthy that the sense organ of the spiracular pouch also resembled the structure of a neuromast, although WRIGHT evidently thought it to be of entodermal origin. It seemed as if one here had the unexpected example of a sense organ of the Chordates, which did not originate from the ectoderm, although it was still supplied by a nerve, belonging to the lateral line system of the epidermal sense organs.

The study of the Dipnoi dispelled the singularity of this phenomenon. In this group PINKUS (1895) discovered in Protopterus annectens a little bladder with a sense organ on its wall, and imbedded in the cartilage of the otic region. The sense organ — evidently a neuromast according to the fig. — is supplied by a caudalwards running branch of the facial nerve, the branch belonging to the lateral line system.

PINKUS still describes two more caudalwards running branches from the lateral line system of the n. facialis. The one forms the well known anastomosis with the ramus lateralis vagi (and glossopharyngei) the other he calls ram. oticus. He, however, draws the origin (l.c. fig. 3) of these branches so close to each other that, according to my opinion, one has to consider them as the strongly developed homologue of the ram. oticus of the Ganoids.

Of this organ PINKUS says (l. c. p. 307) "Das Organ ist zweifellos ein Derivat des Seitenkanales. Ueber seine Bedeutung vermag ich übrigens nichts auszusagen, da vergleichend anatomische und entwicklungsgeschichtliche Thatsachen mir bisher fehlen".

For the knowledge of the development we are indebted to AGAR, (1906) who examined the first stages of the spiraculum in Lepidosiren and Protopterus.

He showed that this sense organ is of ectodermal origin. This seat of origin reaches the top of the solid gut protuberance, which represents the spiracle, and then severs itself from the ectoderm. The organ then naturally gives the impression of having been derived from the entoderm.

AGAR like PINKUS, was not aware of the work of RAMSAY WRIGHT, otherwise he would undoubtedly have mentioned, that the presence of a spiracular sense organ in Holostei was already known. He also

would not have neglected to point out, that, in the Holostei, we have no reason to believe in the entodermal origin of the sense organ, now that in the Dipnoi [1]) its formation from the ectoderm is manifest.

As opposed to PINKUS, AGAR says "This organ has no relation to the lateral line system of sense organs". To my opinion, however, it undoubtedly belongs to this system, because it possesses a neuromast, is supplied by a branch from the lateral line system of the facial nerve, and moreover is clearly of ectodermal origin in the Dipnoi.

The majority of epidermal sense organs, sinks under the epidermis during the ontogenetic period, and finds protection by the subcutaneous connective tissue. Only one organ having its seat of origin in the immediate vicinity of the spiracle, sinks therein, acquiring a considerable development.

In my opinion this not only happens when the spiracle no longer breaks through outwardly, retaining its opening into the gut, as in the Holostei, but also, when it moreover loses its connection with the gut, as in the Dipnoi.

Let us now proceed to the Selachians. In these WRIGHT examined the spiracle of a 60 m.m. long embryo of Mustelus. Here he found two diverticula, situated above each other, on the medial wall. The dorsal diverticulum reached till under the canalis semicircularis lateralis of the auditory organ, and was already discovered in a number of adult Selachians, by JOH. MÜLLER (1841).

The ventral diverticulum did not reach the cranial cartilage, and at one place contained columnar epithelium, which he took for sense organ epithelium, and which according to him, was supplied by the ram. praetrematicus of the facial nerve. This innervation would lead us to expect, that we have here to deal with a different sense organ to that in the Holostei. PHELPS ALLIS, however, in 1901, examined a 122 m.m. long embryo of Mustelus, and was able to trace the nerve from the organ till near the ram. oticus, the same branch which also supplies the sense organ in the Holostei.

Independent of WRIGHT's work, that of VAN BEMMELEN appeared in the same year (1885). The latter, besides in Mustelus, found both the diverticula in a great number of Selachians, in embryos as well

[1]) GREIL (1913) mentions the ectodermal origin of the sense organ ("Hyomandibular organ") in Ceratodus, and its innervation by a branch from the lateral line system ("ram. hypoticus") of the facial nerve. Whether the sense organ in Ceratodus is afterwards also surrounded by the cranial cartilage, I do not find mentioned.

as in the adult fishes. He found both (the dorsal and the ventral) simultaneously in the same animal, in the forms which now-a-days, after TATE REGAN, are called Galeoidei. In rays on the contrary, only the ventral diverticulum of the examined fishes: Raja, Torpedo, Trygon and Myliobatis was found to be present. The dorsal one was absent in concurrence with the results of JOH. MÜLLER, who found it in rays only in the family of the Rhinobatidae.

Vice versa the ventral diverticulum was found missing, while only the dorsal one was present in Acanthias and Heptanchus; each of which is a representative resp. of the groups Squaloidei and Notidanoidei.

On the ventral diverticulum a follicle, resembling an oval bladder, develops in all forms which possess it. It nearly touches the auditary labyrinth, is lined on the inside with columnar epithelium, and is connected to the wall of the spiracle by a pedicle, which may, or may not have a lumen. In an adult Torpedo the bladder was found to be very large.

As regards the morphological significance of the follicle, VAN BEMMELEN thought of the probability of a homologue with the suprapericardial body, which primarily is also a single little bladder. He says (l. c. p. 178) "[später] tritt aber der grosse Unterschied ein: die Suprapericardialkörper entwickeln sich zu drüsenartigen Gebilden [1]) die Spritzlochbläschen treiben nur eine oder zwei acinöse Ausstülpungen oder bleiben wohl ganz einfach."

VAN BEMMELEN further thought of the probability of considering the follicle, even as the suprapericardial body, as the remains of an original gill cleft.

My opinion is that this conception cannot be adhered to any longer, and that the follicle is a spiracular sense organ bladder.

VAN BEMMELEN did not consider this possibility, because he had evidently not observed a supplying nerve.

No mention is made of the appearance of a follicle from the dorsal protrusion of the spiracle in the Galeodei. We may thus accept that it is absent there.

Acanthias and Heptanchus only show the dorsal protrusion. Is the spiracular sense organ now also found missing in them or not?

VAN BEMMELEN speaks of a "dorsale Ausstülpung", but also calls it an "Anhang" of the spiracle. He says: (l. c. p. 176). "Bei erwachsenen Exemplaren von Acanthias endlich konnte ich den Anhang

[1]) Their structure in the Selachians, then has much in common with that of the thyroid gland, from which they, however, totally differ morphologically.

als ein sackförmiges, ungefähr 3 m.m. langes Gebilde aus dem Bindegewebe frei präpariren, seine Wände zeigten sich ausserordentlich dicht und inwendig glatt, das Epithelium hoch und drüsig. Ebenso zeigte sich der dorsale Anhang von Heptanchus, aber relativ noch kürzer". As it will presently be seen, he undoubtedly dissected out the sense organ bladder.

HOFFMANN (1899) inter alia also investigated the development of the diverticulum of the spiracle in Acanthias. He found it to make its appearance first in 28 m.m. long embryos and innervated by a branch from the lateral line system of the facial nerve.

He considers this branch, which also supplies epidermal sense organs, most likely homologous to the ram. oticus of the Ganoids. The diverticulum is soon directed forwards with its blind end, and unites itself there with the nerve. I can confirm this from my material of Acanthias.

HOFFMANN discovered the innervation, well knowing of the work of WRIGHT, from which he quotes in detail. He, however, missed the conclusion that a sense organ had to be present. He was too much under the impression of having here to do with the vestigial part of a branchial pouch, which had disappeared.

Besides the two embryos of Heptanchus, my own investigation also includes a series of sections (15μ thick) through embryos of Acanthias varying in length from 23 to 98 m.m.

In the 23 m.m. long embryo, the anterior wall of the spiracle forms a rostrally directed diverticulum, next to the auditory organ, from which it is separated by the jugular vein (the nervus facialis running under the vein). The diverticulum is to be seen on 7 sections anterior to the external opening of the spiracle, and has the shape of a cone flattened on one side, the axis of which runs parallel to the longitudinal axis, passing through the notochord. The three anterior ones of the seven sections pass through the top of the cone, which is distinguished by its columnar epithelium, so that the lumen appears for the first time on the third section. One also sees the termination of the branch of the ram. oticus connected here to the group of the columnar cells. HOFFMANN already pointed out, that one could stipulate, through this connection the situation of the organ before it is more clearly defined.

A cross section through the anterior margin of the external opening of the spiracle on the skin at the same time passed through the internal opening towards the intestine in an embryo of 39½ m.m. of which I in 1922 described the skull. The diverticulum is to be seen on 21 sections rostralwards. Just as in the embryo of 23 m.m.

it runs forwards along the auditory capsule and is separated from it by the jugular vein and the facial nerve [1]).

If we trace the diverticulum from the base rostrally, we see it after 8 sections already changed into a flat and narrow duct with a lateral and medial wall. The duct is prolonged over 4 sections, and then with nearly no change of lumen, passes over into the top part of the diverticulum, which is perceptable on 9 sections. The medial wall of this part has over its whole length a neuromast, whose posterior end is clearly defined. Near the rostral end (the blind top) of the diverticulum the branch of the ram. oticus unites with the neuromast.

We may now, proceeding from the anterior margin of the spiracle, distinguish three parts, seen resp. on 8, 4 and 9 sections which we shall call vestibulum of the spiracle, excretory duct and corpus of the sense organ bladder.

Excretory duct and corpus are partners, but the vestibulum is nothing more than an ordinary diverticulum of the anterior wall of a visceral pouch, and disappears later, in consequence of the enlargement of the external opening of the spiracle.

The vestibulum is still present in an embryo 69 m.m. long, but in embryos of 78 m.m. or more, it has disappeared. We then only see on a section, passing posterior to the anterior margin of the spiracle, the opening, which meanwhile has become very minute, of the excretory duct. Then the condition of the sense organ bladder principally corresponds to that of the organ which occurs in the adult animal. It then forms an appendix of the spiracle. The description by VAN BEMMELEN of the Galeoidei and rays also applies to the sense organ of Acanthias.

Probably these bladders are homologous in all the Selachians and of ectodermal origin. They have in some forms sunk somewhat deeper into the spiracle, than in others. We shall still examine the little bladder sowewhat closer in a series of cross sections of the Acanthias embryo 98 m.m. long.

The very minute opening in the anterior wall of the spiracle is only to be seen in one section. From here the organ passes rostralwards over 50 sections. It runs along the auditory organ from

[1]) During the translation of this paper I prepared a series of sagittal sections, stained with haematoxylin and eosin, of a 22 m.m. long embryo of Torpedo marmorata. I found the deep neuromast at the inner wall of the spiracle innervated by a branch of the ram. oticus, crossing the outer side of the vena jugularis, just as in Acanthias.

which, — as previously — it is seperated by the jugular vein and the facial nerve.

The corpus of the bladder, with its long neuromast, is visible on the anterior 21 sections. The excretory duct falls in the following 29 sections. The neuromast thus nearly constitutes half the length of the organ, and is much larger than in the lateral line system organs of the skin. Round the corpus one sees the mesenchym in more compact formation, the first stage of a connective tissue capsule. The excretory duct, immediately posterior to the corpus, shows a different construction than further caudalwards.

Fig. 2*a*. Cross section through the otic region of the skull and the anterior wall of the spiracle, from a 98 m.m. long embryo of Acanthias vulgaris.

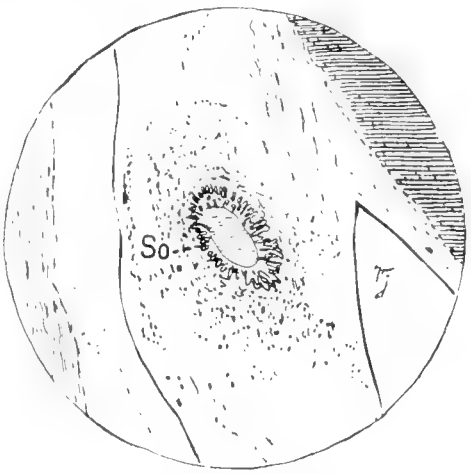

Fig. 2*b* shows the spiracular organ under high power. Its contents, mucus (stained blue in section) are seen as thin striations.

On the first 5 sections behind the corpus, the medial wall of the duct is thickened, as the result of the proliferation of the outer layer of epithelium cells. Here the oval lumen is wider than in other places. The longitudinal axis of the oval is more or less twice as long as in the corpus. On the following 24 sections this lumen continually decreases, the wall consisting of two layers of cells. Those of the inner layer are very flat, those of the outer layer may be called cubic.

It is of importance that the corpus of the sense organ bladder and the proximal part (5 sections) of the duct, should be filled with mucus, which in this stage (and later) allows itself to be stained blue, just as in the ampullary and canal organs of the lateral line system. In the distal part of the duct (24 sections) the mucus is present in lesser quantity.

From this we may see, that the spiracular sense organ shows itself to belong to the lateral line system of epidermal sense organs, which is generally also understood by the term mucus-organs. The direct proof has not yet been given, but may perhaps be found in stages earlier than those which I have studied.

The ram. oticus, in all the studied embryos, arises with a ganglion like thickening from the buccal ganglion of the facial nerve.

In the 39½ m.m. long embryo, it runs along the cartilage of the ear capsule — but not yet surrounded by the cartilage — dorsally and caudalwards. It sends off a few thin branches to the organs in the lateral line canal of the regio otica, and a thick branch, which goes to the spiracular sense organ across the jugular vein.

In the 98 m.m. long embryo, a part of the ram. oticus is overgrown by the cartilage of the ear capsule. This is also the case with the Ganoids. Contrary to the Selachians the sense organ itself is surrounded by cartilage in both Ganoids and Dipnoi.

We shall now pass on to the 63 m.m. long embryo of Heptanchus. The small external opening of the spiracle is here situated far backwards. The fissure like opening in the gut reaches still further rostralwards. If we accept that the beginning — the base — of the vestibulum falls on the section which passes through the anterior margin of this fissure, then the top of the vestibulum lies still 28 sections further forwards. In this top the sense organ bladder opens without an excretory duct. It can be traced in 12 sections rostralwards, along the auditory organ, from which it is separ-

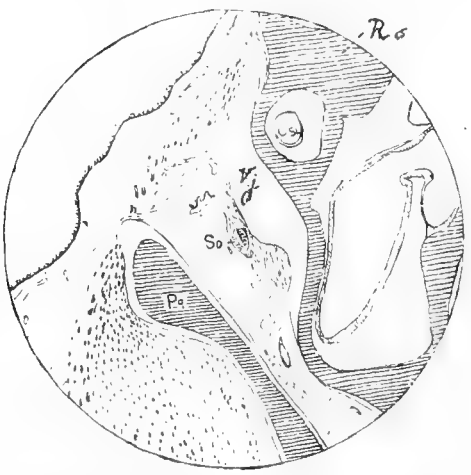

Fig. 3*a*. Cross section through the otic region of the skull of a 63 m.m. long embryo of Heptanchus cinereus.

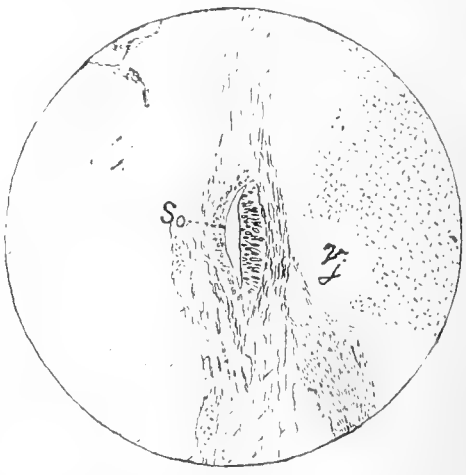

Fig 3*b* shows the spiracular organ under higher magnification.

ated by the jugular vein. The neuromast on the medial wall just projects with its posterior margin from the vestibulum.

I was not successful in finding the supplying nerve. Perhaps it is owing to the intensely stained connective tissue capsule, which is more developed than in the largest of the examined embryos of Acanthias. In the 225 m.m. long embryo the organ was so badly preserved, that nothing of importance can be mentioned [1]).

3. *Fenestra vestibuli.*

In the 63 m.m. long embryo of Heptanchus, the attachment of the hyomandibular to the auditory capsule is brought about by a thin layer of connective tissue, wherein I can find no cavity of

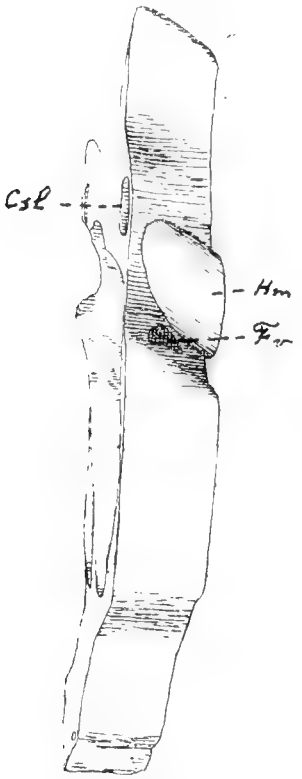

Fig. 4. Lateral surface of the model of a disk from the cartilage of the regio otica of an embryo of Heptanchus cinereus. The disk is placed in such a position that a part of the anterior surface with the canalis semicircularis lateralis, is just visible, and the fenestra vestibuli is not covered by the upper part of the hyomandibular.

[1]) Before the translation of this paper, the work of VITALI (Anat. Anzeig. 1911 and 1912) had escaped my notice, and I am indebted to Dr. BENJAMINS of Utrecht for having called my attention to it. As he remarks, this paratympanic organ in birds must be the homologue of the spiracular sense organ. An interesting referate of the works of VITALI on this organ by RUFFINI "Sull organo nervoso paratimpanico di G. VITALI od organo del volo degli uccelli" is to be found in "Archivio Italiano di Otologia Rinologia e Laringologia" publ. by GRADENIGO. Vol. 31, 1920.

articulation. It is prolonged over 49 sections, 15 μ thick. Immediately ventral to the anterior portion of this place of attachment one sees in the sections 5, 6 and 7 (in antero-posterior sequence) a connection through a small opening in the wall of the auditory capsule, between the mesenchym which in this stage fills the perilymphatic space, and the mesenchym outside the capsule. The posterior margin of the opening is not clearly defined, so that it remains dubious whether the hole is present in the next three sections or not. On the contrary the margins of the opening in the 255 m.m. long embryo, are clearly defined. The attachment of the hyomandibular to the capsule takes place here on about 59 sections 30 μ thick (in all the other embryos the sections are 15 μ thick).

The opening reaches from the 8th to the 25th section (counted antero-posteriorally). It is closed by a deeply red stained connective tissue, which also helps to connect the hyomandibular to the skull, and which is rather conspicuously surrounded by the blue colour of the cranial cartilage. The opening lies in the under part of the fossa for the hyomandibular, which partly covers it.

From the wax model of Mr. P. J. DE VRIES, made according to the method of BORN, one can see that the opening is not truly oval, but rather kidney-shaped, because the under margin forms a re-entering concavity. The mesenchym which formerly filled the perilymphatic spaces, has to a large extent disappeared and been replaced by a liquid, which is prevented from flowing out, by the connective tissue closing the opening.

The opening, owing to its position, has to be considered as the homologue of the fenestra vestibuli, which in Amphibians and Amniotes is closed by the stapes, and which according to general opinion would be absent in fishes.

Owing to the great length of the embryo, it must have been more or less fully developed, and it is improbable that the fenestra would not persist after birth.

I, however, had no opportunity of examining adult material. Irrespective of the autostylic Dipnoi and Holocephali, fishes are as a rule hyostylic. Their powerful hyomandibular functions in the first instance as a suspensorium. This fact evidently has to do with the absence of a fenestra vestibuli. Only two primitive forms viz. Heptanchus and Hexanchus are amphistylic. Their hyomandibular, owing to the firm attachment of the palatoquadrate to the skull, can only feebly function as a suspensorium. It is therefore conceivable, that the hyomandibular, at least in Heptanchus, may still have the function of transferring vibrations to the auditory organ.

The presence of the fenestra in the embryo is in any case a support to the old theory, which in later years has frequently been attacked, the theory namely: that the stapes in higher animals is homologous to the hyomandibular in fishes.

INDEX LETTERS.

Csl. Canalis semicircularis lateralis.
Csp. Cart. spiracularis. Each of the two spiracular cartilages (fig. 2*a*) is sectioned twice.
E. Top of the epibranchial of the first branchial arch.
Ep. Epithelial body.
Fv. Fenestra vestibuli (ovalis).
Hm. Hyomandibular.
K_2'. Second branchial cleft.
Pq. Palatoquadrate.
Ro. Regio otica of the skull.
So. Spiracular sense organ.
Th. Thymus.
Vj. Vena jugularis.

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Physiology. — "*Contributions to an experimental phonetic investigation of the Dutch language. I. The short o*". [1]). By Miss L. KAISER. (Communicated by Prof. G. VAN RIJNBERK).

(Communicated at the meeting of September 29, 1923).

When listening carefully to the pronunciation of the "o" in closed syllables in Dutch, we perceive that — apart from the influence which all sounds undergo from preceding and following vowels or consonants — two completely different ways of pronunciation can be distinguished [2]).

One of these two pronunciations is heard in words like: kok, tot, hol; the other in words like pop, bot, vol, hond. At the suggestion of my former master Dr. PLOMP, I have tried to go further into this question.

I first tried to determine experimentally this difference, suggested by linguistic feeling and observed by simple hearing.

Experimental phonetic analysis of the speech movements.

Several methods used in experimental phonetics were consecutively applied in order to determine the essential movements and positions of the vocal organs during the pronouncing of o̜ and o̤ [3]). In doing so I chiefly made use of one trial person, while the results were afterwards tested to those obtained with other speakers.

1. Observation and measuring of the mouth opening while pronouncing different sounds proved that in this respect a, o̜, oo, o̤, oe form a series in which the mouthopening gradually decreases, the height

[1]) From investigations made at the Physiol. Lab. of the Amsterdam university and at the Phonet. Lab. of the Czech university at Prague.

[2]) I am aware of the fact that so called educated speech varies considerably in different parts of this country. As far as I know facts mentioned here hold good for the pronunciation of Amsterdam and surroundings and probably not or only partially e. g. for that of the Hague and surroundings in which the o̜-sound seems to predominate.

[3]) The o of kok is represented by o̜, that of pop by o̤.

diminishing regularly, while the width also decreases but not so regularly. The latter, namely, shows a sudden decrease between o̤ and oo. In this series the height of the mouth opening was 16 mM., 12 mM., 8 mM., 6 mM., 4 mM. respectively, the width 36 mM., 31 mM., 16 mM., 14 mM., 7 mM. respectively (see fig. 1).

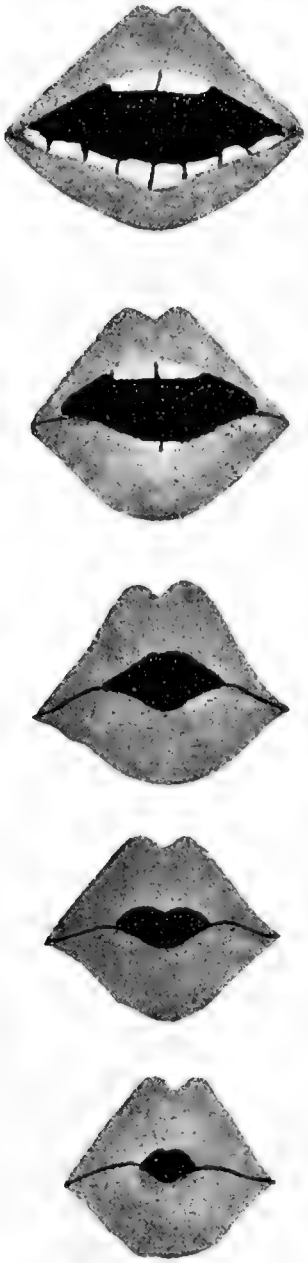

Fig. 1.

Closely connected with this are curves of the lipmovements made with the apparatus of von Wilczewski[1]). This apparatus has been so construed as to have the curves indicate the natural size of the vertical lip-opening. Fig. 2 illustrates this. Fig. 3 shows a curve obtained with the same apparatus by pronouncing alternatively dọl and dọ̤l. The difference is clear; the dimensions are about the same as those mentioned above.

Consequently, if we exclusively consider the shape of the mouth opening, we can imagine that o̤ is an oo that became more or less like an a while ọ is an oo that has acquired some of the qualities of the oe.

2. By means of Zwaardemaker's apparatus[2]) for registering speech movements, the pouting of the upper lip, the movements of the lower jaw relative to the upper jaw, and the contraction of the muscles that form the bottom of the mouth, were recorded. Fig. 4 shows that also as regards jaw opening a, o̤, oo, ọ, and oe form a descending series, while the pouting of the lips increases, (with this trial person there is less pouting of the lips for ọ and oe than for oo in connection with the downward movement of the upper lip, during which the latter is sowewhat flattened). The curve of the mouth bottom is not dealt with here because of its complexity. What interests us most in this curve is that it shows considerable and characteristic dif-

[1]) Vox. Heft 3/6, 1922.

[2]) Onderz. Physiol. Lab. te Utrecht. Ve reeks I 1899—1901 p. 76. Leerb. II p. 98.

ferences between the two o-sounds. These results harmonize quite well with those obtained by EYKMAN [1]) who, working with the same instrument, found an average jaw opening of 7,25 mM for a

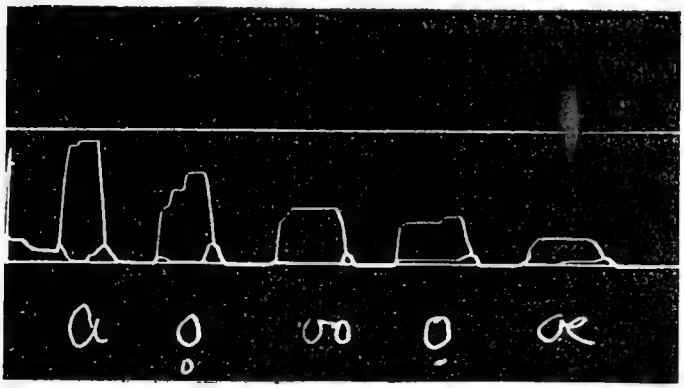

Fig. 2.

in "bat", 5,50 mM for o of "pot", 4,75 mM for oo of "boot", 4,50 mM for o of "bot" and 2,25 for oe of "boet".

Fig. 3.

Fig. 5 also shows curves of lip, jaw, and mouth bottom, but these curves are obtained in another way, viz. by means of a "mouth-funnel" that permits of registering the above mentioned movements at the same time. This instrument, constructed by me for another purpose, will be dealt with elsewhere. As it has no fixed support, it misses the exactness which characterizes ZWAARDEMAKER's apparatus. Still it is very useful to give a provisional impression of something relative. It can be noticed in fig. 5 that in pronouncing "dorscht" there is less pouting of the lips and a larger jaw opening than for „dorst", while the curve of the mouth bottom is almost the same for both words. From the mouth-funnel curve it appears that the air current for o is stronger than for o, as is easily comprehensible. From the above, therefore, it becomes evident again that the two sounds differ considerably.

[1]) Onderz. Physiol. Lab. te Utrecht Ve reeks II 1899—1901 p. 202

3. With the majority of speakers the hard palate is either hardly touched or not touched at all by the tongue in pronouncing oo, o̥, or o̥. Consequently the artificial palate cannot be of much use here. Yet I had the words "po̥p" and "bo̥b" pronounced by two trial persons with whom a rather large part of the palate was touched. The results can be found in fig. 6. The difference between the two sounds is clear with both persons: the surface touched for o̥ being smaller than for o̥, while it is a wellknown fact that for a the palate is not touched at all.

4. Finally the movement of the larynx was registered. It can be easily felt that the larynx assumes a somewhat different position in the two cases, viz. it is advanced more for o̥. However, I did not always succeed in recording this difference. I tried to do so with ZWAARDEMAKER's method [1]). The curves obtained, however, were too unlike in appearance but that definite conclusions could be drawn. Still it appeared from these curves that the larynx was retracted for o̥ (as for a and oo), while it was advanced for o̥ as for oe, though by no means to such a degree. Fig. 7 shows part of a curve in which the difference between o̥ and o̥ can be seen.

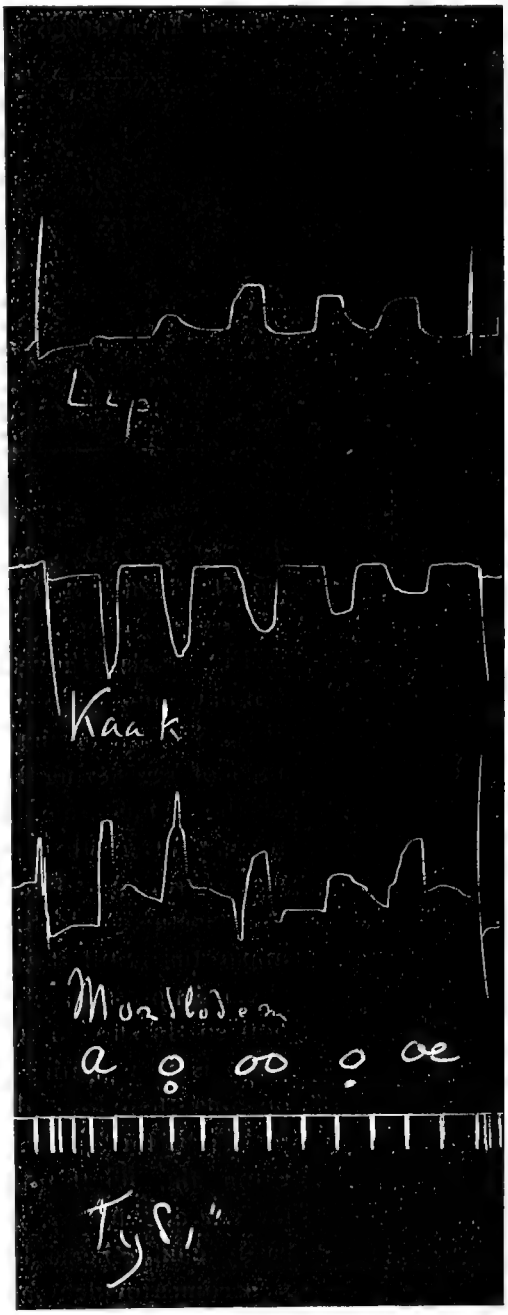

Fig. 4.

[1]) Leerboek der Physiologie II, p. 86.

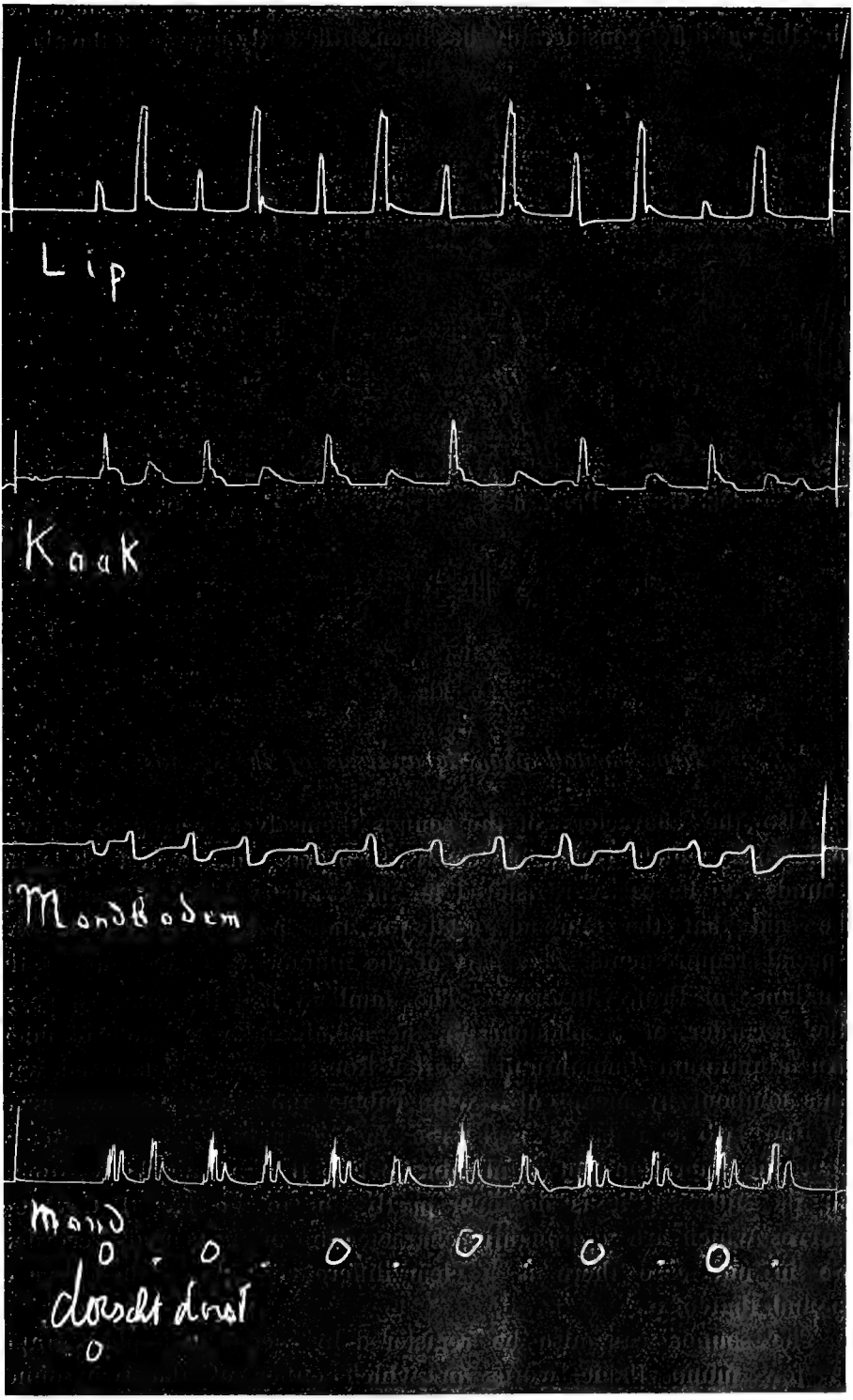

Fig. 5

That the speech movements made to produce ọ and o̤ as distinguished by the ear differ considerably, has been sufficiently proved in the above.

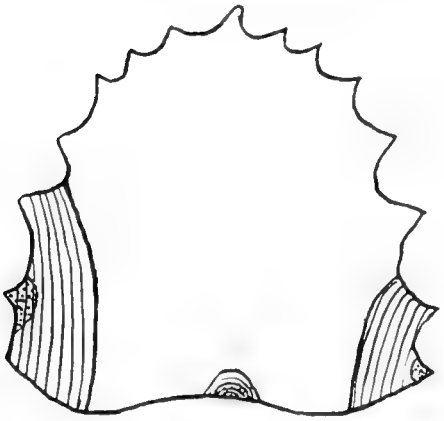

Fig. 6. Pp. v. d. S.

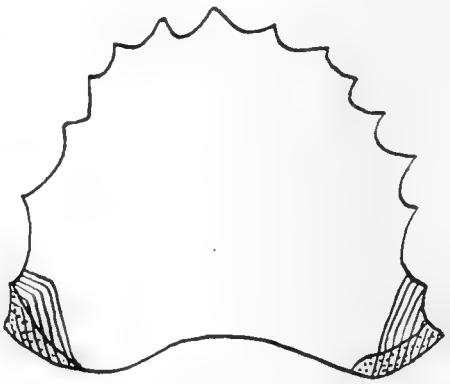

Fig. 6. Pp. R.

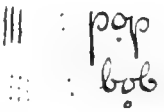

Fig. 6.

Experimental phonetic analysis of the sounds.

Also the characters of the sounds themselves proved to show a difference which could be easily recorded. In the first place the sounds can be easily registered on the kymographion. It need hardly be said that the tambour used for this purpose has to answer special requirements. This part of the inquiry was made under the guidance of Prof. Chlumský. The tambour had the same shape as the recorder of a phonograph, the membrane was made of mica. An aluminium "mouthfunnel" after Rousselot was connected with this tambour by means of a wide rubber tube. Fig. 8 shows curves of the two sounds as registered in this manner. As a matter of fact the vibrations of a membrane like this are not large, owing to its stiffness; it is however partly due to this fact that we get curves which are thoroughly characteristic of the sounds recorded. So in our case there is a clear difference between the curve of o̤ and that of ọ.

The sounds can also be registered by means of a phonograph. A few monosyllabic words in which either of the two sounds occur according to the meaning (e.g. bo̤d and bọt) as well as the

sounds pronounced separately were recorded by means of an Edison phonograph (old type)

The difference between the sounds as recorded by the phonograph can be made much more illustrative and easily measurable by

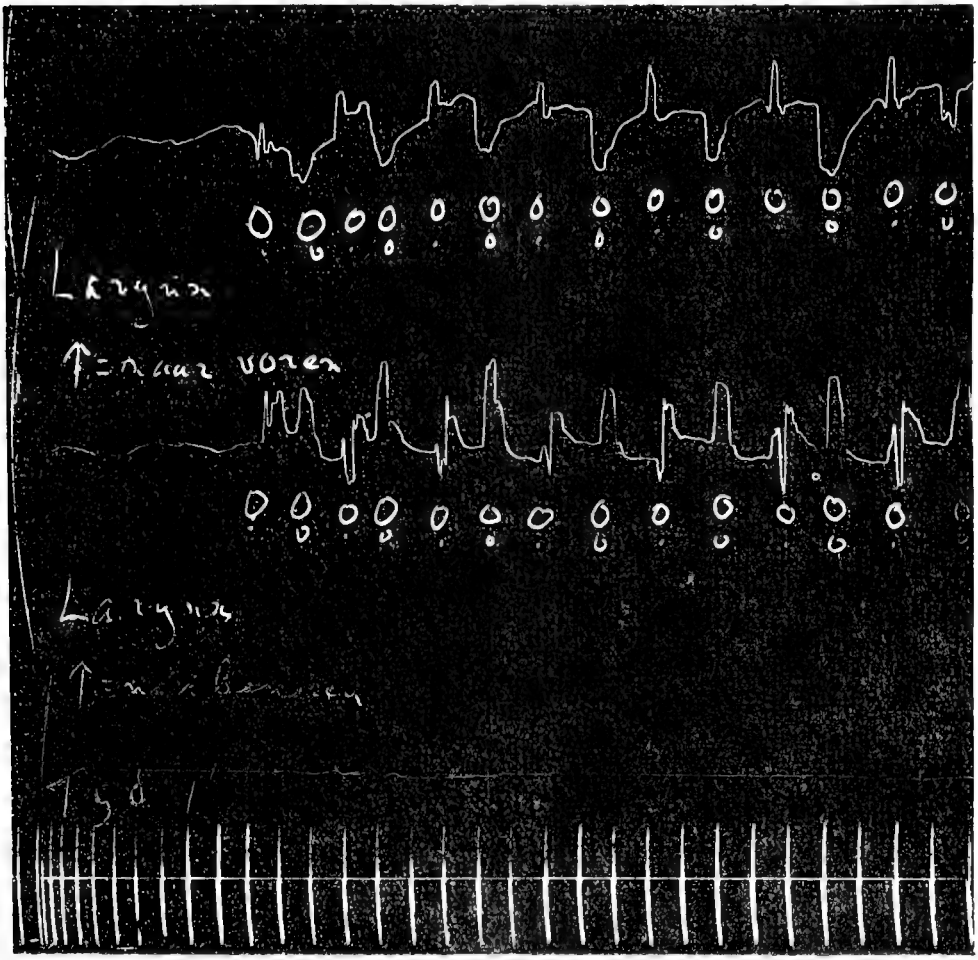

Fig. 7.

transforming the indentations of the wax cylinder into a curve on smoked paper. This is done by the apparatus constructed for this purpose by LIORET. A sapphire follows the groove of the phonographic cylinder; the movements made by the sapphire in doing so are transferred to a writing-lever, recording them on a rotating cylinder. As there is no such apparatus in Holland, as far as I know, also this part of the investigation was made at Prague under the guidance of Prof. CHLUMSKÝ. The words „bo̤d" and „bo̤t" were again recorded phonographically, making use of the apparatus of LIORET. By means

of the same instrument these curves were subsequently magnified 300 times and registered on a smoked cylinder. Fig. 9 shows part

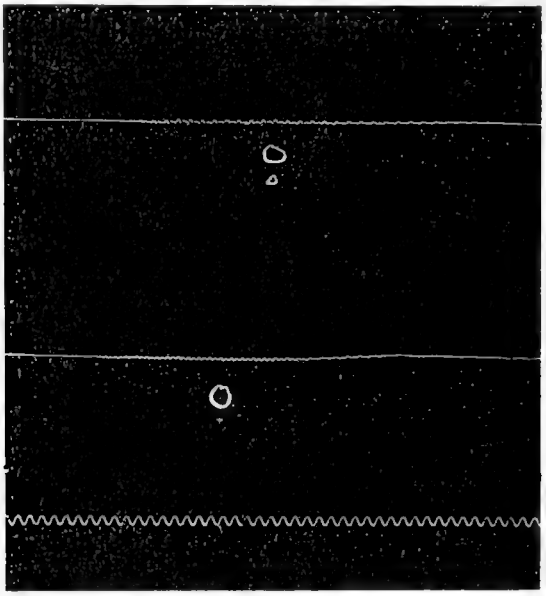

Fig. 8.

of these curves. The upper and lower curves represent the ọ-sound in "bọt", pronounced in a low voice in the former and loud in the latter. The curve in the middle gives o̩ cound in „bod" The difference between the two sounds is clearly revealed in this way and can be easily put into figures. The curve of o̩ is much the same as that which characterizes the aa-sound.

After a considerable and constant difference has thus been ascertained, it may be desirable to get an idea of the circumstances in which ọ and o̩ occur in Dutch.

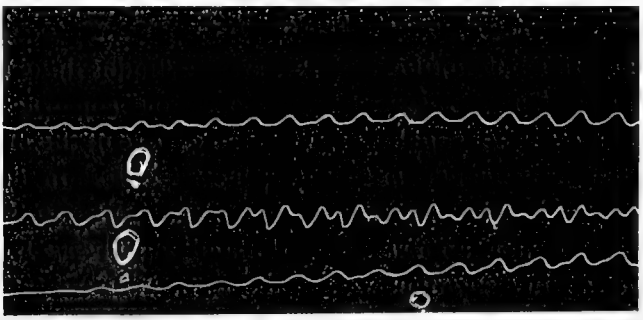

Fig. 9.

Linguistic remarks.

Every Dutch word of one syllable, containing o and also syllables with o, not occurring by themselves but with which influence from other syllables can be safely excluded (e.g. lom(mer)) were considered. Combinations of sounds that can be pronounced quite well, but are not found in the Dutch language, have been omitted.

As regards the influence by the several consonants, a few facts could be ascertained.

The most constant influence is that of following nasals. In this combination, namely, the o̩-sound occurs invariably. This can be easily comprehended, as the narrow mouth-opening and weak current of air passing through the mouth, promote the air current through the nose which follows.

Another influence is that of the lip-sounds; these promote the producing of the o̩-sound especially when preceding it. Also this becomes clear if we consider the narrowness of the mouth-opening. Guttural sounds like h, g, k, etc. are as a rule followed — and as far as they can be final, also preceded — by o̩. This also holds good, though in a less degree, for z, s, l, r, n and j; n of course only when preceding. D and t have no clearly manifested influence.

The r occupies a position of its own; its influence varies according to the way in which it is pronounced, which is different even with one and the same speaker at different times, its place of articulation varying between point of the tongue gums and root of the tongue-uvula. Taking this into consideration we can say that the advanced r promotes the o̩, the retracted r the o̩-sound, both whether preceding or following the vowel.

From this it appears that the adjoining consonants either promote the o̩ or the o̩. It should be borne in mind, however, that the only absolute influence is that of nasals as following sounds. The other influences only work to a limited degree. The fact that the influence of several sounds appears to be inconstant proves that there is at least one factor more playing a part. This becomes evident from the fact that several words, have either o̩ or o̩ according to their meaning, e.g.:

bo̩t (noun = flounder, bone; adjective = blunt)	& bo̩d (noun of bieden = to bid)
do̩l (adjective = mad)	& do̩l (noun, part of a rowing-boat = thowl)
do̩rst (noun = thurst)	& do̩rscht (from the verb dorschen = to thresh)

mo̩tje (dialect of moet-je = must you)	& mo̩tje (noun, dimunitive of mot = moth)
po̩rt (from the verb porren = to stir)	& po̩rt (noun from porto, oporto)
to̩bbe (noun = tub)	& to̩bben (verb = to worry)

It seems to me that the above may induce us to think of etymological influences. Words that have o̩ in Dutch, usually occur in German with u, while those with o̩ either have o, a, or au in German. I do not venture to judge about the value of this phenomenon. Other cognate languages as well may give indications. It may be wortwhile to make an etymological inquiry in this connection.

If etymological influences are ascertained indeed, we can imagine that they be inconsistent to a certain extent with the other influences described above. The word "pols", for instance, may be mentioned in this connection, because the pronunciation is wavering. It appears to me that this word is pronounced "po̩ls" by more careful speakers, while the majority say "po̩ls". Judging by its etymology the former pronunciation would be the right one: the latter may be easier because of the l that follows.

Summary.

There are in Dutch two short o-sounds that can be clearly distinguished both acoustically and phonetically (perhaps also etymologically).

Botany. — "*Ringing Experiments with variegated branches.*" By Prof. TH. WEEVERS. (Communicated by Prof. J. W. MOLL.)

(Communicated at the meeting of September 29, 1923).

For a long time already the transport of carbohydrates and proteins in plants has been considered as a question that seemed fairly set at rest. Of late years, however, the problem has again been brought into prominence.

The well-known ringing experiments, notably the extensive observations made in this field by J. HANSTEIN [1]) had settled the belief that the organic matter was transported along the elements of the phloem. It was left undecided whether the elements of the cribral system (sieve-tubes and companion cells) or those of the parenchymatous phloemsystem (cambiform cells) play the principal part. CZAPEK's [2]) experiments favoured the first view, however, owing to the diametrically opposite conclusions of DELEANO [3]) a decision was impossible at the time.

The primary and the secondary phloem was generally considered as the passage for the conduction of the organic products, which, being formed in the leaves, have to be conveyed to the growing points and the reserve-organs.

In accordance with TH. HARTIG's [4]) conception it was, however, generally received that in the early spring, when the woody plants start new shoots, the organic matter finds its way from the reserve-stores to the shooting parts through the xylem. This hypothesis was based partly upon the results of HARTIG's experiments with ringed plants and partly upon A. FISCHER's [5]) observations regarding the occurrence of carbohydrates in the wood vessels. Researchers refrained from approaching the question as to how this happens in shooting herbaceous plants.

Now the above theory has latterly been impugned from various quarters.

[1]) J. HANSTEIN, Jahrb. f. wiss. Botanik, 1860.
[2]) CZAPEK, Jahrb. f. wiss. Botanik, 1897.
[3]) N. DELEANO, Jahrb. f. wiss. Botanik, 1911.
[4]) TH. HARTIG, Bot. Ztg., 1858.
[5]) A. FISCHER, Jahrb. f. wiss. Botanik, 1890.

On the one side Otis Curtis[1]) made single and double ringing experiments and arrived at the conclusion that the transport of carbohydrates and proteins to the shooting parts may occur through the secondary phloem just as well as the transport in the opposite direction does, when the surplus of assimilates is removed from the place of formation. In my judgment, however, his view has not been sufficiently reinforced by indispensable quantitative examination.

On the other side it is Atkins[2]) and Dixon[3]) in England, and Luise Birch Hirschfeld[4]) in Germany who deny almost any significance to the phloem for the matter-transport. Their arguments consist in the main of indirect evidence. Atkins argues that the bleeding saps are more or less rich in carbohydrates not only in spring but also in other seasons. Luise Birch Hirchfeld and afterwards Dixon base their most cogent arguments upon their belief that an adequate transport of matter along the phloem can hardly be presumed. This difficulty had already been obviated by Hugo de Vries[5]), who made a quicker transport than the law of diffusion admits conceivable by assuming protoplasm-streams in the phloem-elements. Dixon, however, considers the impossibility of a transport of adequate capacity along the phloem as conclusive evidence for denying any significance to the phloem in this respect. Birch Hirschfeld is less positive in her assertion.

That, beside an ascending stream in the wood, there may also be a coinciding transport along it towards the bottom of the stem, may be concluded from various investigations i.a. the above-named by L. Birch Hirschfeld. Then the rate of transport can be much quicker than in the phloem and the capacity of the conducting channels can likewise be greater, as it is a fact that the phloem-production of cambium is invariably smaller than that of the xylem, while the generated phloem is obliterated much sooner.

This conception of Dixon's, however, does not square with the result of the ringing experiments of Hanstein, which result points indubitably to the stream of assimilates being stopped when the ringing wound is made deep enough to reach the cambium. Dixon therefore assumes the transport to pass through the youngest parts of the secondary xylem, which parts being located close to the cambium, are by him believed to be injured and thus rendered inactive by the ringing.

[1]) Otis F. Curtis, American Journal of Botany, 1920.
[2]) W. R. G. Atkins, Some recent researches in Plant Physiology, 1916.
[3]) H. H. Dixon, Pres. Address. Bot. Society, 1922.
[4]) L. Birch Hirschfeld, Jahrb. f. wiss. Botanik, 1920.
[5]) Hugo de Vries, Bot. Ztg., 1885.

To my knowledge this hypothesis has not yet been substantiated by experiments, so that it seems expedient to reconsider the question along what way the carbohydrates and the proteins are transported in plants.

The question can be approached from different sides; in this paper I will confine myself to a discussion of some experiments with ringed branches of variegated plants.

Similar experiments have been made repeatedly with green branches, but then the trouble is that after the buds have opened out, the younger parts above the ring begin to assimilate.

Stripping off the leaves or moving the plant to a dark space involves other difficulties; with variegated shoots it is much easier to state any supply of organic matter.

In consideration of DIXON's hypothesis due precautions should be used in the ringing and the protection of the injured part. A coating of melted butter of cocoa I deem more effectual than one of paraffin. It was applied to the wound at a temperature of 32°—33° C. and can hardly injure the exposed surface, as it does not penetrate into the intact cells.[1]) Moreover, it soon congeals and then affords sufficient protection against outside influences. The parts were then screened from immediate effect of the sun's rays in order to prevent melting.

We performed our experiments with variegated branches of Aesculus hippocastanum L. and Acer Negundo L. The former were derived from a stout specimen, whose green top provided the trunk with abundant food and from this trunk numerous yellow shoots had developed. In about 20 years these shoots attained a length of 1 M. and a thickness of 7—8 mm. in diameter. The specimen of Acer Negundo was provided at the top with green-white variegated leaves and developed from the main stem and side branches perfectly white shoots. In neither specimen did the yellow-white leaves contain any chlorophyl[2]). An iodine test pointed to the absence of starch.

In the spring experiments the branches were ringed (1—2 cm.) just before the buds began to open out and at a distance of 1—2 dm. below the end-bud.

Three series of experiments were always made at a time.

1st *series*: green shoots ringed all round.

2nd *series* variegated (yellow-white) shoots ringed all round.

3rd *series* variegated (yellow-white) shoots partially ringed, viz. so as to leave as trip of bark as a connecting link, 2—4 mm. in breadth.

[1]) R. H. SCHIMDT, Flora Bd. 74, 1891.

[2]) Guard-cells of the stomata excepted.

After rather more than a week a contrast was noticeable between the green and the partially ringed variegated shoots on the one side and the completely ringed variegated shoots on the other. The first two (1st and 3rd series) continued growing normally. The third (2nd series) lagged behind and died off after 2 or 3 weeks, the leaves having previously shrivelled and dried up.

Fig. 1.

That ringing in itself did not injure the plant appeared distinctly from the results of the first and the third series. (See the photos): from left to right we see first 4 completely ringed yellow branches, some brown and dead, others small but still living; the next following are two completely ringed green ones and lastly to the right two partially ringed. The last four have developed normally.

It is clear that with the completely ringed green shoot the supply of water is normal; why then does the completely ringed yellow branch die off under symptoms that point to a deficiency of water?

The reason is obvious. In consequence of too little osmotic pressure the absorptive power of the tissues is too low as compared with that of the other parts.

The researches by Dixon and Atkins [1]) on the determination of the osmotic pressure by lowering the freezing point of the expressed

[1]) Notes Botanical School. Trinity College Dublin, 1912.

sap, clearly show how the osmotic value of the leaf-cells increases with the possibility of assimilation.

Now I endeavoured to determine the suction force by URSPRUNG's [1]) method but the subject appeared to be difficult to experiment on.

A quantitative determination gave in the green leaves of Aesculus an amount of reducing sugars of 3 % [2]), in the variegated (yellow) leaves 1 %, in the ringed variegated (yellow) branches only traces. In general also the amount of extractable salts is trifling; in green and variegated leaves 0,9 % of the fresh weight [3]). SPRECHER finds in yellow varieties lower osmotic values for the cell sap than in the green specimens [4]).

True, the variegated leaves of the ringed branches of Aesculus contain from 18 to 20 % protein and 5 % dextrin (calculated at dryweight) but the influence of these amounts on the osmotic pressure is nothing to speak of. Yet this does not explain all, for in the variegated completely ringed shoots wood and bark above the ringing appeared to contain still a fair amount of starch (6 % of the dryweight, against 9 % in the partially ringed branch), while the leaves were already shrivelling.

Why this starch is not converted into sugar and why, when transported to the leaves, it does not raise the osmotic pressure has not yet been explained.

However this may be, the partially ringed variegated branches do not die off. It appears, then, that there the supply is not cut off and that consequently the young parts are provided with the nutriment that in the green ringed branches is produced by assimilation.

According to HANSTEIN the organic products are conveyed along the bridge of bark, but if this is the case, we must relinquish HARTIG's hypothesis that the transport is effected along the xylem while the branches are budding.

OTIS CURTIS (l.c.) does so and was led by his ringing experiments to regard the phloem exclusively as the path, along which the saps

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[1]) URSPRUNG, Ber. d. d. bot. Ges., 1918.

[2]) Strictly speaking 2 % and 1 % reducing sugars derived from glucosids (calculated at dry-weight).

[3]) The starch determinations were performed by putting the pulverized material immersed in water for 3 hours into an autoclave at 4 atm., and by subsequently boiling the aequeous extract with diluted hydrochloric acid during 60 minutes.

Plasmolytic experiments are objectionable on account of the osmotic pressure in the various cells being unequal. Still, a 10 % saccharose solution plasmolyzes the variegated Aesculus-leaves, not however the green ones.

[4]) A. SPRECHER. Rev. Gen. Bot. 1921.

are transported. From DIXON's point of view, however, it might be objected that in CURTIS's experiments the peripheral woodlayers were injured and thereby the transport along the peripheral xyleem had been suspended indirectly.

This objection can hardly be raised against the above experiments, in which a coating of butter of cocoa was spread on the injured part.

Fig. 2.

Moreover, another series of ringing experiments was carried out. In these experiments the ringing was performed as much as possible aseptically by previously washing the branch bark with 96 % alcohol and then peeling it off aseptically down to the cambium. Subsequently the decorticated surface was covered with sterilized absorbent cotton wool saturated with water; finally the whole was wrapped up with wax taffeta.

These experiments were carried out mid-June in the same way as the others described above, and yielded after four weeks an unequivocal result in connection with the midsummer growth which was very abundant, especially in Aesculus.

With the normal yellow variegated shoots the formation of midsummer growth occurred at the top of the branch and the yellow young leaves contrasted sharply with the others, which had been damaged by the high wind and browned by the sun. (See photo).

It appears then, that here also the yellow leaves suffer under a deficiency of suction force, and under circumstances brought about by stronger evaporation are sooner destroyed than the green ones, although the latter evaporate comparatively more intensely.

With partially ringed variegated shoots the midsummer growth occurred also at the top. With completely ringed specimens, however, it appeared *below* the surface of the wound from lateral or dormant buds. (See photo). This occurred as well when the surface of the wound was covered with butter of cocoa, as when it was dressed with a water-bandage.

The check to the food-supply is apparently as great with Aesculus as with Acer Negundo, in spite of the greatest precaution used in cutting the ring. It follows, then, that the experiments do not yield any evidence whatever, to lend support to DIXON's theory. They rather go against it.

Still conclusive evidence to disprove DIXON's theory cannot be brought forward by this procedure, since in spite of all due precaution the peripheral wood may be prevented by the ringing from performing its function, as far as the transport of the organic products is concerned.

With regard to other inquiries, whose results tell strongly against DIXON's theory, we first of all have to think of HANSTEIN's experiments (l.c.) on the root-growth of ringed branches in water culture.

HANSTEIN finds that detached branches placed in water send out roots chiefly at the basal extremity of the stem, which VÖCHTING ascribes to the polarity of the parts. Leafless branches when ringed develop a large number of roots just above the wound; whether and to what number they will grow at the bottom of the branch, depends on the distance between that extremity and the ringing.

HANSTEIN ascribed this to the check to the transport of nutriment consequent on the removal of the phloem, and established, indeed, in such circumstances a distinct difference in the root-growth, between dicotyledonous plants with an anomalous stem-structure and those with a normal stem-structure, in which the stem derived its thickness form a ring of collateral vascular bundles. With the former the transport of carbohydrates and proteins is believed to be only partially checked. This is ascribed to the fact that the vascular bundles are contained within the xylem (as with Piperaceae and Nyctaginacea) or (as in the case of Apocynacea and some Solanacea) to the fact that there are originally bicollateral vascular bundles or rather medullary phloem strands and consequently phloem remains also within the secondary xylem. Owing to this HANSTEIN stated

in this case only a very slight influence of the ringing upon the root-growth.

This evidently does not fit in with DIXON's view; if the transport is effected along the peripheral parts of the xylem, ringing must in these plants have the same effect. It struck me, therefore, that it would be worth while to repeat some of HANSTEIN's experiments. The Solanacea Cestrum aurantiacum proved to be an unsuitable subject since detached branches sent out roots very sparingly in water culture, but Nerium Oleander yielded quite satisfactory results: all the twelve cuttings presented an aspect, quite in harmony with HANSTEIN's description. The root-growth may be somewhat more abundant above the wound, but the behaviour is quite different from e. g. that with Salix and Cornus spec. In these the roots appear almost exclusively above the wound, unless the stem-piece below it be very long, and the once formed roots are even destroyed when the bark above them is stripped off.

Provisionally all this tells very strongly against the validity of DIXON's conception of a transport of the carbohydrates and the proteins along the peripheral xylem.

If the above-discussed experiments with variegated shoots could also be made with variegated Oleanders, the medullary phloem of these plants would probably cause a quite different result from that yielded by Aesculus and Acer. But unfortunately variegated Oleanders I had not at my disposal, so that now I made a trial with ringed, normal shoots, which, while still attached to the plant, were wrapped up in black paper. The result was rather conclusive. Although some leaves had fallen off, the shoots themselves were still alive ten weeks after the ringing and had increased in length.

We see, therefore, that not only in the formation of the roots of branches in water-culture but also in the budding and the growth of Oleander Aesculus and Acer Negundo in spring, the results of our experiments with ringed branches imply a transport along the phloem.

In a subsequent publication I intend to discuss the question whether the capacity of these paths is sufficient.

For the present the above observations on Aesculus and Acer Negundo, where the detached branches did not bleed, are not applicable to the cases in which this bleeding is so copious, and as with Betula alba the highly sacchariferous sap is exuding directly after the ringing [1]).

[1]) The cases described by MOLISCH, (Bot. Ztg. 1902) as wound-reaction with local bleeding pressure, are of quite a different nature; then the bleeding pressure manifests itself only after days or weeks.

Physiology. – "*Determination of the Power of the Accommodation-Muscle*". By Prof. J. VAN DER HOEVE and H. J. FLIERINGA.

(Communicated at the meeting of September 29, 1923).

The action of the accommodation muscle, the M. Ciliaris, makes itself apparent to us by the increase of refraction of the lens, the so-called accomodation of the eye.

There are still many obscure points in the subject of accommodation; for instance, it is still entirely unknown to us what relation exists between the contraction of the accommodation-muscle and the increase in refraction of the lens

A few ophthalmo-physiologists are of opinion that contraction of the accommodation-muscle increases the tension in the ligament of the lens, the Zonula Zinii, while most of them assume, with HELMHOLTZ, that contraction of the ciliary muscle causes a relaxation of the Zonula Zinii, so that opportunity is given to the lens to curve according to its elasticity. When, through increase of age, the elasticity disappears, contraction of the ciliary muscle does not assert itself by increase of refraction of the lens.

Even if one assumed the last theory, one meets with many unsolved questions, e g.:

a. Is the strongest possible contraction of the accommodation muscle necessary to obtain the greatest possible accommodation? DONDERS and LANDOLT assumed this and find still followers in these days, amongst others CLARKE and DUANE.

FUCHS, HESS and others, on the contrary, are of opinion that the accommodation muscle can contract far more strongly than is necessary to obtain a maximal accommodation.

FUCHS expresses this in the following way: the accommodation-muscle can first contract so far that the lens can follow its elasticity completely, resulting in a maximal accommodation; the eye is then focussed on a point, which is determined by a physical property, viz. the elasticity of the lens. FUCHS therefore calls this point the "physical near point". Now he muscles can contract considerably more, so that the Zonula hangs entirely relaxed and the lens could if only its elasticity were unlimited, increase its refraction considerably, allowing the eye to focus on a point, lying still closer by, and

determined by a physiological property, viz. the power of contraction of its accommodation-muscle: the physiologic near point.

Hess says that it is almost generally assumed that every increase in lens-fraction of one dioptrie exacts an equal increase of the contraction of the ciliary muscle.

Although this simple relation is not self-evident, considering the complicatedness of the accommodation-process, we will accept it for a moment, in order to try and prove it, taking as unit of contraction of the ciliary muscle, the contraction necessary to bring the accommodation from 0 to 1 dioptrie, which unit we can call „myodioptrie".

If Hess' unproved supposition is correct, one will need a contraction of 10 myodioptries in order to be able to accommodate 10 dioptries. We can now also express the total power of the accommodation-muscle in myodioptries, for in an emmetropic person this will be the reciprocal value of the distance of the physiological near point, or otherwise expressed, it will be equal to the number of dioptries one could accommodate if the lens had an unlimited elasticity.

So we stand here before the following two questions:

b. Is the myodioptrie for one person a fixed unit? That is to say: is a contraction of the accommodation-muscle of one myodioptrie necessary for every accommodation-increase of one dioptrie?

c. How great is the power of the accommodation muscle expressed in myodioptries?

Other questions which rise before us are the following:

d. Is it possible to detect the very slightest paresis of the accommodation muscle?

e. Is it possible to make curves of the paralysing influence certain substances exert on the accommodation muscle?

Up to now we were accustomed to determine the action of the accommodation-muscle by finding the nearest point.

Let us now suppose a person (fig. 1) who can accommodate 10 dioptries, and who possesses a power of the ciliary muscle amounting to 24 myodioptries, then the accommodationmuscle can be more than half paralysed, while the nearest point need not have changed its place. In this way we only notice the possible presence of a paralysis of the accommodation muscle, when it is far advanced.

In consequence we know little about the paralysing action of certain substances which only slightly affect the accommodation muscle, even about those substances which we use daily, such as cocaine. We find the most divergent communications in the literature about the paralytic action of cocaine on accommodation.

Some writers assert that it does not act at all on the accommo-

dationmuscle, others say that it acts very strongly, and a third group states that it does work, but only slightly.

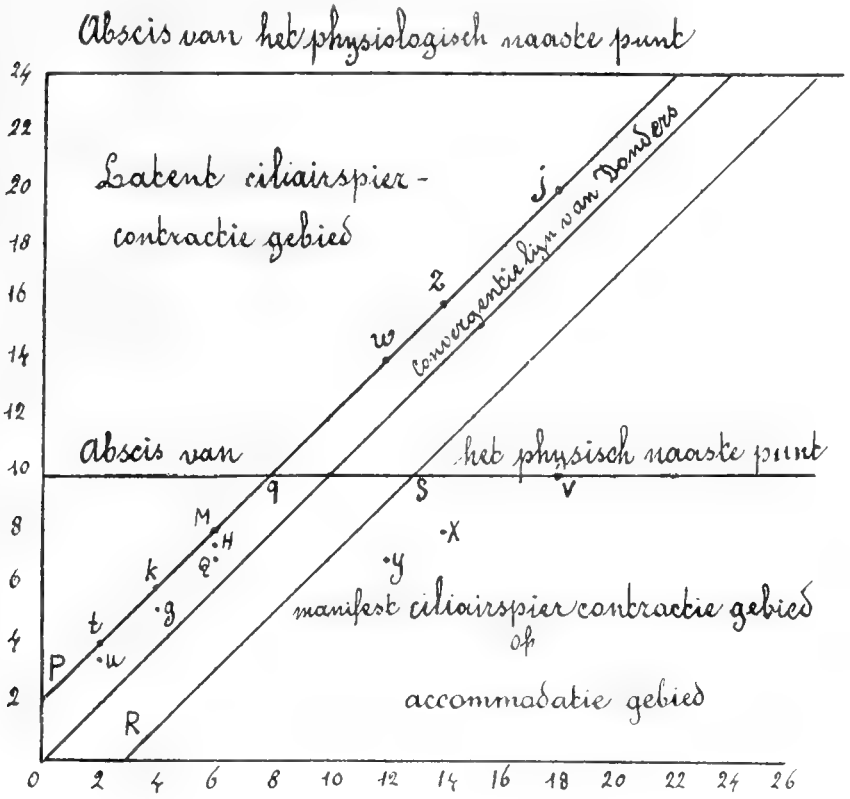

Fig. 1.

Abscis van het physiologisch naaste punt	=	Abscis of the physiologic near point.
Latent ciliairspiercontractie-gebied	=	Area of latent ciliary muscle contraction.
Convergentielijn van Donders	=	Donders' convergenceline.
Abscis van het physisch naaste punt	=	Abscis of the physical near point.
Manifest ciliairspiercontractiegebied of accommodatiegebied	=	Area of manifest ciliary muscle contraction.

Dr. FLIERINGA and I have tried to solve the foregoing questions by a minute study of the relative accommodation.

By relative accommodation we understand the accommodation at a certain given degree of convergence. A certain connection, probably congenital, exists between accommodation and convergence; if a normal emmetropic person wishes to fix his eyes on an object, he must converge as many metreangles as he accommodates dioptries.

If, in Fig. 1, we plot out the myodioptries and dioptries on the ordinate, and the metreangles on the abscis, then we can draw a line through all the points for which accommodation and convergency are alike; if, in our scheme, we take the linear measure for

metre angle and dioptrie the same, then this line divides the right angle between ordinate and abscis exactly in two equal parts. This line, which unites all the points denoting an equal number of metreangles for convergence as dioptries for accommodation, is called: "DONDERS' Convergence-line".

If the relation between accommodation and convergence was absolute and unfringible, then a normal person would only be able to see the points of the convergence-line sharp and single at the same time, and no other points; every person with an *abnormal* refraction or a heterophoria would not be able to see one single point sharp and single at the same time.

Luckily the connection between accommodation and convergence is more or less a loose one, so that at every convergence the acommodation can, to a certain degree, be made stronger or slighter than coincides with the degree of convergence.

If one converges 6 metreangles, then an accommodation of 6 dioptries coincides with this, an accommodation, which one can raise f.i. to 8 dioptries, or decrease to 3 dioptries. This interval between 3 and 8 dioptries is called the relative accommodation for a convergence of 6 metreangles; the interval from 6 to 8 dioptries is called the positive, from 6 to 3 dioptries the negative relative amplitude of accommodation.

The relative amplitude of accommodation differs a great deal in each individual case and can be increased to a certain degree by long practice. It is not necessary that the negative and positive part of the relative accommodation are alike.

One can determine the relative accommodation for all points in the area of manifest contraction of the ciliary muscle and connect the relative near and far points to get the lines of the relative near and for points.

According to HESS the relative accommodation is the same at every convergence, so that for every normal person the lines of the relative near and relative far points run parallel to DONDERS' Convergence-line. (See fig. 1: pq and R.S.)

HESS is of opinion that one can continue these lines in the area of latent ciliary muscle contraction, but could not prove this, as no measuring could be done in the "latent" area.

The next question therefore is:

f. How do the lines of the relative near and far points run in the area of latent ciliary muscle contraction?

Our reasoning is as follows: if the supposed individual of fig. 1 converges 6 metreangles, the unparalysed ciliary muscle can contract

through the stimulus of this convergence, and with the utmost exertion, 8 myodioptries, and can therefore accommodate at M; if however, the muscle is paralysed in the slightest degree, it will contract less strongly through this same stimulus, e. g. only $7\frac{1}{2}$ myodioptries, and will therefore accommodate at H.

By determining the relative accommodation, we can therefore detect the slightest paralysis of the muscle in an individual of whom the accommodation-figure is known (question *d*).

To see if the myodioptrie is a constant value, we paralyse the muscle to a certain degree, for instance so that on converging 6 metreangles, accommodation is only possible as far as Q; the muscle then has an action of 7 instead of 8 myodioptries; if all myodioptries are of equal value, then the muscle only possesses 7/8 of its power and is for 1/8th paralysed. We control this by measuring the relative accommodation and determining the degree of paralysis for the same paralysis and other convergencies too.

If one constantly finds the same degree of paralysis, so that on converging 4 metreangles an accommodation only takes place up to $g = 5\frac{1}{2}$ myodioptries, instead of 6; and on converging 3 metreangles, there is only an accommodation to $U = 3\frac{1}{2}$ myodioptries instead of 4; then the paralysis appears to be constantly 1/8. One can control this with as many degrees of paralysis and convergencies as one wishes, so that question *b*, whether the myodioptrie is a constant value in one particular person, can be definitely solved.

To determine the course of the lines of the relative near and far points in the latent area, one paralyses the accommodationmuscle to a certain degree, say the half, so that one finds by convergence of two metreangles (in fig. 1) a greatest accommodation of 2 D., instead of 4 D.; 3 D. instead of 6 D., on converging 4 m. a.; and 5 D., instead of 10 D., with a convergence of 8 m. d.; if, now, on converging 12 metreangles a greatest accommodation of 7 D. is reached (to Y in fig. 1), then one may say that the half-paralysed muscle contracts 7 myodioptries with this stimulus; the normal muscle would therefore have reacted with $2 \times 7 = 14$ myodioptries, so that the relative near point with a convergence of 12 metreangles would lie at W., on the ordinate of 12 and the abscis of 14.

If, with a convergence of 14 metreangles, one finds a greatest accommodation of 8 D., (to X in fig. 1), then the healthy muscle would be able to contract 16 myodioptries, in answer to this stimulus, thus fixing the point Z on the abscis of 16 and the ordinate of 14.

If with a convergence of 18 metreangles one finds an accommodation of 10 dioptries, then point j on the abscis of 20 myodioptries is

determined. In this manner one can determine in the latent area as many points of the line of relative near points as one wishes, with different convergencies and different degrees of paralysis, and so plot out the entire line.

The course of the line of relative far points in the area of latent ciliary muscle contraction, is determined in the same manner, so that question *f.* is solved.

We determine the strength of the accommodation muscle in the following manner:

When the area of relative accommodation has been completely ascertained, the muscle is paralysed. Supposing that in the individual of fig. 1, the accommodation muscle is paralysed for $^1/_4$th part; the absolute near point is now determined; if this still lies at a distance of 10 cm., then one can say that $^3/_4$ of the muscle-power produces a contraction of at least 10 myodioptries, the total muscle power is therefore at least $4/3 \times 10 = 13\ 1/3$ myodioptries.

If the paralysis is 1/3 while the accommodation remains 10 D., then firstly one may consider question *a.* as answered; for a partially paralysed muscle can evidently give the greatest possible accommodation, so the strongest possible contraction is not necessary, and secondly 2/3 of the muscle-power produces a contraction of at least 10 myodioptries, the muscle-power is therefore at least $3/2 \times 10 = 15$ myodioptries.

If again with a paralysis of $\frac{1}{2}$ an accommodation of 10 D. is reached then the power is at least 20 myodioptries. But if, on paralysing the muscle for one third of its power, only 8 D. accommodation is reached, then the power is $3 \quad 8 = 24$ myodioptries.

Control is obtained by further paralysis; if, after paralysing three quarters of the power, 6 D. accommodation is reached, then the total power is $4 \times 6 = 24$ myodioptries; if there is still an accommodation of 4 D., after the muscle has been paralysed to $^1/_6$, then the power is $6 \times 4 = 24$ myodioptries, etc., so that the result obtained can be controlled by as many observations as one wishes.

If all the values obtained coincide sufficiently, then one has not only determined the muscular power, but has also proved that the myodioptrie has a constant value and that the method must be correct, otherwise the values could not constantly be found to coincide.

A curve of the paralysing action of a substance can be obtained by first determining the total power of the muscle in a certain individual, then dropping the paralysing substance in the eye, and determining the power again, at regular intervals, the results being plotted out in a scheme.

To this purpose we note (fig. 2) the time in minutes on the abscis, the muscle-power in myodioptries on the ordinate.

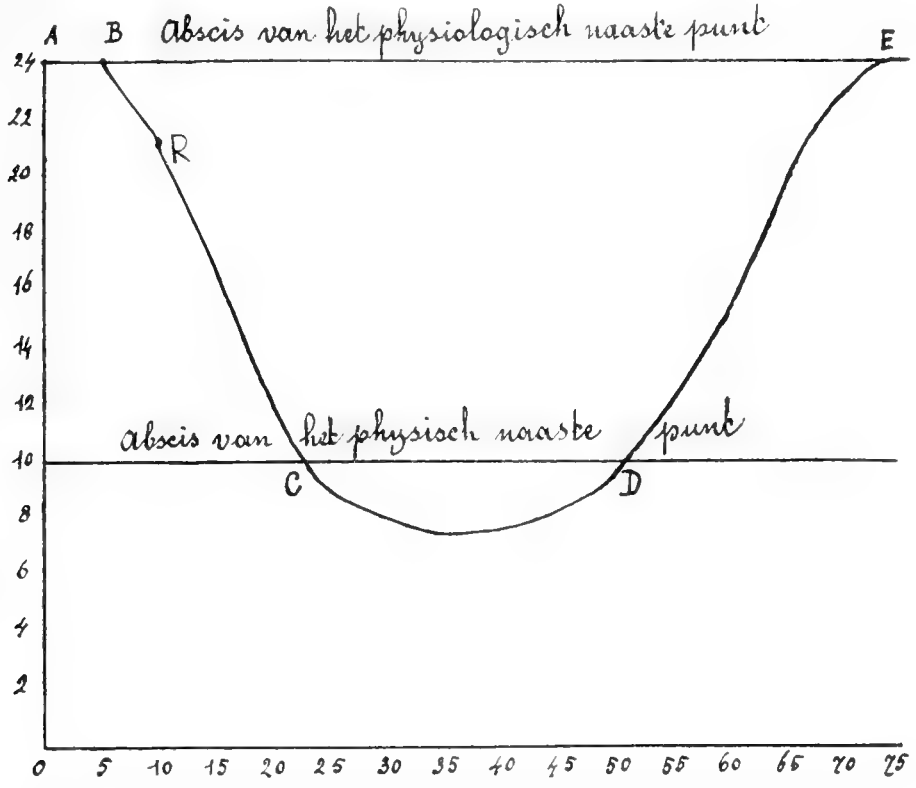

Fig. 2.

Abscis van het physiologische naaste punt = Abscis of the physiologic near point.
Abscis van het physisch naaste punt = Abscis of the physical near point.

Supposing the power, at the beginning of the examination, to be 24 myodioptries, when after 5 minutes there is no sign of paralysis, then one notes point B on the abscis of 24 and the ordinate of 5; if, after 10 minutes, the muscle is paralysed for $^1/_8{}^{th}$ part, then the power is still 21 myodioptries, and one has found a point R on the abscis of 21 and the ordinate of 10. Continuing in this manner, and continually determining the degree of paralysis of the muscle, one can find and plot out the entire paralysis-curve A. B. C. D. E. This examination gives an excellent control of the correctness of the method; for as soon as the curve surpasses the abscis of the physical near point, we can also directly find the degree of paralysis by determining the absolute near point. The part C.D. of the curve can therefore be ascertained in two entirely different ways.

If these two give entirely the same result, or if they agree

sufficiently (taking into consideration the possible errors of the method) then we may look upon this as a proof of the correctness of the method.

We have determined the "accommodation-figures" for a couple of persons, aged respectively 31 and 24 years, and have examined the paralysing action of cocaine and homatropine on the ciliary muscle.

One sees from our curves that the result is such that we feel justified in concluding that the method is good. In one patient we found a power of the ciliary muscle amounting to about 24 myodioptries; in the other 20 myodioptries.

It appeared that total contraction of the ciliary muscle is not necessary to obtain the greatest possible accommodation; that the myodioptrie has a constant value for each of these two persons, that the lines of the relative near and far points in the area of latent ciliary muscle contraction, run parallel to each other and to the convergence line of DONDERS, and that it is possible in persons whose "accommodation-figures" are known, to detect even the slightest decrease in power of the ciliary muscle.

Cocaine has on the accommodation-muscle a cumulative paralysing action, which shows considerable individual difference; it is therefore not at all surprising that one comes across such different reports of its action in the literature; as the possibility of detecting this action was dependent on:

the number of times cocaine is dropped in the eyes; the age of the observer; individual peculiarities; the duration of the observations and from the intervals between the observations.

One can draw still more conclusions from the results obtained, with regard to the influence of heterophoria, condition of refraction, etc. on the "accommodation-figures", and of the influence, which feebleness of the ciliarymuscle has on the power to do our work at short distance.

My only object at present, however, was to draw attention to the fact that the method of examining the relative accommodation enables us to widen our insight into the accommodation, and makes it possible to examine the influence of different substances on the accommodation muscle.

It is a pity that the method itself is so difficult to master, that it will never become a method for clinical examination in the hands of many, but will have to be limited to the laboratory work of a few.

Physiology. — *"On the Influence of the vagi on the frequency of the action currents of the Diaphragm during its respiratory Movements."* [1]) By Dr. J. G. DUSSER DE BARENNE and Dr. J. B. ZWAARDEMAKER. (Communicated by Prof. H. ZWAARDEMAKER.)

(Communicated at the meeting of September 29, 1923).

In a previous paper one of us [2]) was able to show that the frequency of the action currents of the striped muscles, as they occur in the cerebrate rigidity of the cat and in the voluntary contraction in man, undergoes a distinct diminution after elimination of the proprioceptive impulses, originating in the muscles during their contraction. The elimination of these proprioceptive impulses was produced by section of the posterior roots in the animal and by local intramuscular injection of novocain in the human individual.

We then investigated if this experimental fact could also be established in other innervations and first of all in the diaphragm. We will not dwell here on this investigation which gave us similar results for this muscle as in the researches mentioned above. But in the course of these investigations on the frequency of the action currents of the diaphragm, we got results which gave rise to the supposition that perhaps the vagi might have an influence on the action currents of this muscle during its respiratory contractions.

We, therefore, had to investigate this problem separately and propose to deal in this paper with the obtained results. The question to be answered, was therefore the following: Which is the frequency of the action currents of the diaphragm during its respiratory contractions *before* and *after* elimination of both vagi.

At first we made use of the cat; later on, when we had already obtained a definite answer to our question, we did another set of experiments on the rabbit and could show that also in this animal

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[1]) A preliminary communication of this paper was made at the XI[th] International Physiological-Congress at Edinburgh, 25th July 1923.

[2]) J. G. DUSSER DE BARENNE, Untersuchungen über die Aktionsströme der quergestreiften Muskulatur bei der Enthirnungsstarre der Katze und der Willkürinnervation des Menschen. Skandin. Arch. f. Physiol., 1923, Vol. XLIII. (Festschrift für R. TIGERSTEDT), S. 107.

he vagi have the same influence on the action currents of the diaphragm, as found in the cat, this influence in the rabbit being even much more distinct than in the cat.

Anaesthesia of the animal by subcutaneous injection of urethane (ca. 1 gr. pro KG. of body-weight). By means of artificial heating we tried to keep the body temperature of the animal as constant as possible. Incision of the abdominal wall in the linea alba, of about 3 cm., beginning directly caudally of the ensiform cartilage. This processus was kept in upright position by fixing it with a forceps to a support, which was isolated electrically from the table on which the animal was lying. Then we isolated as carefully as possible one of the anterior slips of the diaphragm and put a small piece of celluloid under it, so as to insulate this part of the muscle as well as we could from the other parts of the diaphragm and its surroundings. In this slip were hooked two very small hooks at a distance from each other of about 1—1,5 cm., which served as electrodes, through which the action currents of the muscle were lead off to the string galvanometer (large pattern of EDELMANN). In our earlier experiments these hooks were of copper and therefore polarisable electrodes, in our later experiments we made use of similar shaped silver hooks, galvanoplastically coated with a layer of silver chloride; these electrodes were non-polarisable. As was to be expected we could not find that the use of these different electrodes gave rise to any appreciable difference in our curves, because it cannot be expected that the polarization of the copper hooks has a distinct influence on the weak, frequent and alternating action currents of the muscle. The hooks were connected with very thin copper wires to the thicker wires leading to the galvanometer, so that the movements of the muscle could be followed quite freely by the electrodes and connecting wires. By closing the opening in the abdominal wall with a pad of dry cottonwool loss of heat of the muscle and other disturbing influences were prevented.

The respiratory movements of the diaphragm were reproduced on a kymograph with blackened paper and underneath these tracings we marked electromagnetically during which part of the pneumogram the action currents were registered. The table with the animal was carefully insulated by putting it on large blocks of paraffine.

After these preliminaries we first took the action currents of the diaphragm during normal inspiration, i.e. before the elimination of the vagi. Then both these nerves were carefully prepared at the neck and eliminated without excitation, either through local anaesthesia with ether vapour or through local application on the nerves of a 1 % solution of novocain. When the elimination of the vagi established itself by a change of the respiratory movements of the animal, we again registered the action currents of the insulated anterior slip of the diaphragm. We might draw attention to the fact that by a special devise it was possible to take our electrophysiological records in every desired phase of the respiratory contraction of the muscle.

In all our experiments in which the elimination of both vagi is followed by a distinct change of the mechanical type of respiration, we could establish the fact that the elimination of the nervous impulses *gives rise to a distinct augmentation of the frequency of the*

action currents of the diaphragm during its inspiratory contractions. Only in those few cases, already known to ROSENTHAL, in which the respiratory movements remain nearly unaltered, could we find but a small augmentation. But even in these experiments an augmentation of the frequency was to be seen, though slight. Until now we have not yet met with an experimental result, pointing in an opposite direction, i.e. a diminution of the frequency of the action currents of the diaphragm after elimination of the vagi.

First of all we will give some curves as evidence of our statement.

fig. 1*a*

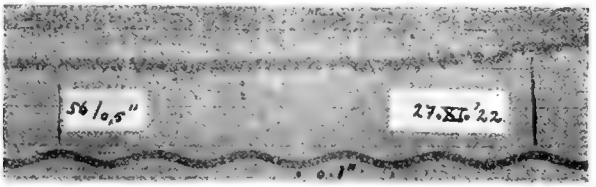
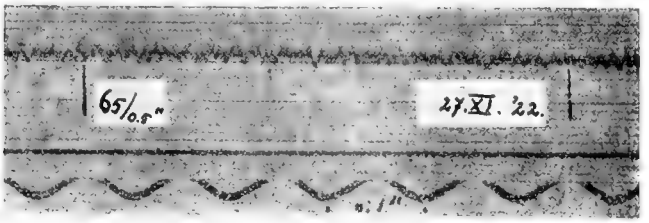


fig. 1*b*



Cat. Experiment of the 27th Febr. 1923. Fig. 1*a* action currents of the diaphragm *before*, fig. 1*b* *after* elimination of the vagi. Time 0.1 sec. On the original photographs in 1*a* 56, in 1*b* 65 action currents could be counted during the marked 0.5". So the frequency was **112** before, and **130** per sec. after the elimination of the vagi. (The date in these figures is wrong.)

fig. 2*a*

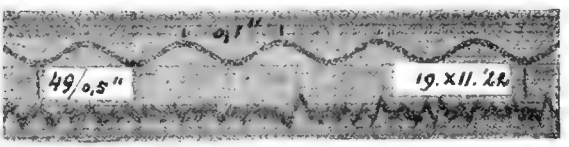
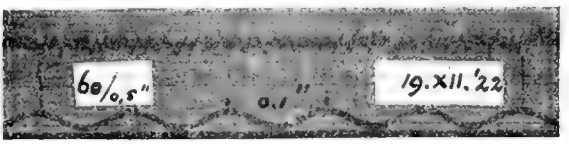


fig. 2*b*



Cat. Experiment of the 19th Dec. 1922. As foregoing figure. Frequency in 2*a* **98** per sec., in 2*b* **120** per sec.

Figures 1*a* and 1*b* show, although unfortunately not quite so distinct as the original photographs, that the frequency of the action currents before and after elimination of the vagi is **112** resp. **130** per second. In fig. 2*a* and 2*b* these numbers are **98** and **120** respective.

The results of our **8** experiments on the cat in the order in which they were performed, are given below in the table. In experiment IV only a slight augmentation of the frequency was found; in this animal the change of the pneumogram of the diaphragm after the anaesthesia of the vagi was not very distinct.

TABLE

of the frequency of the action currents of the diaphragm in the cat.

Number of experiment	Before the elimination of the nervi vagi.	After the elimination of the nervi vagi.	% augmentation.
I.	98	120	22.45
II.	118	130	10.17
III.	102	117	14.7
IV.	95	101	no distinct change in the pneumogram of the diaphragm
V.	118	130	10.17
VI.	112	130	16.07
VII.	113	132	16.81
VIII.	119	133	11.76

On the rabbit **6** experiments were made; in all of which an augmentation of the frequency of the action currents after elimination of the vagi was also found, for the most part still more evident than in the cat. In one of the rabbits this augmentation was even 40 %.

We will now try to answer the question, how one has to look at this experimental fact.

As is already long known the effect of double vagotomia, either by cutting or by local anaesthesia of the nerves, is that the respiratory movements become less frequent and are increased in amplitude. We will for the present confine ourselves to this last point. The muscles which perform inspiration and in the first place the most important, the diaphragm, contract more vigorously, after the elimin-

ation of the vagi. One might consider the most plausible explanation of our experimental fact to be the following, viz.: that this stronger contraction of the muscle might show itself in an augmentation of the frequency of its action currents. This explanation however is not consistent. First of all we know the fact already ascertained by PIPER, that the frequency of the action currents in voluntary contraction of human muscles remains unaltered under various strengths of contraction, a fact which one of us (D. DE B.) lately confirmed. But it might be argued, that this fact, though it may be true with regard to voluntary contraction of the human muscle, might not apply to the diaphragm of the rabbit. We, therefore, tried to get direct experimental evidence on this point by inducing a deepening of the inspiratory movements with other methods, f.i. by letting the animal breath an atmosphere rich in CO_2 or by closing the trachea during a few seconds. It was found that the deepening of the inspiration which follows these procedures is *not* accompanied by an augmentation of the frequency of the action currents of the diaphragm. We could establish this in many experiments; only in one of them we found that after breathing a CO_2-atmosphere there was also an augmentation of the frequency of the action currents. In this experiment we had already performed a local ether anaesthesia of both vagi; it might be possible that the nerves were still functionally slightly damaged; anyhow in all our other experiments, in which the increase of inspiration through CO_2-breathing preceded the vagotomia, we never found an augmentation by CO_2.

Only one objection must still be taken into account.

One of the other results of the elimination of the vagi is an acceleration of the heart. In our experiments, in which the anterior slip of the diaphragm was not detached from the ensiform cartilage for the sake of leaving the muscle in as normal a condition as possible, we generally also found in our curves of the action currents traces of the electrocardiograms of the animal, especially in the cat, where the insulation of the anterior slip of the diaphragm is much more difficult than in the rabbit. These electrocardiograms present themselves under these circumstances as simple, diphasic action currents, which look very much like the action currents of the diaphragm itself and often cannot be distinguished from them. So, when one counts all the peaks during 0.5 a second, as we always did, a few of these electrocardiograms are always included. The objection might now be made that after the vagotomia through the acceleration of the heart, the number of electrocardiograms is aug-

mented, and that this increase in the number of the electrocardiograms might be responsible for the augmentation of the action currents of the diaphragm.

A simple calculation however overthrows this objection. Let us assume that the frequency of the heart in the cat (the same reasoning with somewhat other numbers holds true also for the rabbit) is about 180 per minute [1]), then there will be present in the curve over a length of 0.5 a second, mostly $0.5 \times \frac{180}{60} = 1.5$ and at most 2 electrocardiograms. Supposing that after the elimination of the vagi the heart accelerates from 180 to f.i. 240 or even 360 beats per minute, an acceleration of 100%, which will only be seldom, if ever, present, then we can expect to find in our curves over 0.5 a second $0.5 \quad \frac{360}{60} = 3$ electrocardiograms, i.e. an apparent augmentation of mostly 1 or at utmost 1.5 per 0.5 second. So this would give an apparent augmentation of the frequency of the action currents of the diaphragm of 2 or 3 per second. From this reasoning it is clear that even with these numbers, which we took as unfavourably as possible, this factor, which undoubtedly exists, cannot explain the augmentation present in our experiments.

We think it therefore permissible to conclude that for the greatest part, the augmentation of the frequency of the action currents of the diaphragm after elimination of both vagi is due *to the elemination of the centripetal impulses, which normally travel along the vagi to the central nervous system and obviously exert an inhibitory influence on the respiratory movements, at least in the cat and the rabbit.*

Since the researches of HERING and BREUER it is wellknown that centripetal vagal impulses have an important influence on the respiratory movements, especially on the inspiration. The fact shown by our experiments gives clear and, as far as we know, until now unknown evidence of this influence.

September 1923. *Physiological Laboratory of the University of Utrecht.*

[1]) This assumed number is on the high side; for a smaller number our reasoning becomes yet more conclusive.

Géologie. — *"Description de Raniniens nouveaux des terrains tertiaires de Borneo"*, par V. VAN STRAELEN.

(Présenté par M. le Prof. G. A. F. MOLENGRAAFF à la séance du 24 novembre 1923).

Les Raniniens décrits ci-dessus ont été recueillis par M. J. A. LOHR [1]), au cours d'une exploration effectuée dans la vallée de la rivière Toehoep, affluent de rive gauche du fleuve Barito, au S.E. de l'île Borneo. Ils font actuellement partie des collections du Musée géologique de la "Technische Hoogeschool" à Delft. M. le Professeur G. A. F. MOLENGRAAFF, directeur de ce Musée, a bien voulu attirer mon attention sur ces matériaux et me les confier pour étude.

Famille: *Raninidae* DANA 1852.
1. — Genre: *Ranina* LAMARCK 1818.
Sous Genre: *Hela* MÜNSTER 1840.
Ranina (*Hela*) *Molengraaffi* nov. sp.
(Fig. 1*a* et *b*).

Fig 1*a*.

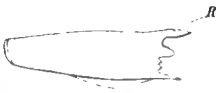


Fig. 1*b*.

Ranina (*Hela*) *Molengraaffi* nov. sp. — Grandeur naturelle.
1*a*. Face dorsale. — 1*b*. Face latérale droite. — R. Rostre.

Cette espèce est connue par les restes d'un seul individu, se présentant par la face tergale. Le céphalothorax dont la longueur dépasse la largeur d'environ $^1/_5$, s'élargit de l'arrière vers l'avant. Sa largeur mesurée au niveau de l'insertion des deux dents marginales et celle mesurée au bord postérieur, sont dans le rapport de 3 à 2. Le céphalothorax est bombé, la courbure s'accentuant dans la région médiane, au point de constituer une crête surbaissée. La région frontale s'incurve vers le haut, de sorte qu'elle semble précédée par

[1]) J. A. LOHR, *Mededeelingen over de Geologie der Doesoen-landen*. Verhandelingen van het Geologisch en Mijnbouwkundig Genootschap voor Nederland en Koloniën, Vergaderingen, N°. 45, 1914, pp. 174—175.

une faible dépression. Une autre dépression plus forte que la précédente, existe dans la région médiane du céphalothorax, au tiers postérieur. La région cardiaque est indiquée par une paire de sillons en arc de cercle, à concavité ouverte vers les bords latéraux.

Le bord frontal sensiblement rectiligne est occupé par un rostre triangulaire, large et long, bordé par des échancrures oculaires limitées chacune latéralement par un lobe triangulaire à base très large. Au delà de ces lobes, se trouvent deux petites épines et enfin une forte dent effilée et incurvée extérieurement, constituant le prolongement des bords latéraux. Ceux-ci sont un peu incurvés et à angle droit avec le bord postérieur. Ce dernier est à peu près rectiligne et bordé par un étroit sillon.

Le test paraissant lisse, est garni de fines granulations, légèrement acuminées, disposées sans ordre apparent.

La face sternale n'est pas connue.

L'attribution au genre *Ranina* pourrait être contestée, en se basant sur la petite taille, la simplicité du bord frontal et surtout le caractère de l'ornementation, fine au point que le test paraît lisse. A première vue, *R. Molengraaffi* se rapprocherait plutôt du genre *Notopus* De Haan, par la forme et l'ornementation du céphalothorax. Cependant, il lui manque entre autres caractères de *Notopus*, la crête épineuse située en arrière du bord frontal et unissant les deux dents latérales. Les autres genres de Raniniens à test lisse, dont ils constituent le groupe le plus nombreux, sont :

Raninoides H. Milne-Edwards Holocène,
Lyreidus De Haan, Oligocène-Holocène,
Notopoides Sp. Bate, Miocène et Holocène,
Cosmonotus Adams et White, Holocène,
Notosceles Bourne, Holocène,
Raninella A. Milne Edwards, Cénomanien-Sénonien,
Raninellopsis J. Boehm, Miocène,
Notopocorystes Mac' Coy, Cénomanien,
Eucorystes Bell, Albien-Cénomanien,
Palaeocorystes Bell, Albien-Cénomanien,
Hemioon Bell, Cénomanien,

et n'entrent pas en ligne de compte, à cause de la forme générale du céphalothorax et des caractères du bord frontal. Par le contour de son céphalothorax, *Notopus* est très voisin de *Ranina*.

M. R. Fabiani [1]) a distingué deux sous-genres dans *Ranina*, établis sur le caractère de l'ornementation. Le sous-genre *Lophoranina* réunit toutes les espèces dont le test est orné de côtes transversales épineuses et flexueuses, le sous-genre *Eteroranina* groupe les formes dont le test est soit à peu près lisse, soit orné de petits granules ou de petits tubercules acuminés, disposés en rangées et quelquefois sans ordre apparent. C'est pour des espèces appartenant à ce dernier groupe que G. zu Münster [2]) avait créé le genre *Hela*, dont le type *Hela speciosa* Münster provient du Chattien de Bünde. Je considère *Hela* comme synonyme de *Eteroranina* sur lequel il a la priorité.

Les *Ranina* décrites jusqu'à ce jour et qui se rapprochent le plus de celle trouvée à Bornéo, sont:

Ranina Ombonii Fabiani, de l'Yprésien des Colli Berici (Vicentin),

R. notopoides Bittner, du Lutétien du Monte Masua (Véronais),

R. budapestinensis Loerenthey, du Bartonien du Kis-Svábhegy (Hongrie),

R. Bouilleana A. Milne-Edwards, du Tongien de Biarritz (Aquitaine) et de Montecchio-Maggiore (Venétie),

R. granulosa A. Milne-Edwards, de l'Oligocène des environs de Dax (Aquitaine),

R. (*Hela*) *oblonga* Münster, du Chattien de Bünde (Hesse).

R. Molengraaffi se distingue:

de *R. Ombonii* par son céphalothorax moins long et plus large, beaucoup plus convexe, son bord frontal coïncidant à peu près avec la plus grande largeur de l'animal, enfin une ornementation beaucoup plus fine;

de *R. notopoides* par son bord frontal et son bord postérieur plus large, la présence d'une seule paire d'épines latérales, un rostre plus long et deux épines situées entre les lobes et l'épine latérale;

de *R. budapestinensis* par une forme beaucoup plus massive, le bord postérieur plus large, le rostre plus développé, les échancrures orbitaires plus profondes et les lobes correspondants plus développés, enfin des épines et des dents plus fortes;

de *R. oblonga* par son bord frontal plus étendu par rapport au bord postérieur et moins profondément découpé et une ornementation plus fine;

de *R. granulosa* par son bord frontal beaucoup moins découpé et le bord postérieur plus large.

[1]) R. Fabiani, *Sulle specie di Ranina finora note ed in particolare sulla Ranina Aldrovandii.* Atti dell' Academia scientifica Veneto-Trentino-Istriana, ser. 3a, t. 3, 1910, p. 85.

[2]) G. zu Munster, *Beiträge zur Petrefactenkunde,* Heft 3, 1840, p. 24.

Parmi toutes les espèces citées, c'est avec *R. budapestinensis* que *R. Molengraaffi* a le plus d'affinités.

Type. Musée géologique de la "Technische Hoogeschool" à Delft, échantillon n° 6 du lot K.A. 6491.

Gisement. Septaria argileux, légèrement calcarifère, coloré par de l'oxyde de fer, d'âge miocène d'après la carte de M. G. L. L. KEMMERLING. [1])

Localité. Vallée de la rivière Toehoep, entre l'embouchure de son affluent Bangkelan et Kampong Brawai (Borneo).

Je dédie cette espèce à M. G. A. F. MOLENGRAAFF, Professeur à la "Technische Hoogeschool" de Delft.

2. — Genre: *Raninella* A. MILNE EDWARDS 1862.
Raninella Toehoepae nov. sp.
(Fig. 2*a* et *b* et *c*).

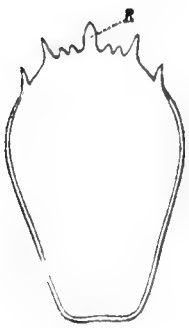


Fig. 2*a*.

Fig. 2*b*.

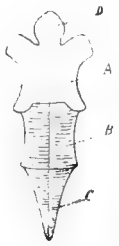


Fig. 2*c*.

Raninella Toehoepae nov. sp.

2*a*. Face dorsale, grandeur naturelle. — 2*b*. Face latérale droite, grandeur naturelle. — 2*c*. Plastron sternal, × 2. *A*, *B*, *C*. Sternites. *D*. Episternum. — *R*. Rostre.

Le céphalothorax est fortement bombé, s'élargissant considérablement vers l'avant, la plus grande largeur se trouvant à peu près à hauteur des sillons cardiaques et correspondant au double de la largeur du bord postérieur. Le bord frontal est faiblement convexe, porte un rostre droit à son origine et se terminant en une pointe triangulaire. De part et d'autre du rostre, le bord frontal présente des

[1]) G. L. L. KEMMERLING, *Geologisch-Topografische Schetskaart van het Stroomgebied der Barito (Borneo).* Tijdschrift van het Koninklijk Nederlandsch Aardrijkskundig Genootschap, 2de ser., Deel 32 (1915), kaart N°. 6.

échancrures limitées par deux faibles épines, au delà desquelles se trouve une forte dent. Une dent marginale plus robuste encore, est insérée un peu au dessus de l'inflexion du bord latéral. Le bord postérieur est à peu près rectiligne, les bords latéraux sont convexes dans la région antérieure, mais rectilignes dans la région postérieure. Le bord postérieur et les bords latéraux postérieurs présentent un sillon marginal limitant une faible carène latérale.

La région cardiaque est marquée par deux sillons cardiaques, ayant à peu près la forme d'arcs de cercle à concavité ouverte vers les bords latéraux, surmontés chacun d'une paire de petits sillons parallèles.

Le plastron sternal est très large tout au moins dans ses parties antérieures. Le premier sternite placé entre la première paire de thoracopodes, est fort large et présente les deux entailles latérales et circulaires habituelles. Il se termine en avant par un épisternum arrondi. Le deuxième sternite est un peu moins large que le précédent, se rétrécissant vers l'arrière et pourvu d'un profond sillon médian. Le troisième sternite est étroit.

Le pléon se recourbe sous la face sternale. Sa largeur à l'origine est égale à celle du bord postérieur du céphalothorax. Ce qui reste des thoracopodes est trop fragmentaire pour permettre une description. L'ornementation du test est constituée par des granules extrêmement fins.

Le genre *Raninella* est un genre essentiellement crétacé. On en connait actuellement les espèces suivantes:

Raninella Trigeri A. Milne-Edwards, du Cénomanien du Mans (Sarthe),

R. elongata A. Milne-Edwards, du Cénomanien du Mans (Sarthe),

R. Schloenbachi Schlüter, du Sénonien (Emsien) de Wöltingerode (Saxe) [1]),

R. baltica Segerberg, du Danien de Faxe et d'Annetorp (Danemarck).

R. Toehoepae se distingue nettement des trois premières espèces citées, par la forme plus ovalaire de son bord frontal. Elle se rapproche de *R. baltica* dont le céphalothorax est également ovalaire, mais elle s'en distingue: 1° par son bord postérieur plus étroit, 2° son élargissement antérieur proportionellement plus considérable et reporté d'avantage vers l'arrière de l'animal.

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[1]) R. Schloenbachi est une espèce imparfaitement connue, basée sur un individu chez lequel la région frontale est en partie détruite et dont on ne connait que le moule interne des régions postérieures. Je la maintiens provisoirement dans le genre *Raninella*.

Jusqu'à présent le genre *Raninella* n'a été rencontré que dans le Crétacé moyen et supérieur. Il présente parmi les *Raninidae* un certain nombre de caractères que je considère comme primitifs: grande dimension du deuxième sternite et rétrécissement relativement faible des sternites postérieurs et du pléon. Il rappelle ainsi les genres *Palaeocorystes* Bell et *Notopocorystes* M'Coy du Gault du Kent, que je rattache aux *Raninidae* [1]).

Type. Musée géologique de la "Technische Hoogeschool" à Delft, échantillon K.A. 6504.

Cotypes. K.A. 6491, 6497, 6504, 6505, 6517, 6522.

Gisement. Septaria argileux, légèrement calcarifères, colorés par l'oxyde de fer, d'âge miocène d'après la carte de M. G. L. L. Kemmerling [2]).

Localités. Vallée de la rivière Toehoep, entre l'embouchure de son affluent Bangkelan et Kampong Brawai (Borneo).

Le nom spécifique est tiré de celui de la rivière Toehoep, affluent de gauche du Haut-Barito.

Les stratigraphes qui ont étudié les couches dans lesquelles la rivière Toehoep a creusé sa vallée, ne semblent pas d'accord sur leur âge. M. G. L. L. Kemmerling [3]) les rapporte au Miocène, M. J. A. Lohr [4]) hésite entre un âge anté — et post — éogène. Les deux Crustacés décapodes qui viennent d'être décrits ne permettent pas de trancher ce différend.

Qu'il soit cependant permis d'attirer l'attention sur le fait que *Ranina Molengraaffi*, forme lisse à bord frontal peu découpé et s'élargissant peu vers l'avant, a un cachet archaïque la rapprochant de ses congénères dont l'âge éogène et même crétacé n'est pas douteux. Quant à *Raninella Toehoepae*, elle appartient à un genre méso- et supracrétacé et présente d'ailleurs également des caractères primitifs accentués.

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[1]) V. van Straelen, *Note sur la position systématique de quelques Crustacés décapodes de l'époque crétacée.* Bulletins de l'Académie royale de Belgique, Classe des sciences, 1923, pp. 116—125, 6 fig.

[2]) G. L. L. Kemmerling, *Geologisch-Topografische schetskaart* etc., loc. cit.

[3]) G. L. L. Kemmerling, *Topografische en Geologische Beschrijving van het Stroomgebied van de Barito, in hoofdzaak wat de Doesoenlanden betreft.* Tijdschrift van het Koninklijk Nederlandsch Aardrijkskundig Genootschap, 2de ser., Deel 32, 1915, pp. 575—641 et pp. 717—774.

[4]) J. A. Lohr, loc. cit.

Mathematik. — *"Ueber die zu einem Punkte und einer Geraden gehörigen Polarkurven inbezug auf eine gegebene algebraische Kurve."* Von F. KÖLMEL in Baden-Baden.

(Mitgeteilt von Prof. JAN DE VRIES in der Sitzung vom 24 November 1923).

1. *Die Aufgabe.* Wird eine algebraische Kurve n-ter Ordnung durch eine Gerade in den n Punkten $R_1, R_2, \ldots R_n$ geschnitten, so ist nach JONQUIÈRES [1]) der harmonische Mittelpunkt r-ter Ordnung R zu diesen n Punkten und einem Zentrum O definiert durch die Gleichung

$$\binom{n}{r}\cdot\left(\frac{1}{OR}\right)^r-\binom{n-1}{r-1}\cdot\left(\frac{1}{OR}\right)^{r-1}\cdot\sum_1^n{}_i\left(\frac{1}{OR_i}\right)_1+\binom{n-2}{r-2}\cdot\left(\frac{1}{OR}\right)^{r-2}\cdot$$

$$\cdot\sum_1^n{}_i\left(\frac{1}{OR_i}\right)_2-\ldots+(-1)^n\cdot\binom{n-r}{0}\cdot\left(\frac{1}{OR}\right)^n\cdot\sum_1^n{}_i\left(\frac{1}{OR_i}\right)_n=0$$

wo $\binom{n}{k}$ Binomialkoeffizienten und $\sum_1^n\left(\frac{1}{OR_i}\right)_k$ die Summe der Produkte der reziproken Abschnitte OR_i zu je k bedeutet, $i=1, 2, \ldots r$.

Beschreibt die schneidende Gerade ein Strahlbüschel mit dem Zentrum Q, wahrend O eine Gerade p durchläuft, so beschreibt der harmonische Mittelpunkt r-ter Ordnung eine algebraische Kurve, die ich die zu dem Zentrum Q und der Geraden p gehörige Polarkurve r-ter Stufe inbezug auf die gegebene Grundkurve n-ter Ordnung nenne.

Allgemein lassen sich die Polarkurven auch auffassen als Erzeugnis des Strahlbüschels Q und des ihm projektiven Büschels der gewöhnlichen Polaren der Punkte der Geraden p.

2. Die vorliegende Mitteilung behandelt zunächst den Fall:

Die feste Grundkurve sei ein Kegelschnitt.

Hier kommt nur die Polarkurve erster Stufe in Betracht, da die zweiter Stufe identisch mit der gegebenen Kurve ist.

[1]) Vgl. JONQUIÈRES. Mémoire sur la théorie des polaires etc. Journal de Liouville. 1857 oder
CREMONA. Geometrische Theorie der ebenen Kurven. Deutsche Ausgabe von Curtze, Greifswald 1865.

I. *Geometrisches.* Es sei f der gegebene Kegelschnitt, P der Pol von p inbezug auf f, q die Polare von Q und y die Polare des Schnittpunktes Y von p und q.

Um auf einem Strahle α von Q den gesuchten vierten harmonischen Punkt zu finden, schneiden wir α mit p (der Schnittpunkt sei $\mathfrak{A}$) und konstruieren zu $\mathfrak{A}$ die Polare a inbezug auf f, die durch P geht. Der Schnittpunkt A von $\alpha \times a$ ist dann der gesuchte vierte harmonische Punkt. Daraus ergibt sich sofort: Jedem Strahl α des Strahlbüschels Q ist die konjugierte Polare a inbezug auf f durch den Punkt P zugeordnet, daher sind diese beiden Büschel projektiv und *der gesuchte Ort des vierten harmonischen Punktes ist ein Kegelschnitt. Diesen nenne ich den Polarkegelschnitt des Punktes Q und der Geraden p inbezug auf den gegebenen Kegelschnitt f und bezeichne ihn mit Φ.*

Aus dem obigen folgt, dass Q und p mit P bezw. q vertauschbar sind.

3. Die Φ geht jedenfalls durch die Schnittpunkte C, D von p mit f, ferner durch die Schnittpunkte U, V von q mit f, durch die Punkte Q und P und berührt die Geraden YQ und YP in Q bezw. P. XYZ ist das gemeinsame Polardreieck für f und Φ. Es ist auch leicht zu entscheiden, welcher Art der Kegelschnitt Φ ist. Soll nämlich Φ einen unendlich fernen Punkt B_∞ haben, so müssen die 2 entsprechenden Strahlen β und b der projektiven Büschel Q und P parallel sein, somit liegt der vierte harmonische Punkt in der Mitte der Schnittpunkte von β mit f.

Verbindet man diese Mitte mit M, so erhält man einen Strahl, der zu β konjugierte Polare ist. Konstruiert man also zu allen Strahlen β von Q die konjugierten Polaren durch M, so erhält man wieder ein zu dem Büschel Q projektives Büschel M und das Erzeugnis dieser zwei projektiven Büschel ist wieder ein $C_2 \equiv \lambda_1$, *der alle Sehnen in f, die durch Q gehen, halbiert.* Schneidet man λ_1 mit p, so geben die Verbindungsgeraden dieser Schnittpunkte mit Q die Richtungen der Asymptoten von Φ an. *Je nachdem also λ_1 die Gerade p in 2 Punkten schneidet, oder berührt oder gar nicht trifft, ist Φ eine Hyperbel, oder Parabel oder Ellipse.* λ_1 geht durch M, Q, U, V. Es gibt noch einen zweiten solchen entscheidenden Kegelschnitt λ_2, der durch M, P, C und D geht und analog wie λ_1 konstruiert wird. Dessen Schnitt mit q gibt dann die Entscheidung. λ_1 und λ_2 bleiben dieselben, wenn Q bezw. P erhalten bleibt, während p bezw. q sich ändert. Für alle möglichen Lagen von p bilden die λ_1 ein Netz von C_2 durch die Punkte Q, V, U; entsprechendes gilt für λ_2.

Auch der vierte Schnittpunt O von λ_1 mit Φ ist leicht anzugeben:

Man verbinde M mit P und schneide MP mit λ_1, der Schnittpunkt ist der gesuchte Punkt O. Denn PM und p sind konjugierte Richtungen inbezug auf f. Zieht man also $QO // p$ so sind QO und PO $(= MO)$ entsprechende Strahlen in den projektiven Büscheln Q und P bezw. Q und M; somit ist der Schnittpunkt von PM und QO sowohl ein Punkt von λ_1 als von Φ. Zieht man ferner QM und $PG // q$, so ist der Schnittpunkt dieser 2 Geraden sowohl ein Punkt von Φ als von λ_1. Entsprechendes gilt für λ_2.

Endlich kann man noch den Mittelpunkt M_3 von Φ bestimmen. Die Verbindungsgerade von V mit der Mitte von QP geht durch M_3; ebenso die Verbindungsgerade der Mitten der Sehnen CD und QO und die Verbindungsgerade der Mitten von UV und PC. Auf den Durchmessern QM_3 und PM_3 lassen sich auch die Endpunkte E bezw. H bestimmen, die Tangenten in H und E sind dann parallel bezw. zu den Tangenten VQ und VP, so dass $VSTR$ ein dem Φ umschriebenes Parallelogramm ist.

4. Von besonderen Fällen je nach der Lage von Q und p seien kurz folgende erwähnt;

a. Ist p die unendlich ferne Gerade, so wird $P \equiv M$, $\Phi \equiv \lambda_1$.

b. Wenn p den f berührt, so berührt Φ den f in P und oskuliert λ_2 in diesem Punkte.

c. Wenn p durch M geht, hat Φ mindestens einen (reellen) unendlich fernen Punkt.

d. Wenn $p \equiv q$, so ist auch $P \equiv Q$ und Φ degeneriert in das von Q an f gehende Tangentenpaar. (Vgl. Analytisches.)

e. Wenn p durch Q geht, so liegt P auf q und Φ zerfällt in p bzw. q. (Gewöhnliche Polare.)

II. *Analytisches.*

Bezeichnungen.

5. Es seien:

$$f(x,y,z) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 = 0 \quad (1)$$

die Gleichung des festen Kegelschnittes f, ebenso

$$g(x,y,z) \equiv b_{11}x^2 + 2b_{12}xy + b_{22}y^2 + 2b_{13}xz + 2b_{23}yz + b_{33}z^2 = 0 \quad (2)$$

die Gleichung eines zweiten Kegelschnittes g; $F(u, v, w)$ und $G(u, v, w)$ die adjungierten Formen zu f, bezw. g;

A und B die Determinanten von f, bezw. g; A_{ik}, B_{ik} die Unterdeterminanten von A und B,

$$3\theta = \sum_{i,k=1,2} a_{ik} B_{ik}, \; 3H = \sum_{i,k=1,2} b_{ik} A_{ik} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

die beiden simultanen Invarianten von f und g,

$$\left.\begin{aligned} H = \Sigma\left(\frac{\partial F}{\partial a_{ik}} . b_{ik}\right) = \Sigma\left(\frac{\partial G}{\partial b_{ik}} . a_{ik}\right) = H(u,v,w) = H_{11}u^2 + 2H_{12}uv + \\ + H_{22}v^2 + 2H_{13}uv + 2H_{23}vw + H_{33}w^2 \end{aligned}\right\} \quad (4)$$

die simultane Contravariante zu f und g, ferner x_0, y_0, z_0 die Koordinaten von Q, $\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$ die von P, u_0, v_0, w_0 die L. K. von q und $\bar{u}_0$, $\bar{v}_0$, $\bar{w}_0$ die L. K. von p, sodass

$$u_0 = f_1(x_0),\ v_0 = f_2(y_0),\ w_0 = f_3(z_0);\quad \bar{u}_0 = f_1(\bar{x}_0),\ \bar{v}_0 = f_2(\bar{y}_0),\ \bar{w}_0 = f_3(\bar{z}_0)$$

und umgekehrt:

$$x_0 = F_1(u_0),\ y_0 = F_2(v_0) \text{ u.s.w.}$$

Dabei ist

$$f_1(x) = \frac{\partial f(x,y,z)}{\partial x},\quad f_2(y) = \frac{\partial f(x,y,z)}{\partial y},\quad f_3(z) = \frac{\partial f(x,y,z)}{\partial z};$$

$$f_1(x_0) = \left.\frac{\partial f(x,y,z)}{\partial x}\right|_{x=x_0,\,y=y_0,\,z=z_0};\quad f_2(y_0) = \left.\frac{\partial f}{\partial y}\right|_{x=x_0,\,y=y_0,\,z=z_0}$$

$$F_1(u) = \frac{\partial F(u,v,w)}{\partial u};\quad F_1(u_0) = \left.\frac{\partial F(u,v,w)}{\partial u}\right|_{u=u_0,\,v=v_0,\,w=w_0},$$

woraus die übrigen Bezeichnungen sich von selbst ergeben.
Dann ist auch

$$\Sigma H_{ik}a_{ik} = 6\theta,\quad \Sigma H_{ik}b_{ik} = 6H, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

und

$$\left.\begin{aligned} B_{11}a_{11} + B_{12}a_{12} + B_{13}a_{13} + H_{11}b_{11} + H_{12}b_{12} + H_{13}b_{13} = 3\theta \\ A_{11}b_{11} + A_{12}b_{12} + A_{13}b_{13} + H_{11}a_{11} + H_{12}a_{12} + H_{13}a_{13} = 3H \end{aligned}\right\} \quad (6)$$

6. Für einen Punkt R der Polarkurve erster Stufe zu O und p gilt dann

$$\frac{2}{OR} = \frac{1}{OR_1} + \frac{1}{OR_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

wo O der Schnittpunkt eines Strahles des Büschels Q mit p, R_1, R_2 die Schnittpunkte mit f sind. Sind ξ, η, ζ die Koordinaten von O, so folgt aus Obigem:

$$\left.\begin{aligned} \frac{\lambda_1}{(x-\xi) + \lambda_1(x_0-\xi)} + \frac{\lambda_2}{(x-\xi) + \lambda_2(x_0-\xi)} = 0, \\ \frac{\lambda_1}{(y-\eta) + \lambda_1(y_0-\eta)} + \frac{\lambda_2}{(y-\eta) + \lambda_2(y_0-\eta)} = 0, \end{aligned}\right\} \quad . \quad . \quad . \quad (8)$$

wobei λ_1, λ_2 die Wurzeln der Gleichung

$$f(x_0,y_0,z_0) + \lambda \cdot \{f_1(x_0) \cdot x + f_2(y_0) \cdot y + f_3(z_0) \cdot z\} + \lambda^2 \cdot f(x,y,z) = 0 \quad (9)$$

sind, und für ξ,η,ζ die Gleichung besteht:

$$\overline{u_0}\,\xi + \overline{v_0}\,\eta + \overline{w_0}\,\xi = 0.$$

Durch Elimination von λ,ξ,η,ζ erhält man als Gleichung für die Polarkurve erster Stufe:

$$\left.\begin{aligned} &2f(x,y,z) \cdot (\overline{u_0}\,x_0 + \overline{v_0}\,y_0 + \overline{w_0}\,z_0) - \\ &- \{f_1(x_0) \cdot x + f_2(y_0) \cdot y + f_3(z_0) \cdot z\} \cdot \{\overline{u_0}\,x + \overline{v_0}\,y + \overline{w_0}\,z\} = 0 \equiv \Phi(x,y,z) \end{aligned}\right\} \quad (10)$$

Unter Anwendung der oben angegebenen Beziehungen zwischen den u_0,v_0,w_0 und x_0,y_0,z_0 kann man der Gleichung noch verschiedene andere Formen geben, von denen wir gelegentlich am passenden Ort Gebrauch machen werden. Erwähnt sei nur folgende Form:

$$\Phi(x,y,z) \equiv \{f_1(x) \cdot x + f_2(y) \cdot y + f_3(z) \cdot z\} \cdot \{f_1(\overline{x_0})\,x_0 + f_2(\overline{y_0})\,y_0 + f_3(\overline{z_0}) \cdot z_0\}$$
$$- \{f_1(x_0)\,x + f_2(y_0) \cdot y + f_3(z_0) \cdot z\} \cdot \{f_1(\overline{x_0})\,x + f_2(\overline{y_0})\,y + f_3(\overline{z_0}) \cdot z\} = 0.$$

aus der die Vertauschbarkeit von Q und P besonders evident ist.

7. Zunächst ersieht man, dass die Φ-Kurve durch den Schnitt von f mit p geht, und dass sie, wenn Q auf p liegt, in p und q zerfällt; auch die Vertauschbarkeit von Q und p mit P und q ergiebt sich mit Rücksicht auf die in (5) gegebenen Beziehungen.

Ist

$$(x\,z_0 - x_0 z) + \lambda\,(yz_0 - y_0 z) = 0 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

das Strahlbüschel Q, so ist das Strahlbüschel der Polaren zu den Schnittpunkten mit p gegeben durch

$$\left.\begin{aligned} &\{f_1(x) \cdot \overline{v_0}\,x_0 + f_2(y) \cdot (-\overline{u_0}\,x_0 + \overline{w_0}\,z_0) + f_3(z) \cdot \overline{v_0}\,z_0\} \\ &+ \lambda \cdot \{f_1(x) \cdot (\overline{v_0}\,y_0 + \overline{w_0}\,z_0) - f_2(y) \cdot \overline{u_0}\,y_0 - f_3(z) \cdot \overline{u_0}\,z_0\} = 0 \end{aligned}\right\} \quad (12)$$

Durch Elimination von λ erhält man wieder $\Phi(x,y,z) = 0$. Aus der Gleichung für Φ lassen sich die Gleichungen für die Kurven λ_1 und λ_2 in (2) ableiten. Diese sind nämlich spezielle Φ-Kurven, wenn p, bzw. q zur unendlich fernen Geraden wird. Nehmen wir Cartesische Koordinaten, sodass $z = 0$ die Gleichung der unendlich fernen Geraden ist, so haben wir:

$$\lambda_1(x,y,z) \equiv 2f(x,y,z) \cdot z_0 - \{f_1(x) \cdot x_0 + f_2(y) \cdot y_0 + f_3(z) \cdot z_0\} \cdot z = 0 \quad (13)$$
$$\lambda_2(x,y,z) \equiv 2f(x,y,z) \cdot z_0 - (\overline{u_0}\,x + \overline{v_0}\,y + \overline{w_0}\,z) \cdot z = 0.$$

Für den Schnittpunkt O von λ_1 mit Φ haben wir:

$$f_1(x_0) \cdot x + f_2(y_0) \cdot y + f_3(z_0)\, z = 0,$$

und

$$\bar{u}_0 x + \bar{v}_0 y + \bar{w}_0 z - (\bar{u}_0 x_0 + \bar{v}_0 y_0) = 0;$$

letzteres ist die durch Q gehende Parallele zu p. Für die Schnittpunkte der beiden Kurven λ_1 und λ_2 findet man:

a) $z = 0$,

b) $\bar{z}_0 (u_0 x + v_0 y + w_0 z) - z_0 \cdot (\bar{u}_0 x + \bar{v}_0 y + \bar{w}_0 z) = 0$,

d.h. b) geht durch den Mittelpunkt M und den Schnittpunkt Y von p und q.

8. Die Φ-Kurve lässt sich auch auf folgende andere Arten erzeugen: Die Gleichung

$$f(x,y,z) + \lambda^2 \cdot (\bar{u}_0 x + \bar{v}_0 y + \bar{w}_0 z)^2 = 0 \quad . \quad . \quad . \quad . \quad (14)$$

stellt ein Büschel von C_2 dar, die f in den Punkten C und D berühren. Das Büschel der Polaren des Punktes Q in Bezug auf dieses C_2-Büschel ist dann gegeben durch

$$\left.\begin{aligned} &f_1(x_0) \cdot x + f_2(y_0) \cdot y + f_3(z_0)\, z + \\ &+ 2\lambda^2 \cdot (\bar{u}_0 x + \bar{v}_0 y + \bar{w}_0 z)(\bar{u}_0 x_0 + \bar{v}_0 y_0 + \bar{w}_0 z_0) = 0 \quad . \quad . \quad . \end{aligned}\right\} \quad (14a)$$

Durch Elimination von λ^2 ergiebt sich wieder Φ; ebenso aus dem Büschel

$$f(x,y,z) + \mu^2 \cdot (u_0 x + v_0 y + w_0 y)^2 = 0 \quad . \quad . \quad . \quad . \quad (15)$$

und dem zugehörigen Polarenbüschel für P in Bezug auf dieses Büschel.

Wenn man endlich die beiden Büschel in den Gleichungen (14) und (15) in Beziehung setzt durch die Relation:

$$2\lambda\mu(\bar{u}_0 x_0 + \bar{v}_0 y_0 + \bar{w}_0 z_0) = 1, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

so erhält man durch Elimination von λ, μ wiederum $\Phi(x, y, z) = 0$, neben einer zweiten, ebenso gebauten Gleichung.

9. Eine wichtige metrische Beziehung für die Punkte der Φ-Kurve ergiebt sich durch folgende Überlegung:

Es seien x, y, z die rechtwinkligen Koordinaten eines Punktes A, dann ist die Polare desselben in Bezug auf $f = 0$:

$$f_1(x) \cdot X + f_2(y) \cdot Y + f_3(z) \cdot Z = 0,$$

wenn X, Y, Z die laufenden Koordinaten sind. Somit ist der Abstand d_1 des Punktes A von seiner Polaren:

$$d_1 = \frac{f_1(x) \cdot x + f_2(y) \cdot y + f_3(z) \cdot z}{\sqrt{f_1^2(x) + f_2^2(y)}} \qquad \frac{2f(x,y,z)}{\sqrt{f_1^2(x) + f_2^2(y)}}.$$

Der Abstand des Punktes A von p ist

$$d_2 = \frac{\overline{u}_0 x + \overline{v}_0 y + \overline{w}_0 z}{\sqrt{\overline{u}_0^2 + \overline{v}_0^2}}.$$

Der Abstand des Punktes Q von der obigen Polaren des Punktes A ist

$$n_1 = \frac{f_1(x) \cdot x_0 + f_2(y) \cdot y_0 + f_3(z) \cdot z_0}{\sqrt{f_1^2(x) + f_2^2(y)}}$$

und der Abstand des Punktes Q von p ist

$$n_2 = \frac{\overline{u}_0 x_0 + \overline{v}_0 y_0 + \overline{w}_0 z_0}{\sqrt{\overline{u}_0^2 + \overline{v}_0^2}}.$$

Setzt man nun $\frac{d_1}{d_2} = \frac{n_1}{n_2}$ und bringt nach Weghebung gemeinsamer Faktoren alles auf eine Seite, so erhält man wieder die Gleichung $\Phi(x, y, z) = 0$.

Somit ist $\Phi = 0$ der geometrische Ort des Punktes x, y, z, für den das Verhältnis der Abstände von seiner Polaren und von einer gegebenen Geraden p gleich ist dem Verhältnis der Abstände eines gegebenen Punktes Q von denselben zwei Geraden, absolut genommen.

10. Das dualistische Gegenbild der Φ-Kurve erhält man auf folgende Art: Die Strahlen des Büschels Q schneiden auf p eine Punktreihe aus, ebenso die des projektiven Büschels der konjugierten Polaren durch P auf q. Die Verbindungsgeraden der entsprechenden Punkte dieser zwei projektiven Punktreihen erzeugen einen Kegelschnitt Ψ; die gemeinsamen Tangenten von f und Ψ sind die Tangenten von f in C, D, U, V; Ψ hat mit f und Φ dasselbe Polardreieck gemeinsam.

11. *Büschel von Grundkurven und zugehörigen Polarkurven.*

Die 4 Punkte C, D, U, V bestimmen ein Büschel von C_2: $kg - f = 0$. Nimmt man für die Φ-Kurve eines jeden C_2 jeweils die in ein Geradenpaar zerfallenden C_2 des Büschels als p- und q-Gerade an, so gehören zu jedem C_2 des Büschels 3 Φ-Kurven, und umgekehrt.

Da diese ebenfalls durch C, D, U, V gehen, so muss $\Phi(x, y, z)$ von der Form $\mu g - f$ sein und es muss sich μ aus k und dem zu den zerfallenden Kurven gehörigen Parameter λ der Gleichung

$$C(\lambda) \equiv B\lambda^3 - 3\theta\lambda^2 + 3H\lambda - A = 0 \quad . \quad . \quad . \quad . \quad (17)$$

bestimmen lassen. Die Beziehung zwischen k, λ, μ erhält man auf folgende Weise. Es ist

$$\Phi(kg - f) = 2(kg - f) \cdot (\bar{u}_0 x_0 + \bar{v}_0 y_0 + \bar{w}_0 z_0) -$$
$$- (u_0 x + v_0 y + w_0 x) \cdot (\bar{u}_0 x + \bar{v}_0 y + \bar{w}_0 z) = 0$$

oder $= 2(kg - f) \cdot (\bar{u}_0 x_0 + \bar{v}_0 y_0 + \bar{w}_0 z_0) -$
$$- \{(kg_1 - f_1) \cdot x_0 + (kg_2 - f_2) \cdot y_0 + (kg_3 - f_3) \cdot z_0\} \cdot$$
$$\cdot (\bar{u}_0 x + \bar{v}_0 y + \bar{w}_0 z) = 0.$$

Zur Berechnung von $\bar{u}_0 x_0 + \bar{v}_0 y_0 + \bar{w}_0 z_0$ haben wir:

$$x_0 = k^2 \cdot G_1(u_0) - k \cdot H_1(u_0) - F_1(u_0)$$
$$= 2k^2 \cdot (B_{11} u_0 + B_{12} v_0 + B_{13} w_0) - 2k \cdot (H_{11} u_0 + H_{12} v_0 + H_{13} w_0)$$
$$+ 2(A_{11} u_0 + A_{12} v_0 + A_{13} w_0)$$

und zwei entsprechende Gleichungen für y_0 und z_0.

Zur Elimination von u_0, v_0, w_0 und $\bar{u}_0, \bar{v}_0, \bar{w}_0$ vergleicht man das Produkt $(u_0 x + v_0 y + w_0 z) \cdot (\bar{u}_0 x + \bar{v}_0 y + \bar{w}_0 z)$ mit $\lambda g - f$.

Dadurch findet man:

$$\left.\begin{aligned}(u_0 x + v_0 y + w_0 z) \cdot (\bar{u}_0 x + \bar{v}_0 y + \bar{w}_0 z) - \\ = 4 \cdot (Bk^3 - 3\theta k^2 + 3Hk - A) \cdot (\lambda g - f)\end{aligned}\right\} \quad (18)$$

und

$$\bar{u}_0 x + \bar{v}_0 y_0 + \bar{w}_0 z_0 = 6H\lambda - 2A - 12\theta k\lambda + 12Hk + 2Bk^2\lambda - 6\theta k^2 \quad (19)$$

und endlich daraus dann:

$$\mu = \frac{A(-k+\lambda) - 3\theta k^2 \cdot (k+\lambda) + 6Hk^2}{Bk^2(-k+\lambda) + 3H(k+\lambda) - 6\theta k\lambda} \quad . \quad . \quad . \quad . \quad (20)$$

oder

$$\lambda = k \cdot \frac{(Bk^2\mu - A) - 3(\theta k^2 + H\mu) + 6Hk}{(Bk^2\mu - A) + 3(\theta k^2 + H\mu) - 6\theta k\mu}, \quad . \quad . \quad . \quad (20a)$$

neben $C(\lambda) = 0$.

Setzt man hierin $k = \lambda$, so erhält man $\mu = \lambda$. D. h. nur dann ist $\Phi(\Phi) \equiv f$, wenn f und folglich auch Φ eine zerfallende C_3 des Büschels sind. Geometrisch erhellt dies, wenn man beachtet, dass die Pole P und Q sich auf den Seiten des dem Büschel gemein-

samen Polardreiecks XYZ bewegen. Die Schnittpunkte der Tangenten in U und V z. B. an f und Φ bestimmen auf der Seite XZ des Polardreiecks zwei coincidente projektive Punktreihen, deren Doppelpunkte eben die Schnittpunkte der zerfallenden Kurven des Büschels sind. Setzt man den für λ in ($20a$) gegebenen Wert in die Gleichung $C(\lambda) = 3$ ein, so hat man eine Relation zwischen k und λ.

12. *Netz und Büschel von Polarkurven bei fester Grundkurve f.*

Hält man f und p fest, so bildet die Gesamtheit der Φ-Kurven ein Netz mit den Stützpunkten P, C, D. Jede solche C_1 ist aber nur Polarkurve für *einen* Punkt auf ihr, nämlich den Pol für die gemeinsame zweite Sehne von f und Φ. Macht man die Tangenten von f in C und D bezw. zur X- und Y-Achse und die Berührungssehne CD zur Z-Achse, so wird

$$f(x, y, z) \equiv xy + z^2 = 0$$

und

$$\Phi(x, y, z) = 2 z_0 xy - y_0 xz - x_0 yz = 0.$$

Die Gleichung einer C_1 durch P, C, D hat dann die Form $\alpha xy + \beta xz + \gamma yz = 0$; daraus folgt für den Pol $x_0 : y_0 : z_0 = \alpha : -2\beta : -2\gamma$. Beschreibt nun der Pol eine Gerade

$$Q(x_0, y_0, z_0) \equiv u x_0 + v y_0 + w z_0 = 0,$$

wo x_0, y_0, z_0 die laufenden Koordinaten sind, so kann man das Büschel der zugehörigen Φ-Kurven in der Form schreiben:

$$x_0 (u x z - v y z) + z_0 (2 v x y + w x z) = 0.$$

Für den vierten Grundpunkt dieses Büschels hat man also:

$$u x - v y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

und

$$2 v y + w z = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21a)$$

woraus folgt

$$u x + v y + w z = 0.$$

(21) ist die lineare Polare des Schnittpunktes von p mit der Geraden $Q(x, y, z) = 0$. Somit liegt der vierte Schnittpunkt auf der Geraden $Q(x, y, z) = 0$ und eben dieser Polaren.

13. Beschreibt der Pol Q einen Kegelschnitt:

$$Q(x_0, y_0 z_0) \equiv c_{11} x^2 + 2 c_{12} xy + c_{22} y^2 + 2 c_{13} xz + 2 c_{23} yz + c_{33} z^2 = 0, \quad (22)$$

während f, P und p festbleiben, so erhält man für die Enveloppe des Büschels der Φ-Kurven die Gleichung:

$$\left.\begin{aligned} E(\xi,\eta,\zeta) \equiv 4\,C_{33}\,\xi^2\eta^2 - 4\,C_{33}\,\xi^2\eta\zeta - 4\,C_{13}\,\xi\eta^2\zeta + C_{33}\,\xi^2\zeta^2 + \\ + 2\,C_{12}\,\xi\eta\zeta^2 + C_{11}\,\eta^2\zeta^2 = 0 \end{aligned}\right\} \quad (23)$$

wo die C_{ik} die Unterdeterminanten zu den c_{ik} sind. Die E ist also eine rationale Kurve vierter Ordnung mit den 3 Doppelpunkten [1]) in den Punkten P, C, D.

Die Q und E berühren sich in den 4 Punkten, die gegeben sind durch die Gleichungen:

$$\left.\begin{aligned} c_{11}\,x + c_{12}\,y + c_{13}\,z &= \varrho\,.\,yz \\ c_{12}\,x + c_{22}\,y + c_{23}\,z &= \varrho\,.\,xz \\ c_{13}\,x + c_{23}\,y + c_{33}\,z &= -\varrho\,.\,2xy \end{aligned}\right\} \quad \ldots\ldots \quad (24)$$

Durch $\Phi \equiv 2\,\xi\eta z_0 - \xi\zeta y_0 - \eta\zeta x_0 = 0$ ist jedem Punkte x_0, y_0, z_0 auf Q ein Punkt ξ, η, ζ auf E zugeordnet und umgekehrt. Der Übergang von x_0, y_0, z_0 auf Q geschieht, indem man, wie oben angegeben, zu E übergeht und den Berührungspunkt von Φ und E bestimmt. Der Übergang von einem Punkte ξ, η, ζ auf E geschieht, indem man diesen als Pol betrachtet und durch das entsprechende Verfahren die Enveloppe der zugehörigen Φ-Kurven bestimmt, wenn ξ, η, ζ auf E wandert. Diese ist dann eben wieder die Q-Kurve und die doppeltgezählten Seiten des Dreiecks PCD (abgesehen von dem auftretenden Faktor Δ der Determinante der c_{ik}).

Zu einer anderen Darstellung dieser Berührungstransformation, die deren Bedeutung erst kennzeichnet, gelangt man durch folgende Überlegung: Es sei $E(\xi, \eta, \zeta) = 0$ gegeben, dann ist die Gleichung der Tangente in einem Punkte ξ, η, ζ:

$$E_1(\xi)\,.\,x + E_2(\eta)\,.\,y + E_3(\zeta)\,.\,z = 0.$$

Soll nun eine C_2 transformiert werden, die E in diesem Punkte berührt und durch die Punkte C, D, P geht, so ist diese C_2 von der Form:

$$\mathfrak{A}_{12}\,xy + \mathfrak{A}_{13}\,xz + \mathfrak{A}_{23}\,yz = 0,$$

---

[1]) Die Schnittpunkte der Tangenten in dem Doppelpunkt $x = y = 0$ liegen auf der C_2:

$$L(x, y, z) = (c_{13}\,x + c_{23}\,y + c_{33}\,z)^2 + 4\,c_{23}\,x\,y = 0.$$

Diese berührt die Tangenten von f in C und D in deren Schnittpunkten mit der gewöhnlichen Polaren des Punktes $x = 0,\ y = 0$ in Bezug auf Q.

daher muss sein:

$$\left.\begin{aligned} \mathfrak{A}_{11} &= \varrho \,.\, \frac{E_3 \,.\, \zeta^2 - E_1 \,.\, \xi\zeta - E_2\, \eta\zeta}{2\,\xi\,\eta\,\zeta} \\ {}_{12} &= \varrho \,.\, \frac{E_2 \,.\, \eta^2 - E_3\, \eta\zeta - E_1 \,.\, \xi\,\eta}{2\,\xi\,\eta\,\zeta} \\ \mathfrak{A}_{13} &= \varrho \,.\, \frac{E_1 \,.\, \zeta^2 - E_3\, \xi\,\eta - E_2 \,.\, \xi\,\zeta}{2\,\xi\,\eta\,\zeta} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (25)$$

Indem man für E_1, E_2, E_3 die Werte einsetzt, erhält man als Abbildungskurve:

$$\left.\begin{aligned} &2\,.\,(-\,2\,C_{33}\,\xi\,\eta + C_{23}\,\xi\,\zeta + C_{13}\,\eta\,\zeta)\,.\,xy \\ &-(-\,2\,C_{33}\,\xi\,\eta + C_{23}\,\xi\,\zeta + C_{13}\,\eta\,\zeta)\,.\,xz \\ &-(-\,2\,C_{13}\,\xi\,\eta + C_{12}\,\xi\,\zeta + C_{11}\,\eta\,\zeta)\,.\,yz = 0, \end{aligned}\right\} \quad . \quad . \quad . \quad (26)$$

oder abgekürzt:

$$2\,U_3\,xy - U_2\,xz - U_1\,yz = 0.$$

Durch Vergleich mit der früheren Form der Φ-Kurve ergiebt sich die Beziehung:

$$x_0 : y_0 : z_0 = U_1 : U_2 ; U_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

woraus durch Auflösung nach ξ, η, ζ folgt:

$$\xi : \eta : \zeta = V_3\, V_2 : V_2\, V_1 : V_1\, V_3, \quad . \quad . \quad . \quad . \quad (27a)$$

wobei

$$V_1 = 2\,Q_1\,(x_0), \quad V_2 = 2\,Q_2\,(y_0), \quad V_3 = Q_3\,(z_0).$$

Somit vermittelt unsere Φ-Kurve eine birationale quadratische Transformation. Daraus erhellt jetzt auch, dass E rational sein muss. Die Gleichung $\Phi\,(x_0, y_0, z_0, \xi, \eta, \zeta) = 0$ und die Gleichungen (27) und (27a) zwischen den x_0, y_0, z_0 und den ξ, η, ζ sind also äquivalent.

14. Es soll hier noch gezeigt werden, dass diese Abbildung eine spezielle Berührungstransformation repräsentiert, ohne auf das zuletzt Auseindergesetzte Bezug zu nehmen. Es seien zwei E-Kurven gegeben

$$\left.\begin{aligned} E^{(1)} \equiv 4\,C_{33}\,x^2\,y^2 - 4\,C_{23}\,x^2\,yz - 4\,C_{13}\,x\,y^2\,z + C_{22}\,x^2\,z^2 + \\ + 2\,C_{12}\,x\,y\,z^2 + C_{11}\,y^2\,z^2 = 0, \end{aligned}\right\} \quad (28)$$

$$E^{(2)} \equiv 4\,D_{33}\,x^2\,y^2 - 4\,D_{23}\,x^2\,y\,z - \ldots + D_{11}\,y^2\,z^2 = 0. \quad . \quad (28a)$$

Soll dann einem Punkte ξ, η, ζ, der beiden E-Kurven gemeinsam ist, derselbe Punkt x_0, y_0, z_0 entsprechen, so muss sein:

$$\left.\begin{aligned} -2C_{33}\xi\eta + C_{33}\xi\zeta + C_{13}\eta\zeta &= \varrho\ (-2D_{33}\xi\eta + D_{23}\xi\zeta + D_{13}\eta\zeta)\\ -2C_{23}\xi\eta + C_{22}\xi\zeta + C_{12}\eta\zeta &= \varrho\,.(-2D_{23}\xi\eta + D_{22}\xi\zeta + D_{12}\eta\zeta)\\ -2C_{13}\xi\eta + C_{12}\xi\zeta + C_{11}\eta\zeta &= \varrho\,.(-D_{13}\xi\eta + D_{12}\xi\zeta + D_{11}\eta\zeta) \end{aligned}\right\} \quad (29)$$

Nun ist:

$$\left.\begin{aligned} 2E^{(1)}(\xi,\eta,\zeta) - E_1^{(1)}(\xi)\,.\,\xi &= -2\xi\zeta\,.(-2C_{13}\xi\eta + C_{12}\xi\zeta + C_{11}\eta\zeta)\\ 2E^{(2)}(\xi,\eta,\zeta) - E_1^{(2)}(\xi)\,.\,\xi &= -2\xi\zeta\,.(-2D_{13}\xi\eta + D_{12}\xi\zeta + D_{11}\eta\zeta) \end{aligned}\right\} \quad (30)$$

und entsprechende Gleichungen bestehen für die Ableitungen nach η und ζ.

Wegen $E^{(1)}(\xi,\eta,\zeta) = E^{(2)}(\xi,\eta,\zeta) = 0$, ergiebt sich also: $E_1^{(1)} = E_1^{(2)}$; $E_2^{(1)} = E_2^{(2)}$; $E_3^{(1)} = E_3^{(2)}$. Dies sind aber gerade die Bedingungen für die Berührung von $E^{(1)}$ und $E^{(2)}$ im Punkte ξ, η, ζ. Ebenso ergiebt sich, wenn man statt der ξ, η, ζ die x_0, y_0, z_0 einführt, dass die den $E^{(1)}$ un $E^{(2)}$ entsprechenden Kurven $Q^{(1)}$ und $Q^{(2)}$ sich in dem Punkte x_0, y_0, z_0 berühren.

Jede C_3, die f in C und D berührt, geht in sich über, indem die entsprechende E-Kurve in diese und die zwei Tangenten an f in den Punkten C und D zerfällt. Wenn die D-Kurve Q im Punkte x_0, y_0, z_0 berührt, so geht auch die E-Kurve durch diesen Punkt und berührt die Q-Kurve daselbst.

Mathematics. — *„Ueber den natürlichen Dimensionsbegriff."* [1] By Prof. L. E. J. BROUWER.

(Communicated at the meeting of November 24, 1923).

Auf Grund der Invarianz der Dimensionenzahl [2]) lässt sich die Dimensionenzahl einer Mannigfaltigkeit [3]) definieren als die Anzahl der Parameter, durch welche sich die Mannigfaltigkeit in der Umgebung eines beliebigen ihrer Punkte eineindeutig und stetig darstellen lässt. Diese „arithmetische" Definition trägt aber nach POINCARÉ [4]) unserer intuitiven Raumanschauung ungenügend Rechnung. POINCARÉ erhebt deshalb die Forderung einer rekurrenten Definition von etwa folgender Form [5]):

„Ein Kontinuum heisse n-dimensional, wenn man es durch ein oder mehrere (n—1)-dimensionale Kontinua in getrennte Stücke zerlegen kann."

Obgleich der n-dimensionale JORDANsche Satz [6]) auf die Möglichkeit einer derartigen Definition deutet, so lässt sich diese in der zitierten Form dennoch nicht aufrecht erhalten.

Zunächst bemerken wir, dass das Wort *„Kontinuum"* hier sicher nicht etwa im Sinne von *„Mannigfaltigkeit"* aufgefasst werden darf; in diesem Falle würde nämlich die Definition erst brauchbar werden, nachdem eine von der Parameterdarstellung unabhängige Charakterisierung der Mannigfaltigkeiten unter den abstrakten Mengen gelungen sein würde. Weil dies aber bis jetzt nicht der Fall ist, so müsste der POINCARÉschen Definition irgendeine allgemeinere abstrakte Charakterisierung des Kontinuums vorausgeschickt werden, z. B. diese: *„Eine Normalmenge* (im FRÉCHETschen Sinne) π *heisse ein Kontinuum, wenn es für je zwei ihrer Elemente* m_1 *und* m_2 *eine zusammenhän-*

[1]) Die vorliegende Mitteilung bildet bis auf den Inhalt von Fussnote [19]) und die in Fussnote [11]) angegebene Berichtigung einen Wiederabdruck meiner in 1913 im Journal für die reine und angewandte Mathematik (Bd. 142, S. 146—152) unter demselben Titel erschienenen Abhandlung.

[2]) Vgl. meinen Beweis in Math. Annalen 70, S. 161—165 und die daran anknüpfenden Entwicklungen von LEBESGUE in C. R. de l'Acad. des sciences, Paris, 27 mars 1911.

[3]) Für die Definition des Begriffes „Mannigfaltigkeit" vgl. Math. Annalen 71, S. 97.

[4]) Revue de métaphysique et de morale, 1912, S. 486, 487.

[5]) a. a. O., S. 488.

[6]) Vgl. den teilweise von LEBESGUE, teilweise von mir erbrachten Beweis in C. R. de l'Acad. des sciences. Paris, 27 mars 1911, und Math. Annalen 71, S. 305—319.

gende, abgeschlossene[7]) *Menge gibt, welche Teilmenge von* π *ist und* m_1 *und* m_2 *enthält.*[8]). Für solche allgemeinere Kontinua, welche keine Mannigfaltigkeiten sind, würde aber unsere Definition zu Schwierigkeiten führen; z. B. würde man einem Kegel des Cartesischen Raumes, der sich ja durch einen Punkt zerlegen lässt, nur *eine* Dimension zusprechen dürfen.

Auch die Worte *„ein oder mehrere"* könnten nicht unverändert beibehalten werden, weil mehrere m-dimensionale Mannigfaltigkeiten zusammen eine $(m + p)$-dimensionale Mannigfaltigkeit bilden können.

Alle diese Mängel lassen sich nun beseitigen, indem wir zunächst die Poincarésche rekurrente Definition wie folgt abändern:

Es sei π irgendeine Normalmenge[9]), π_1, ϱ und ϱ' drei Teilmengen von π, welche innerhalb π abgeschlossen[10]) sind und keine gemeinsamen Punkte besitzen. Alsdann heissen ϱ und ϱ' *in* π *durch* π_1 *getrennt,* wenn π_1 in π eine ϱ enthaltende, aber ϱ' nicht enthaltende Gebietsmenge g bestimmt.[11]) Der Ausdruck: *„*π *besitzt den allgemeinen Dimensionsgrad* n*"*, in welchem n eine beliebige natürliche Zahl bezeichnet, soll nun besagen, dass für jede Wahl von ϱ und ϱ' eine trennende Menge π_1 existiert, welche den allgemeinen Dimensionsgrad $n-1$ besitzt, dass aber nicht für jede Wahl von ϱ und ϱ' eine trennende Menge π_1 existiert, welche einen geringeren allgemeinen Dimensionsgrad als $n-1$ besitzt. Weiter soll der Ausdruck: *„*π *besitzt den allgemeinen Dimensionsgrad Null bzw. einen unendlichen*

[7]) Unter einer *abgeschlossenen* Menge verstehen wir hier eine ihre Grenzelemente enthaltende Menge, in welcher jede unendliche Folge von Elementen mindestens ein Grenzelement aufweist.

[8]) Diese Definition ist der von Schoenflies für die Kontinua des n-dimensionalen Raumes gegebenen nachgebildet (vgl. Bericht über die Lehre von den Punktmannigfaltigkeiten, Bd. II, S. 117).

[9]) Inwieweit die Definition des Textes auch für Mengen allgemeinerer Art einen naturgemässen Sinn behält, soll hier unerörtert bleiben.

[10]) Dieser Ausdruck besagt, dass π_1, ϱ und ϱ' alle ihre in π gelegenen Grenzpunkte enthalten.

[11]) Diesen der Gebietsmenge g auferlegten Bedingungen können natürlich mehrere Gebietsmengen von π genügen. Im in [1]) zitierten Original hat sich an dieser Stelle eine andere, mit dem übrigen Inhalte des Aufsatzes in keinem Zusammenhang stehende Trennungsdefinition eingeschlichen. Dass die obige (übliche) Definition die in der vorliegenden Abhandlung in Wirklichkeit gebrauchte ist, geht aus dem Zusammenhang hervor, insbesondere aus Fussnote[16]) und dem zugehörigen Passus des Textes. Die daselbst eingeführte, von π_2 in τ_1 bestimmte, an die Kante $E_1 E_2$ grenzende Gebietsmenge kann nämlich keinen anderen Sinn haben, als den des Durchschnittes einer schon vorhandenen von π_2 in π_1 bestimmten, an $E_1 E_2$ grenzenden, an $E_1 E_3 \ldots E_{n+1}$ jedoch nicht grenzenden Gebietsmenge mit τ_1. Auf die Berichtigung, welche hier anzubringen war, bin ich von Herrn P. Urysohn in Moskau aufmerksam gemacht worden.

allgemeinen Dimensionsgrad" bedeuten, dass π kein Kontinuum als Teil enthält, bzw. dass zu π weder die Null noch irgendeine natürliche Zahl als ihr allgemeiner Dimensionsgrad gefunden werden kann.[12])

Dieser Definition lässt sich leicht eine von der Rekurrenz unabhängige Form geben. Dazu denken wir uns die Menge π von zwei Personen A und B der „*Dimensionsoperation*" unterzogen, worunter wir folgendes verstehen: A wählt in π zwei innerhalb π abgeschlossene Teilmengen ϱ und ϱ' beliebig aus, worauf B ϱ und ϱ' in π trennt durch eine innerhalb π abgeschlossene Menge π_1. Sodann wählt A in π_1 zwei innerhalb π_1 abgeschlossene Teilmengen ϱ_1 und ϱ'_1 beliebig aus, worauf B ϱ_1 und ϱ'_1 in π_1 trennt durch eine innerhalb π_1 abgeschlossene Menge π_2. Dieser Prozess wird unbeschränkt wiederholt, bis eventuell eine Menge π_h auftritt, welche kein Kontinuum mehr als Teil enthält. Wenn einerseits B unabhängig von den Wahlen der ϱ_ν und ϱ'_ν dafür sorgen kann, dass eine Menge π_h auftritt, deren $h \leqq n$, und andererseits A unabhängig von den Wahlen der π_ν dafür sorgen kann, dass *keine* Menge π_h auftritt, deren $h < n$, so werden wir sagen, dass π *den allgemeinen Dimensionsgrad* n *besitzt*. Wen dagegen keine natürliche Zahl n existiert mit der Eigenschaft, dass B unabhängig von den Wahlen der ϱ_ν und ϱ'_ν dafür sorgen kann, dass eine Menge π_h auftritt, deren $h \leqq n$, so werden wir sagen, dass π *einen unendlichen allgemeinen Dimensionsgrad besitzt*.

Wenn zu einem Punkte P von π Umgebungen, welche den allgemeinen Dimensionsgrad m, aber keine Umgebungen, welche einen geringeren allgemeinen Dimensionsgrad besitzen, existieren, so werden wir sagen, dass π *in* P *den Dimensionsgrad* m *besitzt*. In verschiedenen Punkten kann eine Menge verschiedene Dimensionsgrade besitzen; keiner von diesen kann indes den allgemeinen Dimensionsgrad der Menge übersteigen. Falls in jedem Punkte der Menge der Dimensionsgrad dem allgemeinen Dimensionsgrade der Menge gleich ist, so werden wir sagen, dass die Menge *einen homogenen Dimensionsgrad besitzt*.

Auf Grund der vorstehenden Definitionen soll nun die Poincaré-sche Forderung vollständig erfüllt werden durch die Begründung von folgendem

Dimensionssatz. *Eine n-dimensionale Mannigfaltigkeit besitzt den homogenen Dimensionsgrad* n.[13])

Zum Beweise dieses Satzes zeigen wir zunächst, dass B bei der

[12]) Nach dieser Definition wird sowohl für den Hilbertschen wie für den Fréchetschen E_n ein unendlicher allgemeiner Dimensionsgrad gefunden.

[13]) Weil der Dimensionsgrad offenbar eine Invariante der Analysis Situs ist, so ist im Dimensionssatz die Invarianz der Dimensionenzahl enthalten.

Dimensionsoperation dafür sorgen kann, dass $h \leqq n$. Dazu konstruiert B, nachdem A die Mengen ϱ und ϱ' bestimmt hat, eine simpliziale Zerlegung[14]) ζ von π, und zwar in solcher Weise, dass, wenn wir unter einem ${}_{\pi}s_{\varrho}$ bzw. ${}_{\pi}s_{\varrho'}$ ein entweder in seinem Inneren oder auf seiner Grenze Punkte von ϱ bzw. ϱ' enthaltendes Grundsimplex von ζ verstehen, kein ${}_{\pi}s_{\varrho}$ mit einem ${}_{\pi}s_{\varrho'}$ identisch ist und kein ${}_{\pi}s_{\varrho}$ an ein ${}_{\pi}s_{\varrho'}$ grenzt. Alsdann bilden diejenigen $(n-1)$-dimensionalen Seiten der ${}_{\pi}s_{\varrho}$, welche weder in ihrem Inneren noch auf ihrer Grenze Punkte von ϱ enthalten, ein System von zweiseitigen $(n-1)$-dimensionalen Pseudomannigfaltigkeiten[15]), in welchem übrigens mehrere Elemente oder Elementseiten zusammenfallen können. Die von diesen Pseudomannigfaltigkeiten gebildete Punktmenge wählt B als π_1. Falls darauf A die Mengen ϱ_1 und ϱ'_1 in demselben Teilkontinuum von π_1 wählt, so konstruiert B eine solche simpliziale Zerlegung von π_1, dass kein ${}_{\pi_1}s_{\varrho_1}$ mit einem ${}_{\pi_1}s_{\varrho'_1}$ identisch ist und kein ${}_{\pi_1}s_{\varrho_1}$ an ein ${}_{\pi_1}s_{\varrho'_1}$ grenzt. Alsdann bilden diejenigen $(n-2)$-dimensionalen Seiten der ${}_{\pi_1}s_{\varrho_1}$, welche weder in ihrem Inneren noch auf ihrer Grenze Punkte von ϱ_1 enthalten, ein System von zweiseitigen $(n-2)$-dimensionalen Pseudomannigfaltigkeiten, in welchem übrigens wieder mehrere Elemente oder Elementseiten zusammenfallen können. Die von diesen Pseudomannigfaltigkeiten gebildete Punktmenge wählt B als π_2. In dieser Weise fortfahrend, gelangt B schliesslich zu einer Menge π_m, welche kein Kontinuum mehr als Teil enthält, es sei denn, dass der Prozess schon früher dadurch beendet wurde, dass A einmal ϱ_ν und ϱ'_ν nicht in demselben Teilkontinuum von π_ν wählte.

Wir zeigen zweitens, dass A bei der Dimensionsoperation dafür sorgen kann, dass h nicht kleiner als n ausfällt. Dazu wählt A in π von einem n-dimensionalen Elemente $E_1 E_2 \ldots E_{n+1}$ den Punkt E_1 als ϱ und die $(n-1)$-dimensionale Seite $E_2 \ldots E_{n+1}$ als ϱ'; den zur Elementseite $E_1 E_2$ bzw. $E_1 E_3 \ldots E_{n+1}$ gehörigen Teil von π_1 als ϱ_1 bzw. ϱ'_1; den zur Elementseite $E_1 E_2 E_3$ bzw. $E_1 E_3 E_4 \ldots E_{n+1}$ gehörigen Teil von π_2 als ϱ_2 bzw. ϱ'_2; usw. Um zu beweisen, dass von den Punktmengen $\pi_1, \pi_2, \ldots \pi_n$ keine in Fortfall kommen kann, bezeichnen wir mit τ das Ausgangselement $E_1 E_2 \ldots E_{n+1}$, mit τ_1 die Grenze des von π_1 in τ bestimmten, an den Punkt E_1 grenzenden Gebiets g, mit τ_2 die Grenze der von π_2 in τ_1 bestimmten, an die Kante $E_1 E_2$ grenzenden Gebietsmenge[16]) g_1, mit τ_3 die Grenze der

[14]) Math. Annalen 71, S. 101.

[15]) a. a. O., S. 305.

[16]) Unter einer in τ_ν gelegenen Gebietsmenge verstehen wir eine in τ_ν gelegene Punktmenge, von der kein Punkt Grenzpunkt der durch sie in τ_ν bestimmten Komplementärmenge ist.

von π_2 in τ_2 bestimmten, an die zweidimensionale Seite $E_1 E_2 E_3$ grenzenden Gebietsmenge g_2, usw., konstruieren in τ eine simpliziale Zerlegung von der Dichte ε[17]), bezeichnen mit γ das n-dimensionale Fragment[18]), welches von den mitsamt ihrer Grenze zu g gehörigen Grundsimplexen gebildet wird, mit σ_1 den *innerhalb* τ gelegenen Teil der gleichfalls simplizial zerlegt vorliegenden Grenze von γ, mit ε_1 das Maximum der Abstände, welche die Punkte von σ_1 von τ_1 besitzen, mit γ_1 das $(n-1)$-dimensionale Fragment, welches von denjenigen Grundsimplexen von σ_1, die von g_1 einen Abstand $\leqq \varepsilon_1$ besitzen, gebildet wird, mit σ_2 den *innerhalb* σ_1 gelegenen Teil der Grenze von γ_1, mit ε_2 das Maximum der Abstände, welche die Punkte von σ_2 von τ_2 besitzen, und fahren so fort. Alsdann konvergieren $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$ mit ε gegen Null, so dass die eventuelle Existenz von $\sigma_1, \sigma_2, \ldots \sigma_n$ diejenige von $\tau_1, \tau_2, \ldots \tau_n$, mithin auch diejenige von $\pi_1, \pi_2, \ldots \pi_n$, in denen ja der Reihe nach $\tau_1, \tau_2, \ldots \tau_n$ als Teilmengen enthalten sind, nach sich ziehen wird.

Hiermit ist der Dimensionssatz zurückgeführt auf folgenden

Hilfssatz. Es sei σ ein simplizial zerlegtes n-dimensionales Element mit den Eckpunkten $E_1, E_2, \ldots E_{n+1}$; γ ein aus Grundsimplexen von σ gebildetes Fragment, das alle an E_1, aber kein an $E_2 E_3 \ldots E_{n+1}$ grenzendes Grundsimplex von σ enthält; σ_1 der innerhalb σ liegende Teil der Grenze von γ; γ_1 ein aus Grundsmplexen von σ_1 gebildetes Fragment, das alle an $E_1 E_2$, aber kein an $E_1 E_3 \ldots E_{n+1}$ grenzendes Grundsimplex von σ_1 enthält; σ_2 der innerhalb σ_1 liegende Teil der Grenze von γ_1; γ_2 ein aus Grundsimplexen von σ_2 gebildetes Fragment, das alle an $E_1 E_2 E_3$, aber kein an $E_1 E_2 E_4 \ldots E_{n+1}$ grenzendes Grundsimplex von σ_2 enthält; σ_3 der innerhalb σ_2 liegende Teil der Grenze von γ_2; usw. Alsdann kann von den Punktmengen $\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_n$ keine verschwinden.

Dieser Hilfssatz, auf den schon Lebesgue in Math. Annalen 70 die Invarianz der Dimensionenzahl zurückgeführt hat, dessen Beweis daselbst aber eine wesentliche Lücke aufweist[19]), leuchtet unmittel-

[17]) Math. Annalen 71, S. 101.

[18]) a. a. O., S. 306.

[19]) Die „faits bien évidents", welche dieser Beweis (auf S. 167) voraussetzt, sind nämlich unrichtig, und bilden, wenn sie in eine richtige Form gebracht werden, eine Eigenschaft, welche tiefer liegt, als der Hilfssatz selbst. Nachdem Herr Lebesgue (in 1911) auf dieses Versehen hingewiesen worden war, teilte er mir seine Absicht mit, binnen kurzem im Bull. de la Soc. Math. de France einen neuen Beweis des Hilfssatzes zu bringen, von dem er mir gleichzeitig die Hauptzüge auseinandersetzte. Obgleich diese Auseinandersetzungen mich nicht befriedigten, meinte ich dennoch im in [1]) zitierten Original auf die von Herrn Lebesgue zugesagte Veröffentlichung hinweisen zu müssen. Dieselbe ist indes ausgeblieben und erst in Funda-

bar ein, wenn wir den von mir in Math. Annalen 71 [19]) eingeführten Begriff des *Abbildungsgrades* heranziehen.

Die Eigenschaft, dass die Projektion von σ_ν aus der Elementseite $E_1 E_2 \ldots E_\nu$ die Elementseite $E_{\nu+1} E_{\nu+2} \ldots E_{n+1}$ mit dem Grade 1 bedeckt, lässt sich nämlich von ν auf $\nu+1$ ausdehnen, indem wir zunächst aus ihr folgern, dass die Projektion des in der Elementseite $E_1 \ldots E_\nu E_{\nu+2} \ldots E_{n+1}$ liegenden Teiles der Grenze von σ_ν aus der Elementseite $E_1 \ldots E_\nu$ oder aus der Elementseite $E_1 \ldots E_{\nu+1}$ die Elementseite $E_{\nu+2} \ldots E_{n+1}$ mit dem Grade 1 bedeckt, und sodann σ_ν, indem wir jedesmal ein einziges seiner Grundsimplexe tilgen, stückweise auf γ_ν reduzieren, wobei der in der Elementseite $E_1 \ldots E_\nu E_{\nu+2} \ldots E_{n+1}$ liegende Teil der Grenze von σ_ν schrittweise in $\sigma_{\nu+1}$ übergeht und der entsprechende Projektionsgrad auf $E_{\nu+2} \ldots E_{n+1}$ sich nicht ändern kann. Weil mithin jedes σ_ν ($\nu = 1, 2, 3, \ldots n$) sich mit dem Grade 1 auf eine $(n-\nu)$-dimensionale Seite von σ projiziert, so kann keines der σ_ν sich auf Null reduzieren. W. z. b. w.

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menta Mathematicae, Bd. 2 (1921), S. 256—285 ist Herr Lebesgue auf den Gegenstand zurückgekommen und hat er einen stichhaltigen Beweis des Hilfssatzes gegeben, der, was den Kern betrifft, mit meinem obigen Beweise von 1913 übereinstimmt, davon aber durch eine unnötig verwickelte Darstellung der Einzelheiten abweicht.

[20]) Vgl. daselbst S. 105.

Mathematics. — "*Ueber Invarianten von Bilinearformen*". Von Prof. R. WEITZENBÖCK. (Mitgeteilt von Prof. L. E. J. BROUWER).

(Communicated at the meeting of November 24, 1923).

In der Theorie der endlichen diskreten Gruppen linearer Substitutionen besteht der Satz[1]: Notwendig und hinreichend für die Aequivalenz zweier Gruppen ist die Gleichheit ihrer Charaktersysteme. Von diesem Satze wird hier ein neuer Beweis gegeben, der die Theorie der affinen Invarianten derjenigen Bilinearformen benutzt, die den einzelnen Substitutionen einer Gruppe Γ zugeordnet sind. Im Besonderen wollen wir zeigen, dass die einzigen Invarianten dieser Bilinearformen die Charaktere der Substitutionen von Γ sind.

§ 1. *Bezeichnungen.*

Es sei $\Gamma = E, A, B, \ldots$ eine endliche Gruppe der Ordnung μ von n-ären linear-homogenen Substitutionen

$$(A) \quad \bar{x}_i = a_i^1 x_1 + a_i^2 x_2 + \ldots\ldots + a_i^n x_n \quad (i = 1, 2, \ldots, n). \quad (1)$$

E sei die Einheitssubstitution mit $e_i^i = 1$, $e_i^k = 0$ $(i \neq k)$; $a = |a_i^k|$ sei die Determinante der Matrix $\|a_i^k\|$ von A. $a, b, \ldots$ sind μ-te Einheitswurzeln.

Statt (1) schreiben wir auch kürzer

$$(A) \quad \bar{x}_i = a_i^\lambda x_\lambda, \quad \ldots\ldots\ldots \quad (2)$$

oder auch, symbolisch, für $a_i^k = a_i a'_k$ setzend:

$$\bar{x}_i = a_i (a' x). \quad \ldots\ldots\ldots \quad (3)$$

Der Substitution A ist zugeordnet die n-äre Bilinearform

$$L_A = a_i^k x_k u^i = (a' x)(a u'), \qquad L_E = x_i u^i = (u' x). \quad \ldots \quad (4)$$

Deren einfachste affine Invariante

$$\chi(A) = \sum_i a_i^i = (a' a) = a_1^1 + a_2^2 + \ldots\ldots + a_n^n. \quad \ldots \quad (5)$$

[1]) Vgl. z. B. H. F. BLICHFELD, Finite Collineation Groups, Chicago (1917), p. 129 oder: A. SPEISER, Theorie der Gruppen von endlicher Ordnung, Berlin (1923), p. 116.

heisst der Charakter von A. $\chi(E) = n$, $\chi(A)$, $\chi(B), \ldots$ bilden das Charaktersystem der Gruppe Γ.

Die zu A *inverse* Substitution A^{-1} erhält man durch Auflösung von (2) nach den x_i. Der durch a dividierte Minor von a_i^k in a werde mit A_k^i (Vertauschung der Indexstellung!) bezeichnet. Dann ist

$$(A^{-1}) \quad x_i = A_i^\lambda \bar{x}_\lambda \qquad [A_i^\lambda a_\lambda^\varkappa = e_i^\varkappa \ , \ A_\lambda^i a_\varkappa^\lambda = e_\varkappa^i]. \quad . \quad . \quad (6)$$

Die zu A *transponierte* Substitution A' ist dargestellt durch

$$(A') \quad u^i = a_\lambda^i \bar{u}^\lambda; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

und deren inverse $A_t = (A')^{-1}$ wird gegeben durch:

$$(A_t) \quad \bar{u}^i = A_\lambda^i u^\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

A_t heisst die zu A *kontragrediente* (oder adjungierte) Substitution. Nach (6) ist dann:

$$A_t = (A^{-1})' = (A')^{-1} \quad , \quad (A_t)_t = A \quad . \quad . \quad . \quad . \quad (9)$$

Die Veränderlichen x_i and u^i sind kontragredient zueinander. Die mit Γ homomorphe Gruppe $\Gamma_t = E, A_t B_t, \ldots$ heisst die zu Γ kontragrediente (oder adjungierte) Substitutionsgruppe Γ_t. Es lässt sich leicht zeigen[1]), dass der Charakter $\chi(A_t)$ die zu $\chi(A)$ konjugiert-komplexe Zahl ist.

Analog zu (8) und (4) ist

$$L_{A_t} = A_k^i x_i \, u^k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

die zu L_A kontragrediente Bilinearform; symbolisch wird sie, wenn $a_1, a_2, \ldots$ und $a'_1, a'_2 \ldots$ äequivalente Symbolreihen darstellen, gegeben durch

$$L_{A_t} = \frac{1}{a} \cdot \frac{1}{(n-1)!} \cdot (a_1 a_2 \ldots a_{n-1} x)(a'_1 a'_2 \ldots a'_{n-1} u') \quad . \quad (11)$$

Die Determinante a ist symbolisch gegeben durch:

$$a = \frac{1}{n!} (a_1 a_2 \ldots a_n)(a'_1 a'_2 \ldots a'_n) \quad . \quad . \quad . \quad . \quad (12)$$

Dem Produkte $AB = C$ zweier Substitutionen A und B ist zugeordnet die Bilinearform

$$L_{AB} = a_i^\lambda b_\lambda^k x_k u^i = (u' a)(a' b)(b' x) = c_i^k x_k u^i, \quad . \quad . \quad (13)$$

während der Substitution BA zugeordnet ist:

$$L_{BA} = b_i^\lambda a_\lambda^k x_k u^i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

[1]) Speiser, l.c. p. 110.

Wegen der Gruppennatur führt jede Zusammensetzung der Gestalt

$$a_i^{\lambda}\, b_{\lambda}^{\mu}\, c_{\mu}^{\nu} \ldots g_{\rho}^{k} = h_i^{k} \quad \ldots\ldots\ldots \quad (15)$$

wieder auf eine Substitution H zurück.

§ 2. *Das volle Komitantensystem.*

Wir konstruieren jetzt ein volles System von affinen Komitanten der μ Bilinearformen (4) mit der Einschränkung, dass wir neben den Koëffizienten dieser Bilinearformen nur noch eine Reihe x und eine Reihe u zulassen.

Zur Verfügung stehen dann die Reihen

$$a_1, a_2, \ldots, x \quad \text{und} \quad a'_1, a'_2, \ldots, u'. \quad \ldots\ldots \quad (16)$$

Dabei soll $(a_i)_r\,(a'_i)_s$ gleich dem Koeffizienten a_r^s irgend einer der Formen (4) sein. Aus (16) bilden wir: 1. Faktoren *zweiter* Art der Gestalt:

$$\varphi_1 = (a_1 a_2 \ldots a_n),\ \varphi_2 = (a_1 a_2 \ldots a_{n-1} x);\ \psi_1 = (a'_1 a'_2 \ldots a'_n),\ \psi_2 = (a'_1 a'_2 \ldots a'_{n-1} u'); \quad (17)$$

2. Faktoren *erster* Art der Gestalt:

$$f_1 = (a_i a'_k),\ f_2 = (a\, u'),\ f_3 = (a'\, x),\ f_4 = (u'\, x). \quad \ldots \quad (18)$$

Jede affine Komitante I ist ein Produkt dieser Faktoren. Wir können annehmen, dass in I nicht φ und ψ gleichzeitig auftreten, da das Produkt eines φ und eines ψ durch Faktoren f ausdrückbar ist wegen

$$(a_1 a_2 \ldots a_n)(a'_1 a'_2 \ldots a'_n) = \begin{vmatrix} (a_1 a'_1) & \ldots & (a_1 a'_n) \\ \ldots & \ldots & \ldots \\ (a_n a'_1) & \ldots & (a_n a'_n) \end{vmatrix} \quad (19)$$

Es enthalte nun I einen Faktor φ: $I = (a_1 a_2 a_3 \ldots)\, I'$. In I' suchen wir a'_1 auf, das in einem f stecken muss: $I = (a_1 a_2 a_3 \ldots)(a'_1 a_r)\, I''$. In I'' suchen wir a'_r auf, das wieder in einem f steckt:

$$I = (a_1 a_2 a_3 \ldots)(a'_1 a_r)(a'_r a_s) \ldots$$

Dies geht so fort, bis die Kette $(a'_1 a_r)(a'_r a_s)(a'_s a_t) \ldots$ mit einem Gliede $(a' x)$ abbricht. Mit $a_2, a_3, \ldots$ machen wir es analog und erhalten für I im Falle der Anwesenheit eines Faktors φ_1 oder φ_2 die Gestalt:

$$I = (a_1 a_2 a_3 \ldots)\underbrace{(a'_1 a_r) \ldots (a'_\rho x)}_{K_1} \cdot \underbrace{(a'_2 a_s) \ldots (a'_\sigma x)}_{K_2} \ldots\ldots \quad (20)$$

Die hier mit $K_1, K_2, \ldots$ angedeuteten Ketten können dabei beliebig lang sein.

Eine ganz analoge Gestalt bekommt I bei Anwesenheit von ψ_1 oder ψ_2, nur dass dann die entsprechenden Ketten mit u' endigen. Invarianten (ohne x oder u') erhält man sonach hier nicht.

Es wäre nun nicht schwer bei allgemeinen Bilinearformen die Bildungen (20) auf gewisse einfache Gestalten zu reduzieren. Man kann z. B. bei den Ketten K die Gliederzahl stets $\leqq n-1$ voraussetzen. Doch haben wir dies hier nicht nötig, da unsere Substitutionen $A, B, \ldots$ eine endliche diskrete Gruppe bilden, wodurch sich die Sache sehr vereinfacht. Jede Kette führt nämlich nach (15) wieder auf ein einziges $h_i h'_i$ zurück und diese Reihen müssen untereinander verschieden sein, wenn $J \equiv\!|\!\equiv 0$ ist. Wir erhalten somit im Falle $\mu \geqq n$ je $\binom{\mu}{n}$ Komitanten der zwei Typen:

$$I = (a_{i_1}\, a_{i_2} \ldots . a_{i_n})\,(a'_{i_1} x)\,(a'_{i_2} x) \ldots . . (a'_{i_n} x) \quad . \quad . \quad (21)$$

$$(i_r \neq i_s)$$

$$I' = (a'_{i_1}\, a'_{i_2} \ldots . a'_{i_n})\,(a_{i_1} u')\,(a_{i_2} u') \ldots . . (a_{i_n} u') \quad . \quad . \quad (22)$$

Hier sind auch die Komitenten mit φ_2 und ψ_2 mitaufgezählt, denn es ist z. B. bei φ_2 eines der $a_i a'_k$ gleich $e_i^k = e_i e'_k$.

Wir kommen zu Faktoren erster Art. f_4 ist bereits eine Komitante, nämlich die Bilinearform L_E. Bei den übrigen Faktoren f bilden wir Ketten, von denen zweierlei Typen möglich sind:

$$T_1 \ldots . (x a'_i)\,(a_i a'_k) \ldots . (a_r a'_s)\,(a_s u')$$

$$T_2 \ldots . (a a'_i)\,(a_i a'_k) \ldots . (a_r a'_s)\,(a_s a').$$

Auch diese Ketten reduzieren sich wegen (15) auf einfachste Formen: T_1 auf die Bilinearformen L_A, $L_B, \ldots$ selbst, T_2 auf die Charaktere $\chi(A) = (a a')$. Diese Charaktere sind somit die einzigen affinen Invarianten der Bilinearformen L. Gleichheit der Charaktere bei entsprechenden Substitutionen homomorpher Gruppen Γ und Γ' bedeutet also Gleichheit der affinen Invarianten der entsprechenden Bilinearformen L und L'. Der Homomorphismus garantiert zwischen den Koeffizienten der L dieselben affin-invarianten Gleichungen wie zwischen den Koeffizienten der L'. Die L sind also bezgl. affiner Transformationen den L' äquivalent.

Physics. — "*The Influence of Rotation on the Sensitiveness and the Accuracy of a Pressure Balance.*" (Twelfth communication of results obtained in researches made by the aid of the VAN DER WAALS fund). By A. MICHELS. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of October 27, 1923).

For the accurate measurement of great pressures methods are now of general application, based on the use of the so-called Amagat cylinder. In all these methods the force is studied exerted by a liquid under pressure on a piston of known diameter. The elaboration of this fundamental idea has given rise to different types of pressure balances, as those of WAGNER, STÜCKRADT, SCHÄFFER und BUDENBERG, and HOLBORN [1]).

In order to reach an accuracy as great as possible it is necessary to reduce the frictional forces between the piston and the wall of the hole to a minimum. In this respect WIEBE already obtained good results by tapping the wall of his apparatus with a hammer. Of late a rotation of the piston has pretty generally been applied, though HOLBORN [2]) considers a movement to and fro preferable.

The causes why these operations have such an influence, are only imperfectly known as yet. KLEIN (loc. cit.) tries, indeed, to give a solution of the effect of rotation, but does not succeed.

The purpose of this investigation is to find a solution, and at the same time to determine the circumstances under which the greatest effect is reached.

As there is no room here for an extensive discussion of our results, we shall restrict ourselves in what follows to a brief communication, referring for a fuller treatment to "Annalen der Physik" Bd. 72, 1923, p. 285—320.

It was tried to work theoretically in the direction indicated by the recent theory of bearings lubricated all round [3]). For when the piston revolves in a cylindrical hole, liquid being continually supplied from below, there must certainly be an analogy between the influences of friction to which our piston is subjected and those exerted on an ordinary axle resting in a bearing block.

[1]) I refer for the different types to KLEIN. G. Untersuchung und Kritik von Hochdruckmesser Diss. Berlin 1909.

[2]) Ann. d. Physik 1915, p. 1087.

[3]) SOMMERFELD. Zeitschr. für Math. und Physik 1904, GÜMBEL. Das problem der Lagerreibung Jahresb. d. Schiffbautechn. Gesellsch. 1917.

Undoubtedly there are also points of difference, which must be chiefly owing to this that in our case the so-called bearing-pressure is wanting on account of the vertical position of the piston. Application of the theory taught that if the peripherical speed is sufficient, a liquid layer will be formed everywhere between piston and hole-wall. The number of revolutions at which this takes place, will be called the critical value of the revolutions ω_r. It is dependent on the viscosity of the liquid chosen. In the absence of any metal contact also the axial friction would be a liquid friction above this value of revolutions.

In order to test the validity of this theory the pressure balance of the VAN DER WAALS fund which was at our disposal, was modified in such a way that it had a driving apparatus that could be regulated mechanically.

This alteration was made by the instrument-maker of the laboratory, Mr. J. WASSENAAR.

Characteristic of a liquid friction is its proportionality with the velocity. When a definite initial value of revolutions Ω is given to the piston, after which the motor is cut out, the motion will be retarded, and the angle $\varkappa$ passed over in the time t, will get a value of

$$\varkappa = \frac{\Omega}{A}\left(1 - e^{-At}\right)$$

in which A is a constant. As soon, as the value of revolutions descends below the critical value however, there is metal contact, and the image of the motion changes.

In this way the course is examined all over the measuring scope of the pressure balance, and agreement was found between experiment and theory. As was to be expected, the critical value of the revolutions then appeared to be dependent on the temperature, as this influences the viscosity, but independent of the load.

An electrical determination shows the validity of the suppositions still more clearly. For, when the electrical resistance between axle and wall was measured, it appeared to be about 700 Ohms above a definite number of revolutions, being reduced pretty suddenly to 0.2 Ohm on diminution of the velocity. In these values the resistance of the conducting wires is included.

Conclusion. For a favourable use of the pressure balance experiments should always be made above the critical value of revolutions. This value can be determined experimentally for every liquid and temperature.

Anatomy. — *"The Forebrain of Apteryx Australis"*. By JOHN I. HUNTER, M. B. Ch. M. (Sydney). (From the Central Institute of Brain Research, Amsterdam). (Communicated by Prof. L. BOLK).

(Communicated at the meeting of December 29, 1923).

I. *General Features.*

An examination of the external form of the brain of the New Zealand kiwi (*Apteryx australis*) reveals the presence of distinct differences from the usual condition exhibited by the avian brain. The general shape of the cerebral hemispheres is peculiar in that the frontal extremities are somewhat more pointed than usual, and the lateral surface proceeds backwards by a gentle convexity to the posterior extremity.

The characteristic subdivision of the cerebral hemisphere of birds into a *pars medialis* and *pars lateralis*, of which the *pars lateralis*, may enwrap the *pars medialis* to form the frontal pole, is not visible in this specimen (fig. I). For the *pars medialis* (sagittal-wulst of EDINGER, WALLENBERG and HOLMES, 1903) is not indicated, though there is an ill-defined bulging on the postero-medial part of the dorsal surface of this hemisphere. In consequence of this the *vallecula*,

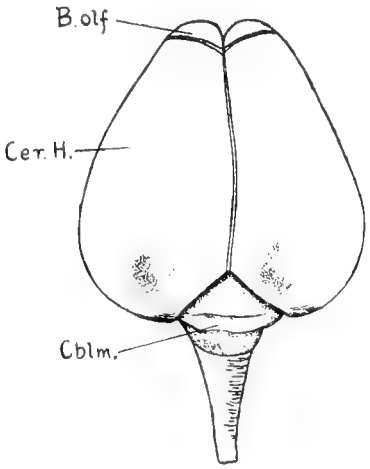

Fig. I.

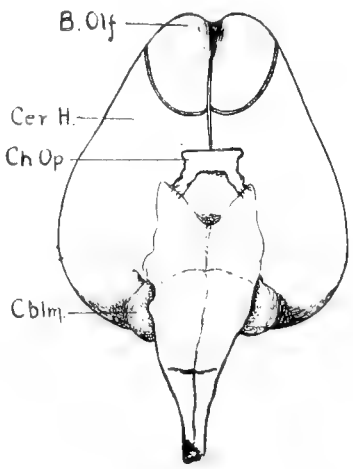

Fig. II.

which usually limits the *pars medialis* laterally, is not conspicuous. (cf. fig. I with fig. 535—537 ARIËNS KAPPERS, 1291. Vide also OWEN, 1872, p. 382).

Another important feature of the brain is the presence of two large olfactory lobes (fig. II). These project for a short distance beyond the anterior extremity of the fore-brain (fig. I and fig. II). Extending posteriorly they receive a very wide attachment to the ventral aspect of the frontal region of their respective hemispheres (fig. II). In marked contrast with this unique degree of development amongst *Aves* of the olfactory lobes, the visual apparatus is very poorly developed compared with a typical avian brain, as is indicated by the smallness of the optic nerves, chiasma, tracts and lobes (fig. II). This enhanced importance of the smell centres and associated reduction in the importance of the visual connections, combined with the presence of an apparently simpler hemisphere than is usually the case in *Aves*, suggest the conclusion that the brain of the kiwi is a comparatively simple and primitive type of avian brain, (cf. OWEN 1841, p. 287). For these reasons Dr. C. U. ARIËNS KAPPERS kindly suggested that I should undertake the investigation of this brain. It is a pleasure to express to him my great indebtedness on this account, and because of the assistance he afforded me in carrying out the comparative investigation necessary to elucidate the somewhat unusual features of the specimen.

II. *Technique.*

The material consists of a transverse series of sections of a single brain. Alternate sections were stained by the VAN GIESON method; the series remaining was treated by the Weigert-Pal-para-carmine method described by ARIËNS KAPPERS and KETJEN (1911). Unfortunately the specimen was in alcohol when received by Professor ELLIOT SMITH from the Zoological Gardens, London, who kindly transmitted it to the Central Institute of Brain Research, Amsterdam, after transference to formalin. The brain was evidently in a bad state of preservation before being hardened. In consequence, the condition of the sections is not good and a final analysis of the cell masses and their fibre connexions is not possible. However many features are so clearly defined that a description of them may be entered upon with confidence. To control the topographical description of the various parts a wax plate-ceresine reconstruction twelve and a half times the size of the original, was prepared. (cf. ARIËNS KAPPERS 1915).

III. *Description of the sections.*[1])

a. Connections of the olfactory nerves.

As already mentioned the olfactory bulbs and lobes are conspicuous structures bilaterally represented. The most frontal sections show a bulbar formation which is arranged in a circular manner (Fig. III) though no extension of the ventricle (rhinocoele) is visible in this region[2]). The fila olfactoria, glomeruli and mitral cells are of the usual structure (cf. EDINGER, WALLENBERG, HOLMES, 1903, p. 403) and call for no special description. The two separate formations right and left, are distinctly seen throughout (Fig. III and IV). TURNER (1891, p. 43) and S. P. GAGE (1893, p. 197) refer to the degree of

Fig. III

Fig. IV

diminution in importance of the olfactory connexions in *Aves*, culminating in the concrescence of two small lobes in some higher forms, as an index of the stage of organisation attained by the brain. The *lobus olfactorius* is spread out upon the ventral aspect of the anterior part of the cerebral hemisphere and is crescentic in cross section (Fig. IV). The second relay olfactory fibres form a distinct tract in the most dorsal lamina of this structure immediately ventral to a small forward prolongation of the lateral ventricle which becomes visible in this region (Fig. IV). These fibres end in the *cortex lobi olfactorii* or *area praepiriformis* of BRODMANN (cf. ROSE, 1914, p. 338). The position of this area immediately dorsal and medial to this ventricular extension can be located in the sections (Fig. IV), though its cell structure is not clearly distinguishable (cf. ROSE, op. cit. p. 339, Taf. III Fig. 8, 9, 61, Taf. I, Fig. 13).

Dorsal and caudal to the *area praepiriformis* the frontal portion of the *septum* is an extremely thin double lamina. (It is shown somewhat crumpled in the diagrams; cf. Fig. V). Somewhat more caudally frontal to and in the region of the anterior commissure

[1]) The sections corresponding to the figures are as follows: III, 21; IV, 58; V, 179; VI, 204; VII, 208; VIII, 212; IX, 235; X, 245; XI, 262; XII, 283 XIII, 293; XIV, 303.

[2]) For the lettering used in the figures see pages 822 and 823.

the *nucleus lateralis septi* is a conspicuous structure. The *nucleus medialis septi* is also visible. The *zona gliosa limitans* separates these nuclei from the Area 28 of Rose (1914) which is well defined. The medial limit of this zone is indicated by the *fissura septo-pallialis*. On the ventricular side of the septum the lateral limit of the zone is marked by the *fissura limitans hippocampi*. Dorsal to the Area 28 slight indentations laterally and medially serve to mark off this area from the cortex.

As the secondary olfactory fibres disappear posteriorly they are replaced by a great fibre field which extends completely across the ventral portion of the hemisphere. Fig. IV shows that the major portion of the hemisphere in this region consists of the corpus striatum (*caput hyperstriati*). The fibres first become distinct on the

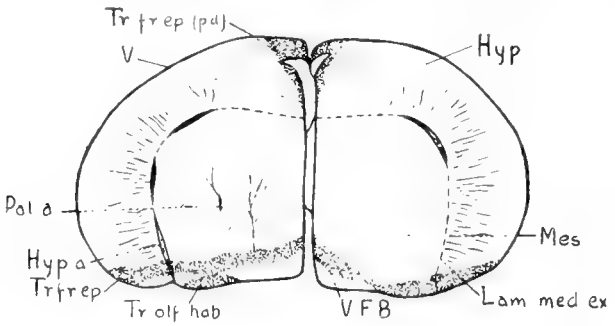

Fig. V

lateral surface of this structure but are soon seen to be arising from the whole ventral region of the hemisphere extending to the medial wall. When this extent has been attained (Fig. V), it becomes obvious, as the connexions of the fibres also show, that there are three main elements in this fibre field.

The most medial bundle arises from the region of the *area praepiriformis* and septum. It forms a conspicuous tract which separates from the remaining fibres in order to enter the diencephalon medial to the *tractus septo-mesencephalicus* (Fig. VI). This tract is the ventral forebrain bundle (*basales Riechbündel*). In the diencephalon it takes up its position lateral to the third ventricle (Fig. VIII) and extends backwards as far as the *nucleus oculomotorius*, (Fig. X). as was shown by Jelgersma (1896).

Lateral to the ventral forebrain bundle and ventral to the main field of fibres a second conspicuous myelinated tract is to be seen. It lies ventral to the base of the mesostriatum and so occupies a superficial position (Fig. V). When the ventral forebrain bundle enters the diencephalon it comes to lie more medially (Fig. VI),

and later bridges across the floor of the fissure separating the telencephalon from the diencephalon (Fig. XI). In this situation it forms a conspicuous oval bundle which is visible in the sections to the

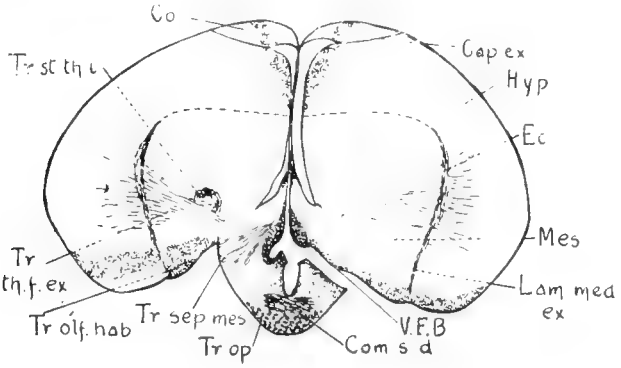

Fig VI

naked eye. This tract is the principal constituent of the *taenia thalami* representing the element called *olfacto-habenular* by Edinger and Wallenberg (1899), p. 251). A similar tract is figured by Edinger, Wallenberg and Holmes (Taf. V) and Schroeder (Fig. 47) but in these cases it is of considerably smaller dimensions than that attained in the kiwi. The condition of preservation of the specimen prevents the identification of a *nucleus taeniae*. When traced medially the tract passes to the lateral aspect of the ganglion habenulae and gradually ends in it. Many fibres cross the median plane returning apparently to the forebrain on the other side forming a very conspicuous *commissura telencephali superior* (Fig. XII). In reviewing

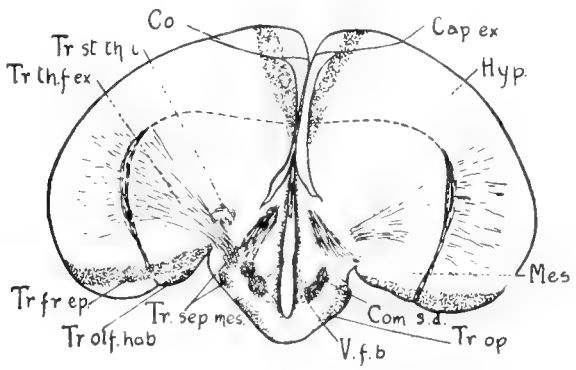

Fig. VII

the fibre tracts of the avian brain, Ariëns Kappers (1921, p. 1046) considers that the presence of this commissure in birds is questionable, though it is clearly present in all animals (cf. Varanus sal-

vator, as figured by DE LANGE, 1911, fig. 25, where the *commissura telencephali superior* is shown). Co-existing with this commissure in Varanus is a well-marked *commissura pallii posterior* or *commissura aberrans* of ELLIOT SMITH (DE LANGE, op. cit. fig. 21) which is absent in *Aves*. The relations of the tract forming the commissura in the kiwi are so precise that there can be no doubt that the commissure present here is not the *commissura pallii posterior*, but the *commissura telencephali superior* (cf. ARIËNS KAPPERS, 1921, p. 797 and footnote; p. 1034) I am unable to recognise the *commissura pallii* (cf. SCHROEDER fig. 42) in the sections under examination.

The remaining fibres of the ventrally situated fibre field constitute

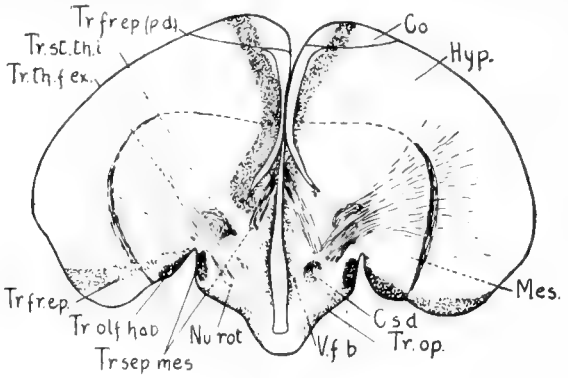

Fig. VIII

the fronto-occipital (fronto-epistriatic or lobo-epistriatic) tract. This arises over a wide area of the ventral aspect of the hemisphere as described by EDINGER, WALLENBERG and HOLMES (1203), (p. 381 and Fig. 5*b*) In the kiwi this tract can be traced to the posterior end of the corpus striatum where its fibres terminate. It is here seen to be augmented considerably by the addition of another great bundle of fibres, the connexions of which are also fronto-occipital. This tract first becomes distinct immediately dorsal to the slight lateral extension of the dorsal part of the lateral ventricle in the cortex region (Fig. V), but it soon appears on the ventricular aspect of the corpus striatum also and increases in size until it is a very extensive tract. It contributes a few fibres to the *commissura anterior* (Fig. IX) and then becomes merged with the fronto-occipital bundle already described to form a conspicuous tract which is oval in cross section (Fig. XIV). Similar fibres to these are described by EDINGER, WALLENBERG and HOLMES (op. cit. p. 383) who figure the fronto-occipital bundle divided into a dorsal and ventral part in the sparrow (Taf. II, Fig. 4). SCHROEDER, (1911, p. 145) in his excellent

account of the order of myelinisation of the fibre tracts in the chick, demonstrates the presence of a band of fibres on the ventricular aspect of the dorso-occipital part of the corpus striatum. Some of these fibres in the kiwi enter into the formation of the inter-epistriatic commissure. It is probable that these commissural fibres are comparable to the *fibrae marginales* found on the ventricular aspect of the striatum of Varanus and crossing to the opposite side in the *commissura anterior* (cf. DE LANGE, op. cit. Fg. 19, 20). The further description of the inter-epistriatic commissure will be deferred until the discussion of the subdivisions of the corpus striatum is undertaken.

b. The Corpus Striatum.

Notwithstanding the unusual external features of the brain of *Apteryx* to which reference has already been made the outstanding features of the sections are definitely avian. In 1891 Professor T. J. PARKER observed that his investigations of the development of the brain of the kiwi though very imperfect owing to lack of material "prove conclusively what might have been inferred from adult anatomy, that the brain of *Apteryx* is simply a typical avian encephalon with reduced optic lobes." (p. 107).

As is usual in *Aves*, the forebrain of the kiwi consists, for its greater part, of the corpus striatum. This body appears on each side as a great ventricular bulging. Frontally it forms the frontal pole of the hemisphere. Caudally its posterior extremity projects freely into the ventricle in close proximity to the hinder pole of the hemisphere. In the more frontal sections (Fig. V—VIII) the lateral ventricles form two vertical slits separated from one another by the two thin laminae constituting the septum and the corpora striata form the vertical lateral boundaries of the ventricles in this region.

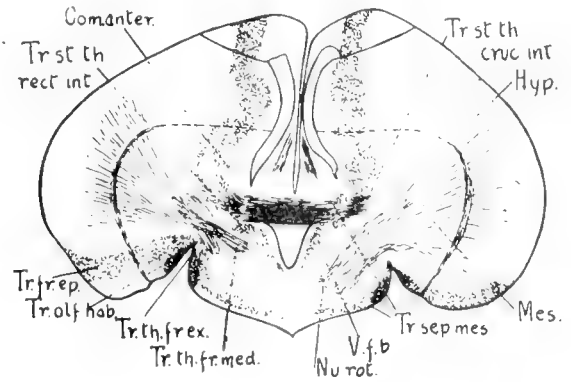

Fig. IX

Further caudally the ventricles increase in size being less reduced by the encroachment of the corpora striata laterally (Fig. IX—XIV).

The criterion of cell structure (cf. Rose, 1914) cannot be employed in analysing the constitution of the corpus striatum in this instance on account of the poor state of preservation of the specimen. For this purpose it becomes necessary to rely upon fibre connections and topographical relations in differentiating its various parts. In naming these the nomenclature of Edinger (1896) will be followed.

Fortunately the *nucleus entopeduncularе* can be recognised at the junction of the telencephalon and diencephalon. Surrounding this nucleus is a large-celled area forming the *nucleus basalis* constituting the *palaeostriatum primitivum* (Ariëns Kappers, 1908; 1922). This is surrounded by a larger part of the striatum which is an extension of the palaeostriatum — the mesostriatum or *palaeostriatum augmentatum*, (Ariëns Kappers, 1922, p. 140). The *lamina medullaris*

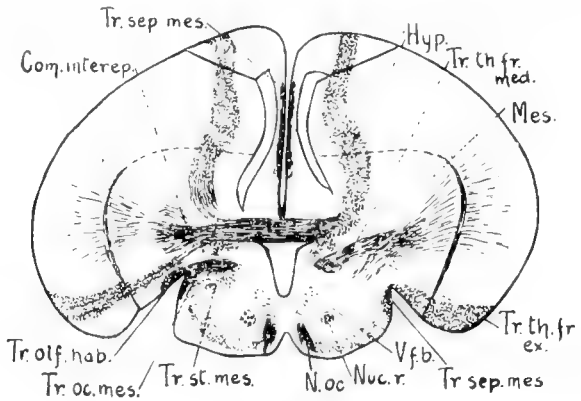

Fig. X

ventralis (*lamina medullaris interna* of Kappers, op. cit.) which separates these two parts from one another is not to be seen in the sections at hand.

If the sections showing these areas be examined (Fig. VI—VII) it is very evident that dorso-lateral to the mesostriatum there lies a mass of considerable size constituting the hyperstriatum. Its extent is as follows. Dorsally it is separated from the cortex by an ill defined lamina of fibres (*capsula externa* of Edinger, Wallenberg and Holmes, op. cit. p. 365; Taf. V). Traced laterally and ventrally it becomes continuous with the external or pallial surface. This mass is separated from the mesostriatum by a layer of fibres which constitute the *lamina medullaris dorsalis* (cf. Edinger, Wallenberg and Holmes op. cit. p. 390; Schroeder, op. cit. p. 141). This subdivision is exceptionally clear in *Apterix*. This is partly due to the

fact that this lamina is richly provided with blood-vessels, a point to which I shall return later. It extends from the ventricular surface

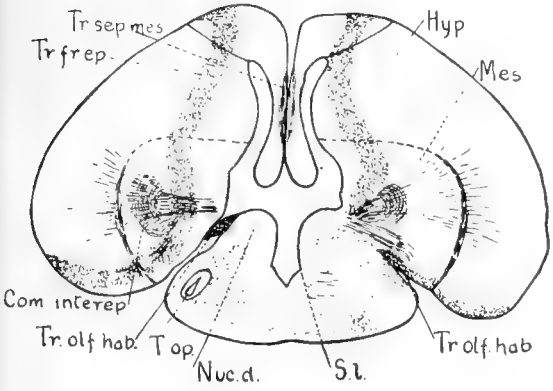

Fig XI

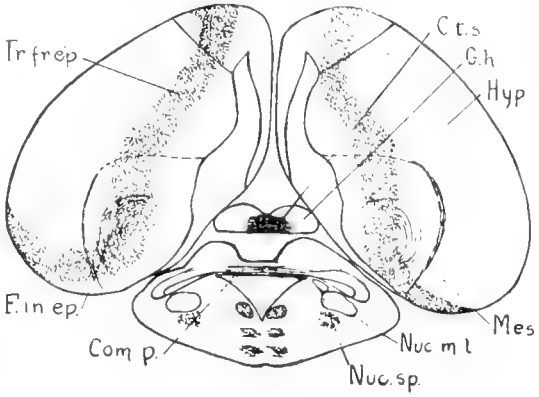

Fig XII

of the corpus striatum medially to an external groove marking the interval between the hyperstriatum and mesostriatum ventrally. Therefore this lamina separates the two parts not only dorsally but laterally. For this reason as Ariëns Kappers has suggested (1922) it is preferable to employ the term *lamina medullaris externa* in referring to this fibre zone. The fibres constituting it, which are connected on the one hand with the thalamus (vide infra), radiate laterally between columns of cells of the hyperstriatum. This confers a striated appearance upon this structure especially in its ventrolateral part.

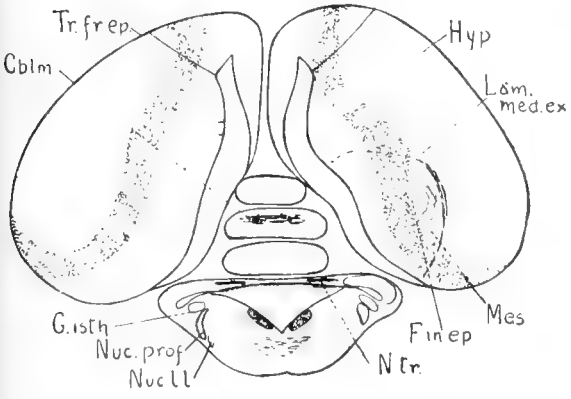

Fig XIII

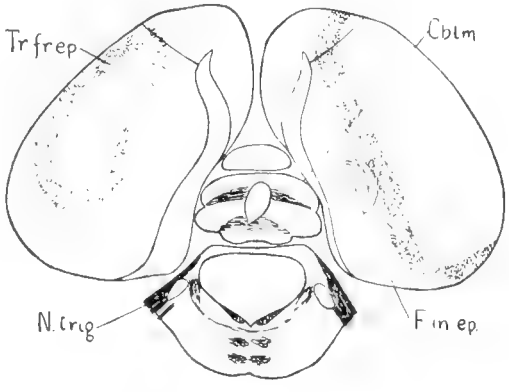

Fig XIV

The ectostriatum which lies between the mesostriatum and hyperstriatum and is recognisable by the naked-eye on account of its infiltration by fibres, and microscopically by the presence of large

cells, is not conspicuous in this specimen although the thickened fibre zone lateral to the external medullary lamina (Fig. VI) is comparable to the area figured by EDINGER, WALLENBERG and HOLMES (op. cit) in *Lothrix lateus* (Taf. III, fig. 5 and 6) and *Sylvia hortensis* (Taf. IV fig. 6).

In the sections under examination the hyperstriatum shows no clear sign of subdivision. In most birds the *lamina medullaris hyperstriati* divides the hyperstriatum into the *hyperstriatum superius* and *hyperstriatum inferius* of ARIËNS KAPPERS (1922). As this author rightly points out (op. cit. note 1, p. 23) PARKER's figures, in the work already mentioned on the development of the brain of the kiwi, show only two intraventricular primordia which probably represent the palaeostriatum and hyperstriatum inferius, the hyperstriatum superius being apparently absent, (PARKER 1891, Plate 19, Fig. 304). This point however needs re-investigation upon material in a better state of preservation than that at present available.

Frontally the hyperstriatum covers the frontal pole of the mesostriatum and forms the frontal extremity of the cerebral hemisphere (*caput hyperstriati*, Fig. IV).

The caudal part of the corpus striatum receives the fronto-epistriatic tracts (Fig. XIV). This region constitutes the secondary epistriatum or archistriatum (ARIËNS KAPPERS 1908). It is the area called epistriatum in the memoir of EDINGER, WALLENBERG and HOLMES. The *fissura strio-archistriatica* is not visible in the bird's brain. The archistriatum is connected to its fellow of the opposite side by a great strand of fibres (*commissura interepistriatica*, EDINGER). This bundle forms the main constituent of the *commissura anterior* (vide Fig. X) which is large and conspicuous in this brain (Fig. IX—X). Some of these fibres accompany the fronto-epistriatic tracts; the majority form a distinct fibre field in the ventral part of the mesostriatum (Fig. IX—XIII).

c. The significance of the blood vessels accompanying the lamina medullaris externa.

In his recent work on the morphology of the corpus striatum, ELLIOT SMITH (1919*b*) has emphasised his contention that the great ventricular eminence which forms such a conspicuous feature of the cerebral hemisphere of *Sphenodon* is pallial in origin. He has introduced the term hypopallium to designate this structure because "it is pallial in origin; it lies below the main portion of the pallium which forms the roof of the hemisphere; and morphologically and

functionally it is analogous to but upon a lower plane of usefulness than the neopallium", (op. cit. p. 272). He has established the truth of this statement for *Reptilia* in general by reference to examples of the *Ophidia*, *Lacertilia*, and *Chelonia*. Moreover, he points out, that the evidence now goes to show that every mammalian brain passes through a stage of development in which the corpus striatum is clearly divisible into hypopallium and palaeostriatum. Subsequent development shows that the hypopallium in man gives rise to the putamen and most of the caudate nucleus (together constituting the neostriatum of ARIËNS KAPPERS), the claustrum, and the hypopallial element of the *nucleus amygdalae*. The palaeostriatum forms the *globus pallidus* and according to ELLIOT SMITH a small part of the caudate nucleus (op. cit. p. 291; vide however, ARIËNS KAPPERS 1922, p. 153).

In *Sphenodon* the boundary line between the hypopallium and palaeostriatum is indicated by the course of large arterial vessels and emerging veins, the former constituting the lateral striate artery of reptiles, (ELLIOT SMITH op. cit. p. 272).

SHELLSHEAR (1920) has identified this artery in the adult human brain immediately lateral to the palaeostriatal area and has called it the claustral (or hypopallial) artery. It seems that, in conformity with the phylogenetic and ontogenetic history of the pallial origin of the hypopallium, pallial vessels have become hypertrophied at the site of intilting of the pallium to supply this new pallial development. Deeply penetrating into the hemisphere they form in man a lateral group of the antero-lateral basal vessels of the middle cerebral artery.

The vascular supply of the corpus striatum of *Apteryx* presents some remarkable features. In the first place a series of large vessels enter the base of the hemisphere in the *fissura ventralis* (*fissura limbica*, EDINGER) and penetrate deeply into the corpus striatum (Fig. V). The course of these arteries follows very closely that of the external medullary lamina; in other words they form a clear line of seperation between the hyperstriatum laterally and the mesostriatum medially. This arrangement is constant even in the most posterior region in which the external medullary lamina can be identified and the separation of the mesostriatum from the hyperstriatum distinguished. Medial to this fissure many smaller vessels penetrate the corpus striatum in the region of the palaeostriatum and the blood supply is considerably from this source, (cf. OWEN 1872, p. 381).

In contrast with this the blood supply entering the corpus striatum

on its lateral surface is very small. In this respect this great surface area which is formed by the hyperstriatum is sharply differentiated from the pallium which receives a relatively rich supply of vessels entering from the surface. The result of this arrangement is that the blood supply of the lateral part of the hyperstriatum is derived from a series of vessels penetrating deeply into the hemisphere along the line of the external medullary lamina and sending frequent branches laterally. Such a deep penetration of vessels demands an explanation which is to be sought on phylogenetic grounds. The explanation which suggests itself is that the hyperstriatal artery of this avian brain represents the lateral striate artery of reptiles and the hypopallial (claustral) artery of man. That, in effect it is a greatly hypertrophied vessel originally in series with the pallial arteries which are in the bird's brain mainly confined to the dorsal aspect, the lateral series having been greatly reduced in importance; and further that this hypertrophy has occurred because the hyperstriatum is pallial in origin. The hyperstriatum and the archistriatum together represent the hypopallium of reptiles and therefore also the hypopallial elements of the corpus striatum (hyperstriatum) and of the *nucleus amygdalae* (archistriatum) of the mammalian brain, (cf. Elliot Smith 1919a; Dart 1920).

The mesostriatum can be excluded from this complex on account of the difference in the origin of its blood supply alone. For the vessels situated more medially (palaeostriatal arteries) supply not only the basal nucleus or *palaeostriatum primitivum* (Ariëns Kappers 1922) but the extensive mesostriatum which surrounds it, (fig. 5). If the criterion of blood supply is to be applied (cf. Shellshear op. cit. p. 35) in this case, the mesostriatum must be regarded as an augmentative homology of the palaeostriatum so forming the *palaeostriatum augmentatum* of Ariëns Kappers (1922).

An examination of a series of sections of *Pratincola rubra* (figured by Ariëns Kappers in his text book, 1921) *Casuaris*, *Athene noctua*, *Paleornis*, *Ciconia alba*, reveals the fact that the same vascular arrangement as described in *Apteryx* holds for *Aves* in general. But in *Apteryx* the arrangement is displayed with the greatest clearness.

It follows from the above discussion that the *lamina medullaris externa* of birds is the line of separation of the neostriatum from the palaeostriatum and that the point where this lamina reaches the ventricle (e. g. fig. 6) represents the site of the *fissura neopalaeostriatica* which is clearly seen if embryonic stages of the chick's brain be studied, (cf. Ariëns Kappers, 1922, p. 140).

d. Ariëns Kappers' *Studies on the ontogeny of the corpus striatum of birds.*

In a recent paper Ariëns Kappers (1922) reported the results of his investigations upon the ontogenetic development of the different parts of the striatum complex in birds. He concludes that apart from the archistriatum "at least two chief divisions of the striatum may be distinguished: the palaeostriatum, which is enlarged to a *palaeostriatum augmentatum* (or meso-striatum) and which arises entirely from the base of the brain in front of the recessus praeopticus, and the hyperstriatum of which the upper part arises entirely from the mantle (*hyperstriatum superius*), while the underpart (*hyperstriatum inferius*), arises from the mantle (laterally) as well as from the base of the brain in front of the palaeostriatum. Both parts of the hyperstriatum thus show the fact, that intraventricular protrusions of striatal type may originate from the pallium as well as from the base of the brain, as I already pointed out for the primary epistriatum in bony fishes, and as was pointed out by Elliot Smith for the neostriatum of reptiles". (op. cit. p. 148).

The arrangement of the blood vessels in the adult bird's brain is in accord with these results based upon ontogenetic studies. Moreover the material employed demonstrates the fact that in the embryo the *lamina medullaris externa* "is a place of predilection for blood vessels", (op. cit. p. 146, cf. figs. 11, 12, 13).

e. Summary of the Fibre-Tracts of the Fore-Brain.

The following tracts have already been discussed.

1. Ventral forebrain bundle.
2. Olfacto-habenular tract.
3. Superior telencephalic commissure.
4. Pallial commissure.
5. Fronto-epistriatic tract.
6. Interepistriatic commissure.

Three bundles connect the forebrain with the mesencephalon.

1. *Tractus strio-mesencephalicus.* This tract, which connects the mesostriatum with the *nucleus spiriformis* (fig. XII), can be recognised in its course through brain stem (fig. X).

2. *Tractus occipito-mesencephalicus.* The occipito-mesencephalic tract takes origin in the archistriatum and ends in the *nucleus spiriformis* and neighbouring gray matter of the mesencephalon.

It enters the brain stem ventral to the anterior commissure arching

over the strio-thalamic and strio-mesencephalic bundles (Fig. X; cf. SCHROEDER fig. 42, 47).

3. *Tractus septo-mesencephalicus.* This tract forms a very conspicuous bundle in the kiwi. Arising from the cortex and septum (Fig. IX, X, XI), it passes forwards to turn laterally in front of the tractus thalamo-frontalis externus (Fig. VI). Trace deandally it occupies a superficial position in the brain stem (Fig. VIII). In this situation it may be traced as far as the *tectum opticum* (Fig. VIII, IX, X). The details of its connexions with the nucleus of the septo-mesencephalic tract, with the *tectum opticum*, the oculomotor nucleus and the caudal portion of the brain stem cannot be followed in the sections.

Tracts of considerable size connect the corpus striatum and diencephalon as follows.

1. *Tractus thalamo-frontalis externus.* This bundle originates from the *nucleus rotundus* (Fig. VIII, IX) of the diencephalon. It proceeds to the lateral part of the hyperstriatum forming a compact fibre tract in its passage through the mesostriatum (fig. VII—X).

The fibres help to constitute the *lamina medullaris externa* before entering the hyperstriatum. The striated appearance of the hyperstriatum is in great measure due to its infiltration by fibres of this tract. It is probable that a neurobiotactic principle is here exemplified. The presence of this afferent tract from the nucleus rotundus of the thalamus would tend to determine the origin of the lateral part of the hyperstriatum as an infolding of the pallium into which the tract originally poured the impulses carried by it.

Commissural fibres accompany the *tractus thalamo-frontalis externus* constituting the *commissura supra-optica dorsalis.* Though they are not heavily myelinated, the decussation of these fibres is clearly to be seen (Fig. VI). On each side the tract proceeds dorsally and caudally to merge with the external thalamo-frontal (Fig. VII, VIII).

2. *Tractus thalamo-frontalis medius.* This is a second afferent thalamo-striate tract situated medial to the external thalamo-frontal tract (Fig. IX). It arises from the *nucleus dorsalis* of the thalamus (Fig. XI) which lies dorsal to the *sulcus limitans* of His. Passing frontally and laterally the fibres of this tract mingle with those of the external thalamo-frontal tract and proceed to the frontal and occipital region of the hyperstriatum.

3. *Tractus strio-thalamicus internus.* The internal strio-thalamic tract is the main efferent tract from the corpus striatum to the brain stem. It can be recognised in the mesostriatum (Fig. VII) from which it passes medially (Fig. VIII) to take up a position medial to the afferent tracts to the corpus striatum. Some of the fibres join the anterior

commissure and cross to the other side as the *tractus strio-thalamicus cruciatus internus* (Fig. IX). Here they join the homolateral fibres of the opposite side (*tractus strio-thalamicus rectus internus*). The destination of these fibres in birds, as shown by EDINGER and WALLENBERG (1899) is the ventral thalamus and mid-brain.

f. Hypophysis and Epiphysis.

The hypophysis extending ventrally contains a funnel shaped prolongation of the median ventricle (fig. 8).

The epiphysis though damaged is clearly recognisable in a series of the sections.

IV. *General summary.*

The brain of the kiwi, for that of a bird, is remarkable for the great development of its olfactory lobes. In contrast with this the visual connexions are much reduced. Externally the usual subdivision of the avian cerebral hemisphere into *pars medialis* and *pars lateralis* cannot be seen.

A study of sections shows that the olfactory bulbs and lobes present a typical bulbar formation. The *area praepiriformis*, the nuclei of the septum, and the Area 28 of ROSE are recognisable. In accordance with the great development of the smell apparatus the ventral forebrain bundle, the fronto-epistriatic tract, and the olfacto-habenular tracts are well developed. Accompanying the olfacto-habenular tract is the *commissura telencephali superior* which is usually not seen in birds.

As is usual in birds the corpus striatum forms the major part of the cerebral hemisphere. The natural subdivison of the striatum is clearly revealed in the kiwi. The archistriatum (secondary epistriatum) can be recognised by the fact that it receives the fronto-epistriatic tract and is connected to its fellow of the opposite side by the inter-epistriatic commissure which is a conspicuous constituent of the *commissura anterior*. The palaeostriatum consists of the basal nucleus (*palaeostriatum primitivum*) and mesostriatum (*palaeostriatum augmentatum*). The mesostriatum in separated from the hyperstriatum by the external medullary lamina which extends from the ventricle medially to the ventral surface of the hemisphere, the line of separation here being indicated by the *fissura ventralis*. Vessels (hyperstriatal artery) enter this groove and accompany the external medullary lamina. These vessels are homologised with the lateral striate artery of reptiles (ELLIOT SMITH) and the claustral artery of man (SHELLSHEAR). The

fact that it deeply penetrates the hemisphere to supply the lateral part of the hyperstriatum indicates that this structure is pallial in origin, as this vessel represents a greatly hypertrophied pallial vessel. The basal nucleus and mesostriatum are supplied by palaeostriatal arteries indicating that together these masses form the palaeostriatum. This is in accordance with the ontogenetic studies on the bird's brain of Ariëns Kappers.

In this way the subdivision of the bird's brain may be linked up with those of the reptile and so, from work already published, homologised with the constituents of the corpus striatum of mammals. The palaeostriatum of birds, represented by the *palaeostriatum primitivum* and the *palaeostriatum augmentatum*, is homologous with the globus pallidus of mammals. The hyperstriatum corresponds to the putamen and caudate nucleus (neostriatum of Ariëns Kappers). Though the hyperstriatum in most birds is divided by the *lamina medullaris hyperstriati* into the *hyperstriatum superius* and the *hyperstriatum inferius* the sections under review do not exhibit this subdivision. Ariëns Kappers believes that the *hyperstriatum inferius* corresponds with the putamen and caudate nucleus of mammals and that a possibility exists that the *hyperstriatum superius* represents the claustrum which is also hypopallial in origin. The archistriatum forms the hypopallial part of the *nucleus amygdalae*.

The forebrain acts upon the brain-stem by the ventral forebrain bundle, and upon the ganglion habenulae by the olfacto-habenular tract. The strio-mesencephalic, occipito-mesencephalic, and septo-mesencephalic tracts connect it with the mesencephalon. The corpus striatum receives the external and medial thalamofrontal tracts from the *nucleus rotundus* and *nucleus dorsalis* of the thalamus respectively. Accompanying the external thalamo-frontal tract is the dorsal supra-optic commissure. The efferent mechanism of the corpus striatum consists of the direct and crossed internal strio-thalamic tracts which terminate in the ventral thalamus and mid-brain.

LETTERING USED IN THE FIGURES.

A. prp.	Area praepiriformis.
B. olf.	Bulbus olfactorius.
C. t. s.	Commissura telencephali superior.
Cap. ex.	Capsula externa.
Cap. h.	Caput hyperstriati.
Cblm.	Cerebellum.
Cer. H.	Cerebral hemisphere.
Co.	Cortex.

Com. anter.	Commissura anterior.
Com. interep.	Commissura interepistriatica.
Com. p.	Commissura posterior.
Ec.	Ectostriatum.
G. h.	Ganglion habenulae.
G. isth.	Ganglion isthmi.
Gl.	Glomeruli.
Hyp.	Hyperstriatum.
Hyp. a.	Hyperstriatal artery.
Lam. med. ex.	Lamina medullaris externa.
Lob. olf.	Lobus olfactorius.
M. C.	Mitral cells.
Mes.	Mesostriatum.
N. oc.	Nervus oculomotorius.
N. tr.	Nervus trochlearis.
N. trig.	Nervus trigeminus.
Nuc. d.	Nucleus dorsalis.
Nuc. l. l.	Nucleus lemnisci lateralis.
Nuc. prof.	Nucleus mesencephali profundus.
Nuc. m. l.	Nucleus mesencephali lateralis.
Nuc. r.	Nucleus ruber.
Nuc. rot.	Nucleus rotundus.
Nuc. sp.	Nucleus spiriformis.
Pal. a.	Palaeostriatal artery.
S. l.	Sulcus limitans.
T. op.	Tectum opticum.
Tr. fr. ep.	Tractus fronto-epistriaticus.
Tr. fr. ep. (p. d.).	Tractus fronto-epistriaticus (pars dorsalis).
Tr. oc. mes.	Tractus occipito-mesencephalicus.
Tr. op.	Tractus opticus.
Tr. olf.	Tractus olfactorius.
Tr. olf. hab.	Tractus olfacto-habenularis.
Tr. sep. mes.	Tractus septo-mesencephalicus.
Tr. st. mes.	Tractus strio-mesencephalicus.
Tr. st. th. int.	Tractus strio-thalamicus internus.
Tr. st. th. cruc. int.	Tractus strio-thalamicus cruciatus internus.
Tr. st. th. rect. int.	Tractus strio-thalamicus rectus internus.
Tr. th. fr. ex.	Tractus thalamo-frontalis externus.
Tr. th. fr. med.	Tractus thalamo-frontalis medius.
V.	Ventricle.
V. f. b.	Ventral forebrain bundle.

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Histology. — *"The histopathology of Lyssa in respect to the propagation of the lyssavirus"*. By Mistress E. WINKLER-JUNIUS and J. A. LATUMETEN. (From the Psychiatric Neurologic Clinic at Utrecht). (Communicated by Prof. C. WINKLER).

(Communicated at the meeting of December 29, 1923).

Thanks to the kindness of Professor DE BLIECK and Dr. WINCKEL of the Veterenary University, we were in the opportunity of examining the nervous system of some dogs and rabbits inoculated with lyssavirus. The inoculationtime of the different cases diverged from seven weeks to three months. The animals were killed and submitted to an autopsy as soon as the first symptoms of the illness appeared. The questions, which after histopathological examination of the first case came to the front, diverged too much for our limited material to answer them all. We restrained our investigation therefore to one single question, a question that was given us by the clinical and experimental facts concerning lyssa. The clinical point of view, that the unknown virus of lyssa reaches the central nervous system by the peripherical nerves is often defended by the fact, that the duration of incubation is in direct proportion to the distance of the entrance spot from the spinal cord or medulla oblongata.

Experimental researches have established this point of view and proved that, the segment of the central nervous system corresponding to the inoculated limb, first becomes virulent, whilst from that segment the virulence spreads proximally and distally through the nervous system (SCHÄFFER).

According to DI VESTEA and ZAGARI, the lyssavirus does not propagate along the sheats of the nerve, but chooses the nerve-substance itself as a medium for its growth, viz. after inoculation with lyssavirus in the nervus ischiadicus the propagation of the virus is stopped, if directly after inoculation a more central part of this nerve is sectioned and cauterized.

However there remains a divergence of opinion on these points, in detail discussed in the *Handbuch* VON KOLLE und WASSERMANN by Professor Jos. KOCH. This author himself holds the opinion that

the lyssavirus reaches the central nervous system along the nerves as well as along the blood- or lymphvessels.

For the spreading of the virus along the nerves pleads:

1st the experiments of DI VESTEA and ZAGARI

2nd the experiments of SCHÄFFER

3rd that the blood has never been virulent

[„Fast alle Forscher sind der Meinung dasz das Virus im Blute nicht vorhanden ist" KOCH, Band VIII, KOLLE und WASSERMANN, pag. 835].

4th that subdural inoculation of blood from animals affected with lyssa never causes the lyssadisease.

Against the propagation of the lyssavirus along the nerves pleads:

1st the experiments of ROUX and MARIE by which is proved that only intradural and intracerebral inoculation of lyssavirus gives 100 % positive results, whilst endoneural injection remains uncertain.

The fact that the saliva of animals inoculated with lyssavirus is most times virulent, would also plead against the propagation exclusively along the nerves, if there were not the experiments of BERTORELLI. This author proved that one sided section of the nerves innervating the salivary glands, shortly before an subdural injection with lyssavirus prevents the infection of the saliva at the operated side. Moreover the fact that the saliva becomes virulent in the latter part of the incubation, when the central nervous system is already infected, makes it possible that from the medulla oblongata along Nervus facialis and Nervus trigeminus the infection of the salivary glands takes place.

We especially intended to see how far the histopathology of lyssa agreed with the clinical and experimental facts mentioned above. So:

first: *„whether peripheric nerves have altered and whether these alterations may prove in favour of a propagation along the nerve.*

secondly: *how far a similar propagation of the lyssavirus persists in the central nervous system and*

thirdly: *whether the nervous path to the salivary glands has altered in such a way that these nerves may be considered as the medium through which the virus reached the glands.*

To answer these questions and to make the histopathological examination at the same time as complete as possible we subdued our material to different fixation and staining in a way as gives the annexed scheme. (See Table following page).

As soon as in the cornu *Ammonis Negri*bodies were found and besides the clinical fact there was also a histopathological proof,

Fixation.	Staining and purpose of the method.	Part of the nervous systeem.
I. Alcohol 96 %.	Toluidin cellstaining NEGRIbodies (LENZ)	Cornu Ammonis, med. spinalis, cerebellum, brainstem, cortex cerebri
II. Sublimate trichloretic acid.	Neuroglia staining FIEANDT.	Cornu Ammonis, tela, gyrus centralis
III. Alcohol 96 % + ammonia.	Neurofibrils RAMONY CAJAL	med. spin, cortex, stem
IV. Formaline 10 %.	BIELSCHOWSKY neurofibril method. CAJAL's neuroglia method, fat staining	cortex, med. spinalis, cerebellum
V. FLEMMING's fluid.	ALZHEIMER's method for the myelin sheaths	cortex, med. sp., med. oblongata, peripheric nerves, brainstem, cerebellum
VI. ORTH-MÜLLER.	DOINIKOW's nervesheaths method	med. spinalis, peripheric nerves.
VII Cobaltnitrate.	DA FANO method for GOLGI apparatus	Cortex cerebri

that the animal had succumbed by lyssa, the nerves innervating the inoculated limb were examined on alteration.

Fig. I shows that the nerve everywhere has lipoid lumps. Although

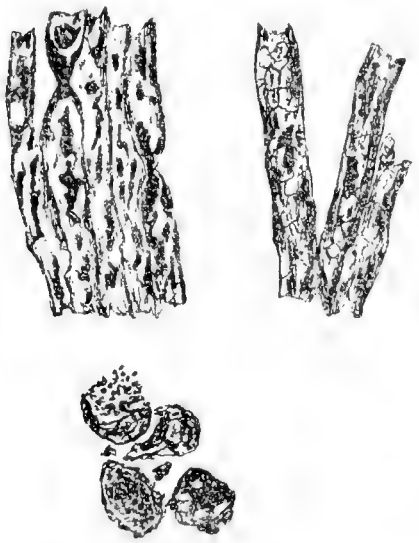


Fig. I.
Degeneration of the lumbal nerve at the inoculated side.

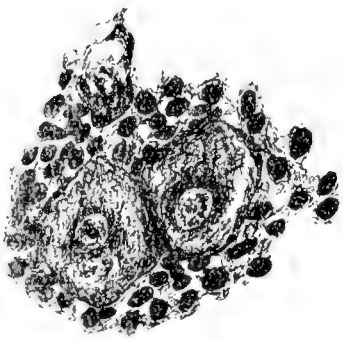


Fig. II.
BABES' Knötchen in the spinal ganglion of the inoculated side.

the degeneration of the nerve was not equally divided through the whole breadth of the nerve, there was sufficient proof that the nerve had been submitted directly or indirectly to a noxious influence. Therefore it was necessary to pursue the examination up to the spinal ganglion. It then appeared that the whole proximal part of the nerve was degenerated and the nervefibres in the spinalganglion as well were swollen and rich in lipoid lumps.

The nervecells in the ganglion had altered and were surrounded by phagocytes. Here and there „Babes Knötchen" (an accumulation of small cells in the midst of which were lying some skeletons of nervecells) were visible. (Fig. II).

Pursuing the radices up to the medulla, we got nearly the same image of a degenerated nerve as was shown to us by the more peripherical parts of the nerve. The swollen cylindre axis is lying in a hollow tube from which the myelin has totally disappeared (Fig. III). This degeneration of the roots could be followed into the white matter of the medulla spinalis, although between the degenerated tubes there were always fibres quite normally built. The nervecells in the segment of the medulla corresponding with the examined nerve had partly lost their *Nissl* lumps, partly were they recognizable only by a pale nucleus, surrounded by a mass of small gliacells (Fig. IV) *Negri*bodies were not found here.

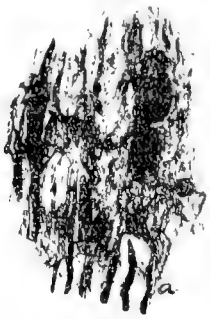


Fig. III.
Lumbal root of the inoculated side with degeneration and loss of the myelin sheaths.

Fig. IV.
Neuronophagy.

Microscopical slices impregnated with silver showed that the intracellular fibrils either had disappeared or had clotted together to big threads (*Golgi* alterations fig. V).

Just as it was found in the spinal ganglion the degenerated cells lay scattered among quite normal cells, so that the examination

with a slight magnification at first gave the impression that the cellgroups had hardly suffered any loss.

Fig. V.
GOLGI alteration in a nervecell of the medulla spinalis.

Transversal sections of the adjacent proximal part of the medulla showed the following histopathological changes. All the nervefibres of the lateral columns of the inoculated side are destroyed, the anterior columns as well as the posterior columns are partly destroyed, which destruction is continuous with the group of degenerated fibres of the healthy side.

Pursuing the medulla spinalis proximally, it appears that mainly both the lateral columns have a fatty degeneration of their myelin but the posterior and anterior columns too have some destroyed fibres

To value the alternations of the bloodvessels in the peripheric lumbar nerves and in the medulla spinalis was extremely difficult. Prepared as we were by the description in literature to find large infiltrations round the vessels and in the tissue, to find hyalin lumps in the walls of the bloodvessels, we were disappointed, when searching for these changes.

The bloodvessels in our slices showed a growing of their endothel and were studded with blood corpuscles, we found little haemorrhages in the peripheric nerve and somewhere round the vessels. The blood corpuscles however showed no trace of fatty degeneration, the haemorrhages proved to be of very recent date. So, although the alterations of the bloodvessels were in accordance with the result of other authors, as far as the degree of these alterations was concerned, it was impossible for us to place them above the changes of the nervecells and nervefibres. The serious degeneration of the myelin sheaths, the loss of the intracellular fibrils in the cells made it very probable that these changes had preceded the very recent infiltrations round the vessels.

Toluidin preparations of the lower part of the medulla oblongata,

where the nuclei Nervi XII, Nervi XI and Nervi X are to be found, offered a similar aspect of the state of their cells as is given by us for the cells of the medulla spinalis. In this part of the nervous system *Negri*bodies in the nervecells were found. Plate I, Fig. 1.

The more proximal part of the medulla oblongata, the brainstem, the cerebellum and some parts of the braincortex were submitted to different methods.

In all those parts we found degenerated myelin sheaths and cylindreaxes. The degeneration often consists in a tumefaction of the myelinsheath and a loss of myelin, whilst the cylindreaxis is preserved lying in the middle of the hollow tube. Fig. VI. No growing of neurogliacells along the destroyed sheaths was to be seen. Changes of nervecells of those parts were of different degree. The nuclei often had the most intensive changes. They partly had lost their membrane, were pale and swollen with only a single nucleolus stained red with eosine. Some times the nucleus was diffuse stained, and scarcely any structure was to be seen.

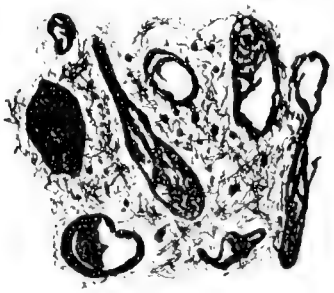


Fig. VI.
Degeneration of myelin sheaths.

Silverpreparations showed a granulation of the intracellularfibrils, some times alterations as described by GOLGI, the clotting together of the fibrils were present. The impregnation of the extracellular fibrils was some times very coarse just as it is to be found in ALZHEIMER's disease. This argentophily of the tissue round the fibrils is perhaps to be explained by the presence of a large quantity of demolition products, reducing the silver nitrate.

The *Purkinje* cells in the cerebellum had lost their *Nissl* lumps and their intracellular fibrils, and were often only recognizable by a partly destroyed nucleus in the neighbourhood of which *Negri*-bodies were lying. Plate I, fig. 2.

The cells of the cornu *Ammonis* were the principal seat of the *Negri* bodies, but our preparations gave no proof of these bodies, being generally found in relatively healthy parts. In cells which had scarcely undergone any change, as well as in cells totally destroyed, these *Negri*bodies were present. For instance there was a lack of *Negri*bodies in the cells of the medulla spinalis; in the brainstem and in the medulla oblongata, however we found a large number of these bodies although the medulla spinalis as well as the brainstem and medulla oblongata were the seat of serious alterations of the nervecells and nervefibres

Benedek and Posche in their monography as well as Francesco di Felice in his paper dispute the parasitic nature of the *Negri*bodies and both these authors are convinced that fragments of the nucleus of the nervecell take part in the formation of these *Negri*bodies.

Touching our material in this question we could state that often *Negri*bodies were lying next to a quite normally built nucleus, on the other hand we found through the whole nervous system changes of the nuclei, that pleaded for the opinion of both authors. So we found coarsely granulated nuclei, some of which granules entering into the cytoplasma (Fig. VII). The *Lenz*method, for the staining of *Negri*bodies too demonstrated big granules in the nucleus assuming the same stain as the *Negri*bodies do, whereas in normal brains we did not find such granules. Sometimes the whole nucleus had assumed a diffuse red stain (with Lenz method) as if the different granules were dissolved.

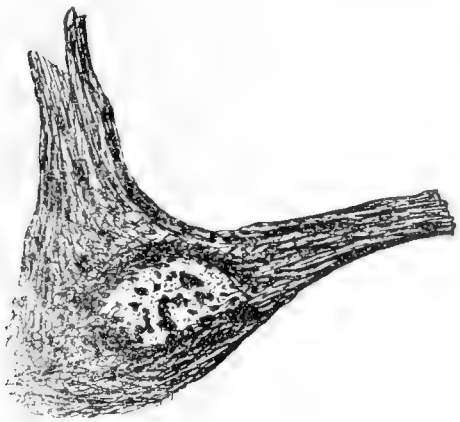


Fig. VII.
Granules from the nucleus entering the cytoplasm.

It was not our intention to enter deeply into the question concerning the nature of the *Negri*bodies; curiosity however stimulated us, when we treated our slices of the cornu *Ammonis*, rich in *Negri*bodies, with nuclease, in order to see whether these corpuscles as well as the nuclei could be destroyed.

Dr. van Herwerden was so kind as to teach us how to arrange these experiments. The result was as shows Plate I, fig. III that only skeletons of the nucleus were visible after the experiment, but that the *Negri*bodies had obstinately resisted the resolving power of the nuclease. They had kept the same form and stain as the *Negri*bodies which were treated with boiled nuclease that had lost its destructive power. Plate I, Fig. IV.

But these negative results do not exclude that the *Negri*bodies are built from material of the nucleus, especially by oxyphile elements, which are not destroyed by the nuclease. Our experiments only pleaded for the fact that the chromatine elements of the nuclei probably do not take part in the formation of these bodies.

As we had at our disposal section as well as fixation of our material, we succeeded in the impregnating of the Golgi apparatus

of the nervecells and as changes of this apparatus in pathological cases are seldom described, we think it justified to demonstrate them in this essay, although the significance of this apparatus and its changes is not at all known and has apparently nothing to do with the questions we treat of in this essay.

WILDER PENFIELD in his paper "Alterations of the GOLGI apparatus in nervecells" writes: "The GOLGI apparatus presents normally in the great majority of cases a complete attenuated reticulum with many varicosities or lacunae... The structure is confined to the cytoplasma never encroaching on the nucleus or the periphery of the cells.... The whole structure appears rarely in one half of the cytoplasma only. It may be hypertrophied or meagre but under normal conditions the general pattern is *surprisingly constant* [1]) for each type of cells Fig. VIII shows the apparatus in nervecells of normal brains.

According to CAJAL the reticulum should be more resistant to pathological agents than are the neurofibrils.

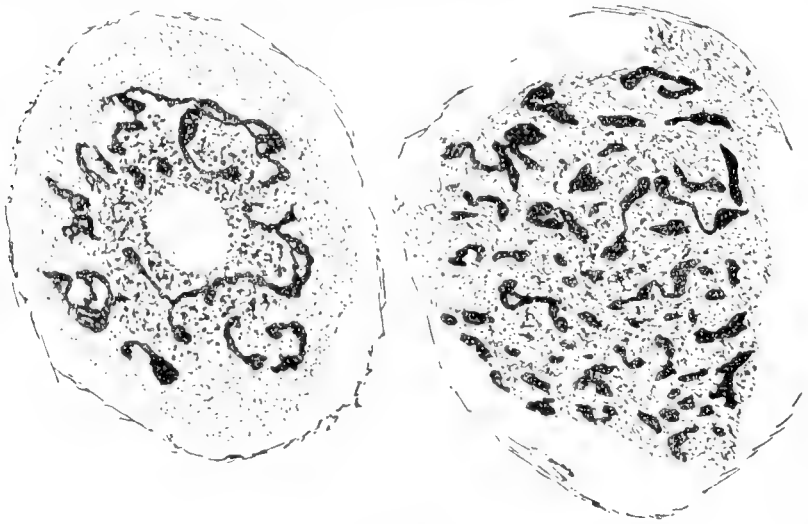


Fig. VIII.
GOLGI-apparatus in normal cells.

PENFIELD divides the reaction of the apparatus in the nervecells after sectioning the nerves into three stages.

1st Displacement of the unbroken apparatus to the periphery of the cell and away from the axonehillock *retispersion.*

2nd Dissolution of the reticulum *retisolution.*

3rd *Reconstruction.*

[1]) The italics are ours.

In the cases of lyssa the most conspicuous change of the GOLGI-apparatus was the retispersion as shows fig. IX. Round the nucleus

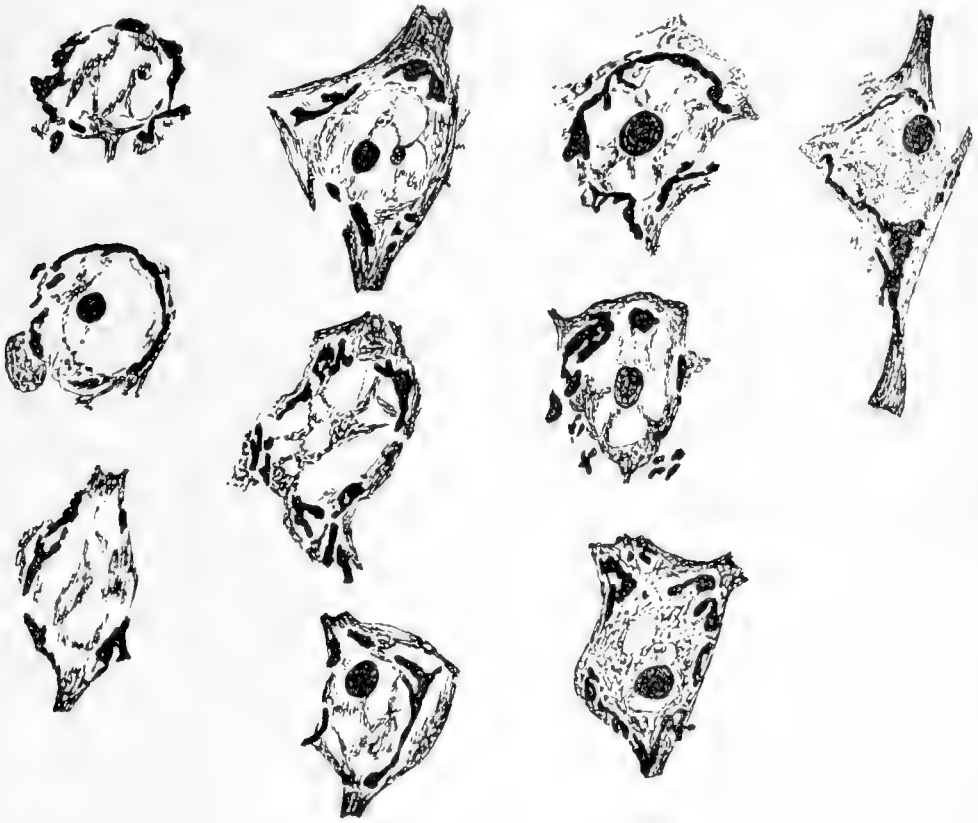


Fig. IX.
Retispersion and *stretching* of the GOLGI apparatus.

the apparatus has disappeared. The second constant change was the *stretching* of the apparatus. Instead of the small curvations as seen in normal cells, the meshes of the changed reticulum are bordered by stretched threads. Perhaps this stretching is due to the retispersion viz. to the fact, that the apparatus has to occupy a larger sphere round the nucleus than in normal cells.

Changes of the neuroglia were only of a slight degree.

Round the changed or destroyed nervecells the number of neuroglia-cells had augmented, amöboid gliacells were scarcely found along the degenerated nervefibres. There was a total lack of neurogliafibres. Resuming the changes of the different elements described above, we get in the first place.

1st *Changes of the nerve fibres.*

Serious degeneration of nervesheaths, swollen cylindre-axes in the

peripheric lumbar nerve of the inoculated side, in the lateral columns of the medulla spinalis and everywhere scattered among normal fibres in parts of the cerebellum brainstem and cerebrum.

2nd *Changes of nervecells.*

a. Destruction more or less of the nucleus

b. Dissolution of *Niss*l lumps

c. Granulating of nervefibrils

d. GOLGI alterations of the intracellularfibrils

e. Presence of *Negri* bodies

f. Retispersion and stretching of the GOLGI apparatus

g. BABES' Knötchen mainly in the ganglion spinalis

3rd *Changes of the bloodvessels.*

a. Vessels studded with cellular elements

b. Small infiltrations round the vessels of haematogenous elements

c. Growing here and there of the endothelcells

4th *Changes of the neuroglia.* Very slight.

Trying to answer our first and second question with these facts it is obvious, that of all the changes those of the nervefibres, especially of the myelinsheaths are the most conspicuous.

As to the degeneration of the peripheric nerve it is certainly not a WALLER degeneration for:

1st. the cylindre-axis are much less destroyed than are their myelinsheaths.

2nd. there is scarcely any reaction from the side of the cells of SCHWANN.

As to the degeneration of the fibres in the medulla spinalis there is no question of a system degeneration. The destroyed myelinsheaths and swollen cylindre-axes can not be pursued up to their nervecells, viz. in the neighbourhood of the different cellgroups, in the medulla scarcely any destruction of fibres is to be found. So the destroyed myelin can be explained only by a direct influence of the virus.

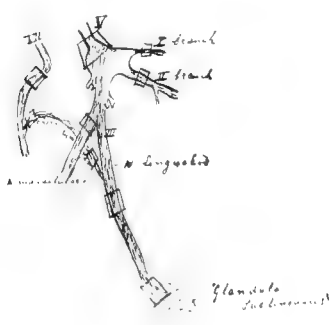
Admitting that the lyssavirus propagates along the lymph or bloodvessels and the virus itself or its toxines entering the nerve on different spots destroys the myelin, than the question arises why precisely the nerve of the inoculated muscles has degenerated in its whole length without any interruption whilst other peripheric nerves either have a degeneration of their roots and the most central part of the postganglionic part or have no degeneration at all.

Admitting therefore that the virus from the inoculated spot reaches the central nervous system, it does not in the least exclude that along the vessels as well the virus is transported. In the medulla

spinalis the destroyed fibres are mainly lying in the peripheric part of both the lateral columns, but they are not entirely lacking in the peripheric zones of the dorsal and ventral columns. As to the degeneration of the right (inoculated) side of the medulla, the supposition lies at hand that the virus as soon as it has reached the spinal chord by the anterior or posterior roots, takes the most peripheric lying myelinsheaths as a medium for its growth. But then it has to be explained that there is scarcely any difference between both the lateral columns and even in a transversal section of the lumbal medulla, it is difficult to see which side is the inoculated side.

However, this histopathological fact fully agrees with the experiments of Roux, which teach us, that the virus directly after its arrival in the medulla infects the opposite side, so that the peripheric nerve of this not inoculated side becomes virulent, before the proximal and distal parts of the medulla are infected. Supposing, the neuroglia reticulum, in which meshes the nervefibres are lying, undertakes the transport of the virus, then it is not explained why the more centrally lying myelinsheaths have not altered. Our opinion in this question is, that probably the liquor cerebrospinalis surrounding the medulla as well as the anterior and posterior roots undertakes also the transport. (The experiments of Daniel Konradi recently proved the virulence of the liquor cerebro-spinalis, a fact that hitherto was denied in literature).

The answer to the third question, whether the virus reaches the salivary gland by growing along the afferent or efferent nerves, required a complete examination of a larger part of the brainstem of Nervus facialis with chorda tympani, and of the ganglion Gasseri with Nervus trigeminus, especially its ramus lingualis. Annexed scheme demonstrating the innervation of the salivary gland of rabbits and dogs shows what nerves were submitted to examination.



Scheme of the examined parts of N. V. and V. VII.

The brainstem, fixated in Flemmings fluid was sectioned in a series of transversal slices in order to stain them with Fuchsin Lichtgrün.

Fig. X representing a section through the brainstem and the roots of N. V. shows the degenerated fibres especially in the portio minor, though also the portio major has swollen or destroyed fibres.

The nervecells in this section and through the whole brainstem have more or less changed and many *Negri*bodies are found here. Pursuing the roots up to the ganglion *Gasseri* partly the fibres have degenerated though by far not the greater part. Longitudinal sections through the third branch of this ganglion consisting of N mandibularis with N lingualis show a serious degeneration especially of the N. lingualis. Also the cells of the ganglion oticum and the fibres

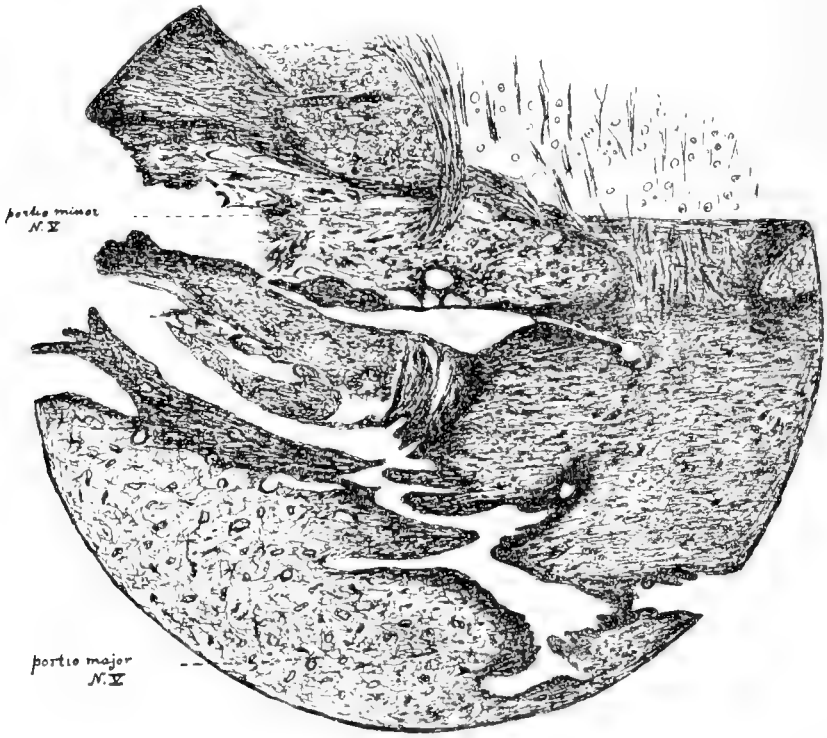


Fig. X.

Section through the roots of *N.V.* Degeneration of the portio minor- and portio major *N.V.*

passing that ganglion show a serious degeneration. Fig. XI. Pursuing the N lingualis up to the salivary gland we saw this nerve degenerated along its whole length (Fig. XII). So there is no doubt that the nervous path connecting the central nervous system with the salivary gland is more or less destroyed.

A section through the brainstem and the roots of N facialis shows a degeneration of the facialis roots, though of a slight degree and more or less of the vestibularis roots. Also the part of the corpus trapezoides lying between both roots is partly destroyed, so that it seems that especially the lateral peripheric part of the brainstem

has been influenced by the virus (Fig. XIII). A more distal section of N facialis and its branch the chorda tympani proves, that both these nerves have degenerated, so that the second nervous path too, connecting the brainstem with the salivary gland prove to be degenerated. As to the cellular elements connected with both these pathes we found in the ganglion *Gasseri* as well as in the ganglion oticum serious alterations of nervecells. We did not succeed in getting sufficient slices of the ganglion geniculi.

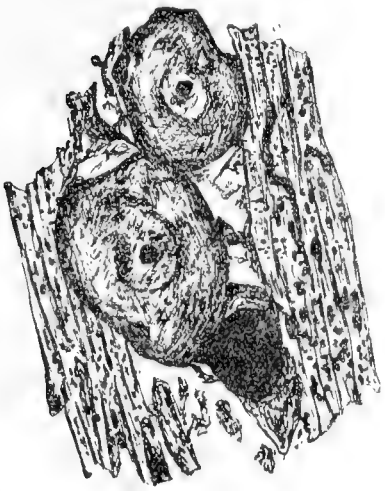


Fig. XI.
Degenerated lingualis fibres passing the ganglion oticum.

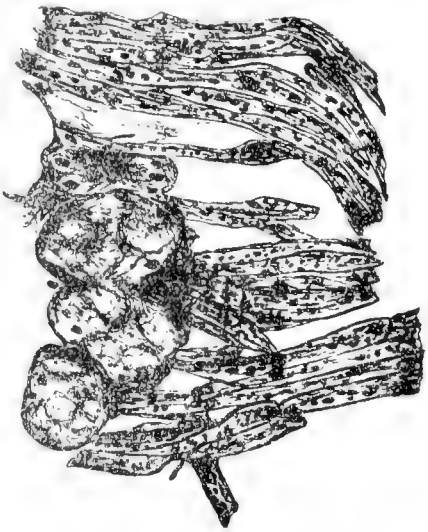


Fig. XII.
Nervus lingualis entering the glandula sublingualis.

On the other hand the N abducens, the more distal part of the N. mandibularis, N. ophtalmicus, and N. maxillaris proved to contain for the greater part normal fibres as well as nerves more proximally entering or leaving the brain e.g. the N. opticus and N. oculomotorius had no changes at all.

Resuming we found the brainstem seriously changed, the alterations of the nervecells with their *Negri*bodies, the destroyed myelinsheaths especially in the peripheric lateral parts of the stem, proved that this part of the central nervous system had been strongly influenced by the noxious virus.

Pursuing the different roots leaving the brainstem we found both roots of the N. trigeminus degenerated especially the portio minor. The facialis roots also showed some degenerated fibres, but in a far less degree than the roots of the N. trigeminus. Of the branches of the ganglion *Gasseri*, the N. lingualis was the most destroyed nerve.

This result was of some importance in connection with the question, whether the lyssavirus chooses the nervous path to reach the salivary gland.

Suppose we did not find any degeneration in the nerves, innervating the salivary gland in casu the N. lingualis with the fibres

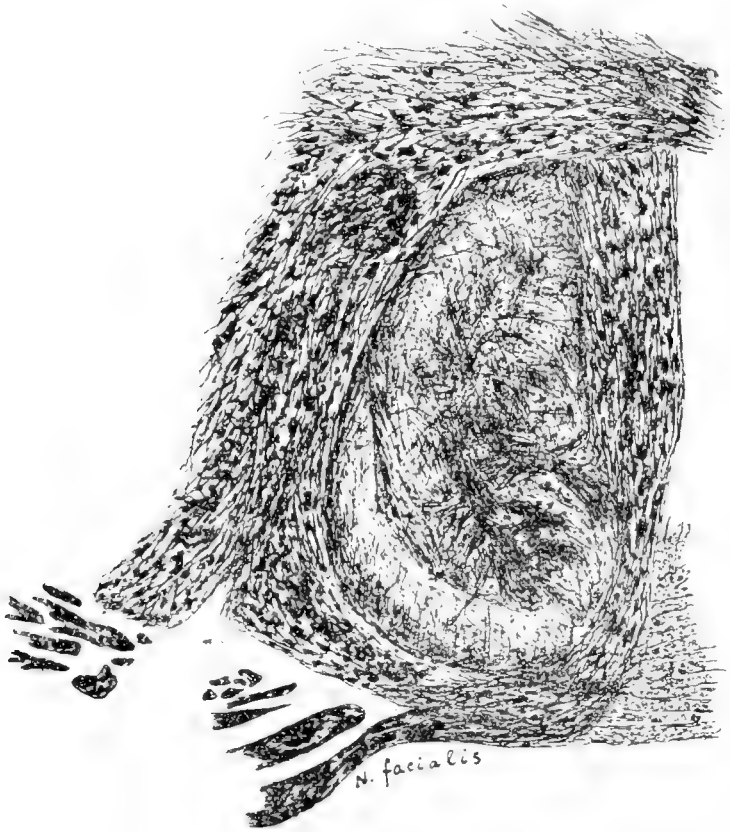


Fig. XIII.
Degeneration of the roots of N vestibularis and N facialis.

of the chorda tympani, it would be evident that the virus had not reached the gland along the nerves, because the histopathology of lyssa gave sufficient proof that the virus has a noxious influence especially on the nervefibres and its myelinsheaths.

However as the nervefibres, connecting the nervous system with the salivary gland have indeed changed and these changes seem to be of older date than the changes of the side of the bloodvessels, it is most probable that these changes of the fibres are directly brought about by the lyssavirus.

Therefore the histopathological changes of the brainstem and the

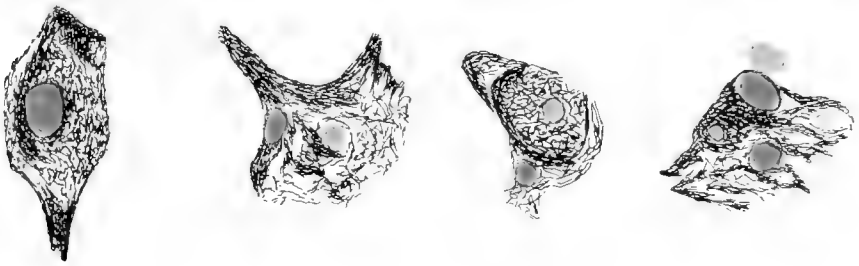


Fig I

Fig II

Fig III

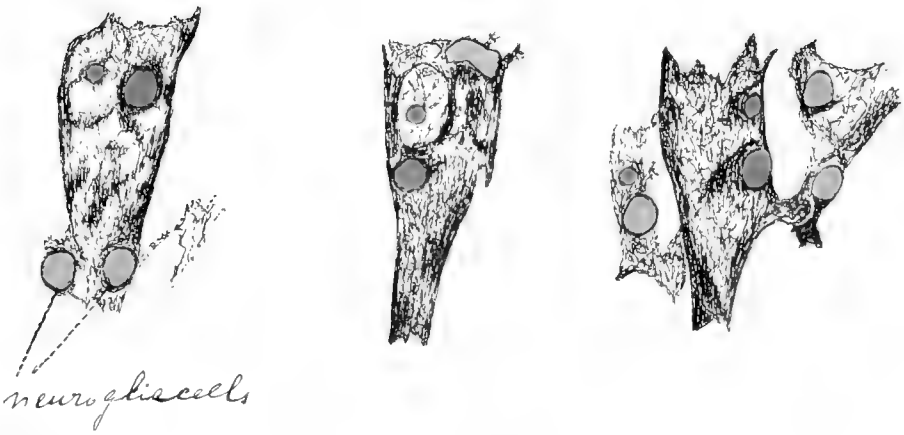


Fig IV

nerves innervating the salivary gland make it most probable that the lyssavirus reaches the salivary gland along the nervous path.

The histopathology of lyssa fully agrees with the experimental results teaching 1st that only in the second part of the incubation the saliva becomes virulent, 2nd that the virulence of the central nervous system precedes that of the salivary gland, (experiments of *Bertorelli*).

As to the other side of the question whether the nervous path is chosen *exclusively* by the virus, we think the most detailed histopathology of the lyssa brain incompetent to solve this problem.

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DESCRIPTION OF PLATES.

Plate I.

Fig. I. Negri bodies in the nervecells of the medulla oblongata. Lenz staining.

Fig. II. Negri bodies in the Purkinje cells.

Fig. III. Negri bodies having resisted the influence of nuclease; Neurogliacells having lost the greater part of their nuclei surrounding the nervecells.

Fig. IV. Negri bodies in slices treated with boiled nuclease.

Physics. — "*Magnetic Researches.* XXVI. *Measurements of Magnetic Permeabilities of Chromium Chloride and Gadolinium Sulphate at the Boiling Point of Liquid Hydrogen in Alternating Fields of Frequency 369,000 per Second.*" By G. BREIT, National Research Fellow, U.S.A., and H. KAMERLINGH ONNES. (Communication N°. 168c from the Physical Laboratory at Leiden.)

(Communicated at the meeting of December 29, 1923).

§ 1. *Introduction.* It has been suggested by EHRENFEST [1]) that at very low temperatures paramagnetic substances may show phenomena of hysteresis. The experiments reported on in this communication were made in order to see whether this effect is present at reasonably high frequencies. The quantities measured were magnetic susceptibilities. The measurements were made on samples previously used in steady field determinations to as to enable a direct comparison. The measurements made do not give one sufficient confidence to claim great numerical accuracy of the results. However, they seem to indicate definitely that the order of magnitude of the susceptibilities for steady and alternating fields is the same. The numerical values obtained for both salts are smaller than the values obtained in direct fields and the apparent consistency of trial measurement given below suggests that this discrepancy may be not due to experimental error.

§ 2. *Methods and Apparatus.* The method was similar to that described by BELTZ [2]). Two electron tube circuits (Nr. 1, Nr. 2) were set up to generate sustained oscillations of high frequency. The frequencies of the two were adjusted so as to be nearly integral multiples of each other. A two stage audio frequency amplifier was coupled loosely to both. The audible beats produced in the amplifier were made to give beats with an audible note produced by a third electron tube circuit, say Nr. 3. The paramagnetic sample was put into the inductance of circuit Nr. 1. The cryogenic apparatus surrounding the sample was placed inside the same coil. The coil was shielded on its inside by means of tinfoil. The tinfoil was cut into 8 segments so as to allow the magnetic field to pass to the inside of the shield. The cryogenic

[1]) P. EHRENFEST, these Proc. 23, p. 989; Leiden Comm. Suppl. N°. 44*b*.

[2]) Phil. Mag. 44 (1922) p. 479.

apparatus consisted of two non-silvered vacuum glasses — the outside being used for air and the inside for hydrogen in the usual manner. Fig. 1 shows the shielded box B into which circuit Nr. 1 was put. C is the top of the inductance coil shield, L is the lid of the box, P is the packing. Under the hood circuits Nr. 2 and Nr. 3

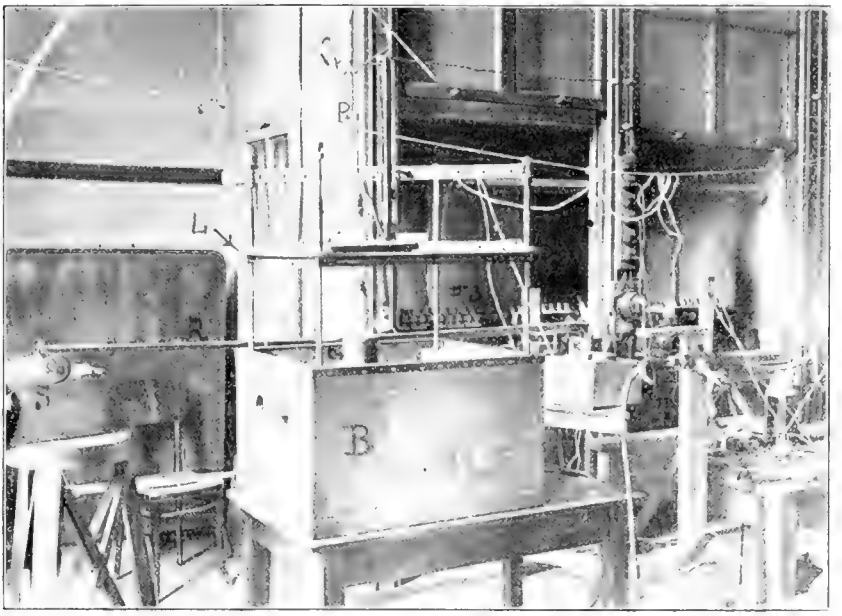

Fig. 1.

are arranged as shown. The amplifier A is on the table to the right. The small condenser K (used at a place where it has 313 $\mu\mu f$ capacity) is connected in series with a larger fixed condenser (capacity 2847 $\mu\mu f$) between the filament and grid of circuit Nr. 2. The rod R 270 cms. long controls the motion of K, being attached to a screw adjustment S. Adjusting S changes the frequency of circuit Nr. 2 by small amounts. On fig. 2 a view is given of the inside of the box B, the shield of the coil C and its windings W being plainly visible. The shield segments are connected by the wire D and when the lid is lowered the wire D is connected to the box shield by a short wire and a clip. The windings of the coil W are supported by a piece of glass tubing to which they are well fastened with paraffine. The upper part of the shield is made of pasteboard tubing thoroughly boiled in paraffine and hardened by several coats of shellac. The filament and plate batteries are also shown on this picture. The filament rheostat which is shown on the right wall has been short circuited in the final measurements so as to eliminate fluctuations in the

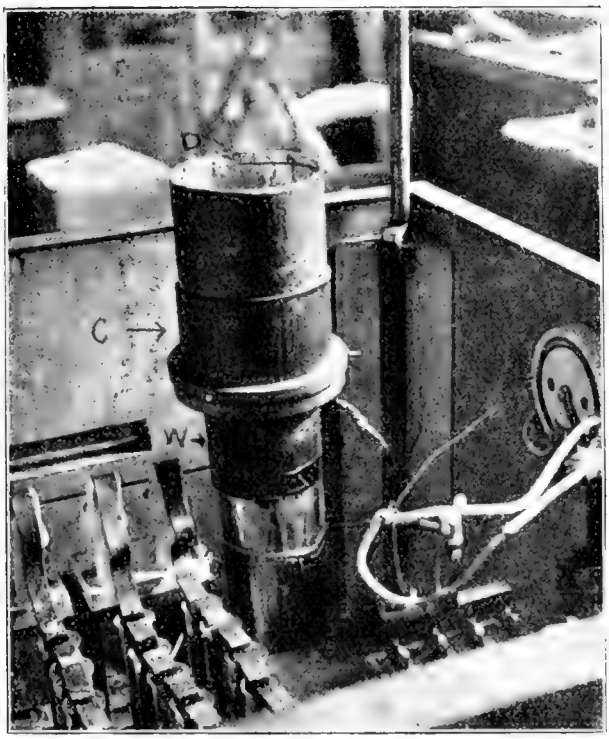

Fig. 2.

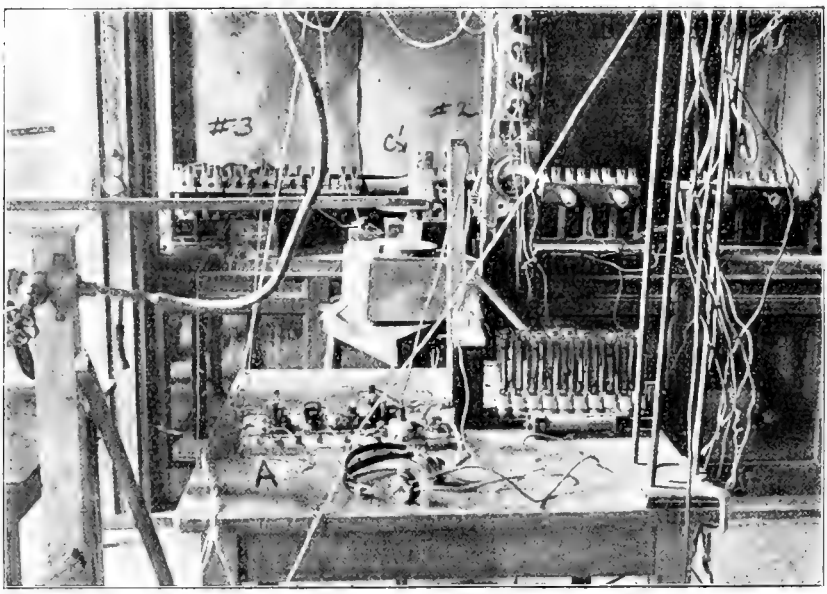

Fig. 3.

filament current. Fig. 3 shows circuits Nr. 2, Nr. 3 and the amplifier in more detail. C' is the inductance of Nr. 2. The valves used have been MULLARD or PHILIPS receiving valves. The circuit connections in Nr. 1 and Nr. 2 have been those of the usual HARTLEY circuit a tap off at the middle of the coil leading to the filament and the extremities going to the grid and plate battery terminals. A condenser was connected across the coil terminals. All the parts of circuit Nr. 1 were thoroughly shielded. The coupling to the amplifier A was accomplished by a wire entering the box B through a hole on the side opposite to that shown Fig. 1. This wire was connected to the input of A either directly or through a transformer. The input terminal of A was also coupled by a wire to circuit Nr. 2. The loose end of this wire was stretched in over a rope towards circuit Nr. 2. Circuit Nr. 3 was controlled by a pulley arrangement allowing one to turn its condenser plates without introducing capacity due to the observer's body. The observer was situated at the outstanding corner of B in Fig. 1. The telephones were used on the observer's head. The coupling in this apparatus was sufficiently loose to make the capacity effect through the phones negligible for the small motions of the observer's body during the measurement. The liquid air in the vacuum glass inside C produced sufficient cooling of it to cause unsteadiness. It thus was necessary to have a rapid method of measurement.

For this purpose a system Q (Fig. 1) was attached to the rod sliding through P. Q consists of a horizontal metal rod carrying three collars. The central one is attached to a vertical guide passing through a collar almost directly above P. The motion of this guide can be controlled by the lever shown. The collar on the right is connected to P by a vertical rod and the collar on the left carries a glass rod from the lower end of which is suspended a glass tube by means of a thread. The metal rod passing through P ended well inside the tube of German silver supporting P and was thus shielded from the action of the magnetic field of C. To the lower end of that rod a glass tube was fixed with sealing wax and this tube supported the paramagnetic specimen. The paramagnetic specimen and the suspended tube could be thus moved in and out of C simultaneously. Inside the glass tube a single or several copper wires were put and these were selected in such a way as to make the number of beats per second in the upper and lower positions the same. The combination was recorded and a subsequent calibration determined the magnitude of the effect. The range of motion was fixed by permanent stops on the sliding rods. This was necessary

because balance in the end positions was found not to mean always a balance in the intermediate ones.

§ 3. *Preliminary Tests.*

a. Necessity of shield for coil. The fact that the coil had to be shielded on the inside was ascertained by first trying the arrangement without the shield. A glass bottle lowered inside the coil by a thread produced an appreciable effect on the beats. This effect was absent when the inside of the coil was covered with thin tinfoil strips.

b. Bending of box lid. Many blank tests have been made to see whether the strains in the lid due to the up and down motion of the rod system Q affected the beats. No such effect was observed.

c. Interaction between circuits has been observed to be generally very small. Thus by adjusting the beat frequency between Nr. 1 and Nr. 2 to a low value like 100 no tendency of pulling into one has been noted. Putting a paramagnetic sample in and then out again under these circumstances did not alter the beat frequency noticeably. The beat frequency employed in the measurements was of the order of 500 or 1000 per second and the interaction must have been still smaller. A further test of this was made by first compensating a sample by changing K, say with set Nr. 1 going at a higher frequency than Nr. 2; then Nr. 2 was set to the higher frequency and the measurement repeated. The following table gives the results of the changes of the scale S.

	First Side.	Second Side.	Difference.	
Iron wire Nr. 7 in suspended tube	22.3	22.5	— 0.2	— 1%
Iron wire Nr. 8 in box . . .	13.9	13.2	0.7	5.4%
Combined action	34.3	33.6	0.7	2%

The iron wire Nr. 8 was here put in a tube fastened to the paramagnetic sample in the warm condition with thread and sealing wax. This tube was approximately at the axis of the coil while Nr. 7 was in the suspended tube almost at the shield. There appears to be no true systematic difference between measurements made with the two frequencies reversed.

d. Direct test of balance. It so happened that a copper wire (2 N) when used in the suspended tube balanced by its diamagnetic eddy current effect the paramagnetic effect of iron wire (Nr. 8). When the

effect of 2 N was balanced by means of K, 13.4 scale divisions were necessary while Nr. 8 took —14.2 scale divisions. The residual effect of the combined action was estimated at —0.5. The equation $14.2 = 13.4 + 0.5$ is satisfied to about 2 %.

e. Effect of length of paramagnetic sample. To test this four iron wires of different lengths were cut from one piece. They had lengths: 6.73 cm., 7.80 cm., 10.00 cm. Two glass tubes were fastened to the tube of $CrCl_3$ in the warm condition. Tests with these were made on May 12 and on July 13, 1922. In the first set of tests the paramagnetic action of the iron wires used in one of the tubes at the axis was balanced by the diamagnetic action of suitably chosen combinations of copper wires put into the suspended tube and besides a copper wire was put into the other tube at the axis so as to take up the bulk of the paramagnetic action of the iron wire. The measurements were then repeated with the iron wires turned upside down. No difference of using one end up rather than the other was found thus showing that the iron wires are homogeneous. Both sets of measurements agreed in showing that the effects of the wires having lengths 7.80 cm. and 8.80 cm. are nearly equal and slightly greater than the effects of wires having lengths 6.73 cm., 10.00 cm. The effect of the wire having 10 cm. length was only slightly smaller than the maximum. The copper and iron wires used at the axis were made to change places and the observations were confirmed the effect of the interchange being very small. On July 13 further confirmation of the observed effect of length was obtained the effect of the iron wires being this time compensated by changes in K. The number of scale divisions of S required for compensation was for the same wires in increasing order of length 18.4, 22,6, 22.5, 21.1.

The reason for the decrease in the paramagnetic effect at 10 cm. is not clear. The increase in the region of 7 cm.—8 cm. must be due to the simple increase in the length in a fairly homogeneous field because $\frac{22.6}{18.4} = 1.17$ while $\frac{7.80}{6.73} = 1.16$. It may be that the shield causes a peculiar distribution of magnetic field resulting in the upper portion of the 10 cm. wire being in a stronger field in the "up" position than in the "down".

f. Effect of criterion of compensation. When the beat frequency of circuits Nr. 1 and Nr. 2 becomes nearly equal to the frequency of Nr. 3, it becomes at times difficult to distinguish them. Also in theory one cannot deny a possible action of Nr. 3 on the combined system Nr. 1 and Nr. 2 and thus an effect on their beat frequency. If these effects

are present, one should expect the result of making settings by adjusting beats to zero and by adjusting beats to a fixed number to be different. The results however indicate that this effect is absent. Thus 13.2 and 13.1 are the readings obtained on the scale S by the two methods.

g. Calibration. Since all the changes in the frequency are very small the change in the frequency is very nearly proportional to the change in capacity or to the change in inductance that causes it. Therefore changes in K necessary to compensate for two different changes in inductance are proportional to these changes. If one change in inductance is known or if its meaning as a susceptibility is known the other is derived by multiplication into the ratio of the two settings of K. This is the principle of the calibration employed. The calibration divides itself into the following parts:

a. *To produce a change in inductance which has a direct interpretation as a susceptibility.*

For this purpose two glass tubes were attached to the paramagnetic sample at opposite sides of a diameter — while the sample was in the warm condition. Copper wires accurately drawn and measured could be inserted into these The length of the wires was nearly equal to the length of the column of paramagnetic substance employed. Roughly the wires may be said to exclude the high frequency magnetic field from their interior. To a first approximation they are therefore equivalent to a material of susceptibility $-\frac{1}{4\pi}$. If the positive effect of a paramagnetic sample is equal to the effect of a wire of a certain size, its susceptibility must be then equal to $\frac{1}{4\pi}$ times the ratio of the volume of the wire to the volume of the substance. Since the field is not quite excluded from the interior of the wire, its diamagnetic action is not quite as large as we have just supposed but a correction for this may be applied. Taking the field to be a homogeneous one along the axis of the wire the correction factor is $-\mu\beta(q)+1$ where

$$\beta(q)=\frac{2}{q}\,\frac{ber\,q\;bei'\,q-ber'\,q\;bei\,q}{ber^2\,q+bei^2\,q}$$

where $q=\sqrt{\frac{4\pi\mu\omega}{\sigma}}\,a$, *ber* and *bei* are the KELVIN functions, and μ, σ, a are respectively the permeability, resistivity, and radius of the wire used at the frequency $\frac{\omega}{2\pi}$.

b. Now the paramagnetic sample was balanced against wires put not in its immediate neighborhood but at the shield. A correction factor must be applied for this. The factor was determined experimentally by balancing the effect of the same wire by K, first at the tubes attached to the paramagnetic sample and second in the suspended tube used in the compensation of the paramagnetic salt. The ratio of the readings on K gave the correction factor. The determinations of the correction factor vary somewhat and a considerable part of the experimental uncertainty is due to this.

c. Finally a determination had to be made of the ratios of the changes in K which had to be made in order to compensate the change in inductance produced by the combination of wires which compensated the paramagnetic sample and the changes produced by the standard accurately drawn wires.

§ 4. *Results for anhydrous Chromium Chloride at the boiling point of hydrogen.*

At a frequency of 3.69×10^6 the sample of chromium chloride was balanced by a combination of wires the effect of which was soon afterwards compared with the effect of one of the standard wires ($2N$). The ratio of the effects of the combination of wires to the effect of $2N$ as measured by K was $\frac{6.0}{6.3}$. The correction factor due to the inhomogeneity of the field as measured on $2N$ was $\frac{6.3}{4.6}$. Finally the correction for the length (the sample of chromium chloride was larger than $2N$) was $\frac{18.3}{21.5}$. (This was determined from the results on iron wires cited above). The resultant correction is then $\frac{6.0}{6.3} \frown \frac{6.3}{4.6} \frown \frac{18.3}{21.5} = 1.11$.

Again at the frequency $3.69 \cdot 10^6$ the wire $2N$ has a $q = 9.01$ and hence $1 - \beta(q) = 0.841$. Thus the volume susceptibility of the wire is $\varkappa = -\frac{0.841}{4\pi} = -0.0669$. Now the radii of the wire and of the sample were 0.705 mm. and 3.5 mm. respectively. Hence the volume susceptibility of the sample is $\varkappa = 1.11 \times \left(\frac{0.705}{3.5}\right)^2 \times 0.06692 = 0.00305$. The weight of chromium chloride was 3.192 grams and its length 9.7 cm.; the density is $\frac{3.192}{\pi \times 9.7 \times (0.35)^2} = 0.85$

and the specific susceptibility $\chi = 0.0036$. The value obtained in direct fields, according to unpublished results of Dr. H. R. WOLTJER is 0.0048 [1]).

§ 5. *Results for Gadolinium Sulphate at the boiling point of hydrogen.*

At the same frequency of 3.69×10^5 the gadolinium sulphate was balanced against a different combination of wires which when compared with $2\,N$ had an effect smaller than $2\,N$ in the ratio $\frac{1.48}{10.1}$. The length of the sample was 8.74 cm. and the corresponding correction $\frac{18.3}{22.7}$. The weight of the sample is 2.897 grams. The specific susceptibility is hence $\chi = 0.00051$. The value obtained at the same temperature for the same sample in a steady field was 0.0010 [2]). Measurements on manganic and nickel sulphate have been also made and gave results of the same order of magnitude as those for steady fields.

§ 6. *Conclusions and Discussion.*

A. The order of magnitude of the susceptibility is unchanged if the frequency is increased to 3.69×10^6.

B. The results seem to indicate that the susceptibility is smaller than for direct fields. The values obtained in alternating fields for $CrCl_3$ and Gd sulphate are 0.75 [3]) and 0.51 [4]) respectively of what

[1]) This value is obtained by the method of weighing a rod of the material in an inhomogeneous magnetic field (KAMERLINGH ONNES and PERRIER, these Proc. 16, p. 689, Leiden Comm. N°. 139*a*). However, the susceptibility seems to depend on the field strength, decreasing with increasing magnetic force. The value given relates to a field ranging from 4500 gauss at the top of the rod to 220 gauss at the bottom. The limit for very weak magnetic fields may be about 20% higher (as found by *extra*polating the susceptibility-magnetic force curves), so the ratio 0.75, given in § 6B for the susceptibilities in alternating and direct fields, may be too large.

[2]) KAMERLINGH ONNES and OOSTERHUIS, these Proc. 15, p. 322, Leiden Comm. N°. 129*b*, § 6. However, it has to be pointed out, that it was not sure the sample was really at the temperature of the bath, as it appeared afterwards, that in the experiments of KAMERLINGH ONNES and OOSTERHUIS it took some 4 hours before the susceptibility had taken a definite value, probably owing to lag in the temperature equilibrium. Even in order to avoid this difficulty and to ensure a better temperature equilibrium of the powder and the bath, the tubes for the magnetic investigations were later on not evacuated but filled with a small quantity of non condensing gases (hydrogen or helium). The value 0.00051 obtained with the present tube is probably too low.

[3]) See note § 4.

[4]) See note § 5.

they are in direct fields. However, it would be preposterous to conclude that the susceptibility is actually decreased by the amount found. Further work will be necessary for that. The choice of the place of the suspended tube was rather unfortunate. It was situated rather close to one of the slits in the tinfoil. Even though capacity effects appear to be absent this is dangerous because the magnetic field in the neighborhood of the slit is not homogeneous. It is possible that the divergence between the values for alternating and direct fields is due to insufficient caution in the manipulation of the suspended tube and a slight displacement of it during the experiment. This would hardly explain, however, the similarity [1]) of the results for the two substances investigated.

The writers wish to express their sincere thanks to Dr. H. R. WOLTJER for help in comparing the results with those in steady fields and for making unpublished results of his measurements available.

[1]) Especially, if the value 0.75 is too high and 0.51 too low (see notes §§ 4 and 5, this similarity is perhaps not only qualitative, but also more or less quantitative.

Mathematics. — "*On a non-symmetrical affine field theory.*" By Prof. J. A. SCHOUTEN. (Communicated by Prof. H. A. LORENTZ.)

(Communicated at the meeting of October 27, 1923).

1. *Introduction.* In his last publications[1]) EINSTEIN has given a theory of gravitation which only depends on a symmetrical linear pseudo-parallel displacement ("affine Uebertragung") and a principle of variation. From the equations, that result in this case, we see that the electromagnetic field only depends on the curl of the electric current vector, so that the difficulty arises that the electromagnetic field cannot exist in a place with vanishing current density.

In the following pages will be shown that this difficulty disappears when the more general supposition is made that the original displacement is not necessarily symmetrical.

The equations which define such a displacement are

$$\nabla_\mu v^\nu = \frac{\partial v^\nu}{\partial x^\mu} + \Gamma'^\nu_{\lambda\mu} v^\lambda$$

$$\nabla_\mu w_\lambda = \frac{\partial w_\lambda}{\partial x^\mu} - \Gamma'^\nu_{\lambda\mu} w_\nu ,$$

in which the parameters $\Gamma'^\nu_{\lambda\mu}$ (with an accent to distinguish them from the $\Gamma^\nu_{\lambda\mu}$ of a symmetrical displacement) are not symmetrical in λ, μ.

EINSTEIN[2]) has defended the use of symmetrical parameters with the remark that in the non symmetrical case not only

$$\frac{\partial w_\nu}{\partial x^\mu} - \Gamma'^\nu_{\lambda\mu} w_\nu$$

but also

$$\frac{\partial w_\lambda}{\partial x^\mu} - \Gamma'^\nu_{\mu\lambda} w_\nu$$

can be regarded as the covariant differential quotient (Erweiterung)

[1]) Berliner Sitzungsberichte 1923 p. 32—38, 76—77, 137—140.

[2]) L.c. p. 33.

of a covariant vector, and thus the unambiguous character of this quotient would vanish. But when the second expression is used the transvection $v^\lambda w_\lambda$ of two vectors v^ν and w_λ is no more an invariant with a pseudo-parallel displacement, so that the differential quotient of the first formula occupies a well defined preferred position.

We will not consider the most general case, but the *semi-symmetrical* case in which the alternating part of the parameters has the form

$$^1/_2(\Gamma'^{\nu}_{\lambda\mu} - \Gamma'^{\nu}_{\mu\lambda}) = {}^1/_2(S_\lambda A^\nu_\mu - S_\mu A^\nu_\lambda) \quad ; \quad A^\nu_\lambda = \begin{cases} 1, \nu = \lambda \\ 0, \nu \neq \lambda, \end{cases}$$

in which S_λ is a general covariant vector[1]). It will be shown that already with this simplified supposition the above mentioned difficulty can be made to disappear.

About the *special form* of the world function Ω, nothing will be supposed, so that the resulting expressions are quite general.

2. *Deduction of the field equations.* The $\Gamma'^{\nu}_{\lambda\mu}$ of a semi-symmetrical displacement can always be divided into a symmetrical and an antisymmetrical part:

$$(1) \qquad \Gamma'^{\nu}_{\lambda\mu} = A'^{\nu}_{\lambda\mu} + S_{[\lambda} A^\nu_{\mu]} \quad ; \quad A'^{\nu}_{\lambda\mu} = A'^{\nu}_{\mu\lambda}\ ^2).$$

Be $R'^{\cdots\nu}_{\omega\lambda\mu}$ the curvature quantity belonging to $\Gamma'^{\nu}_{\lambda\mu}$:

$$(2) \qquad R'^{\cdots\nu}_{\omega\mu\lambda} = \frac{\partial}{\partial x^\mu}\Gamma'^{\nu}_{\lambda\omega} - \frac{\partial}{\partial x^\omega}\Gamma'^{\nu}_{\lambda\mu} - \Gamma'^{\nu}_{\varkappa\omega}\Gamma'^{\varkappa}_{\lambda\mu} + \Gamma'^{\nu}_{\varkappa\mu}\Gamma'^{\varkappa}_{\lambda\omega},$$

$A'^{\cdots\nu}_{\omega\mu\lambda}$ the curvature quantity formed in the same way with the parameters $A'^{\nu}_{\lambda\mu}$, $R'_{\mu\lambda}$ the quantity obtained from $R'^{\cdots\nu}_{\omega\mu\lambda}$ by contracting, $\omega = \nu$:

$$(3) \qquad R'_{\mu\lambda} = \frac{\partial}{\partial x^\mu}\Gamma'^{\alpha}_{\lambda\alpha} - \frac{\partial}{\partial x^\alpha}\Gamma'^{\alpha}_{\lambda\mu} - \Gamma'^{\alpha}_{\varkappa\alpha}\Gamma'^{\varkappa}_{\lambda\mu} + \Gamma'^{\alpha}_{\varkappa\mu}\Gamma'^{\varkappa}_{\lambda\alpha}$$

and $R^*_{\omega\lambda}$ the quantity obtained in the same way from $R^{*\cdots\nu}_{\omega\mu\lambda}$, then we can easily deduce the relation

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[1]) That the differences $\Gamma'^{\nu}_{\lambda\mu} - \Gamma'^{\nu}_{\mu\lambda}$ always are the components of a quantity of the third rank may be supposed as known. Cf. the author's paper in Math. Zeitschrift 13 (1922), p. 56—81, Nachtrag 15 (1922) p. 168.

[2]) In this paper the symbol $v_{[\lambda} w_{\mu]}$ means ${}^1/_2(v_\lambda w_\mu - v_\mu w_\lambda)$.

$$(4)\quad R'_{\mu\lambda} = R^*_{\mu\lambda} - {}^1/_2\left(\frac{\partial S_\nu}{\partial x^\mu} - \frac{\partial S_\mu}{\partial x^\nu}\right) + {}^1/_2(n-1)\left(\frac{\partial S_\lambda}{\partial x^\mu} - A^\nu_{\lambda\mu} S_\nu\right) - {}^1/_4(n-1) S_\lambda S_\mu =$$

$$= R^*_{\mu\lambda} - \nabla^*_{[\mu} S_{\lambda]} + {}^1/_2 (n-1)\, \nabla^*_\mu S_\lambda - {}^1/_4 (n-1)\, S_\lambda S_\mu,$$

in which ∇^* is the covariant differential operator belonging to $A^\nu_{\lambda\mu}$. We suppose that the determinant $R' = R'_{\lambda\mu}$ does not vanish. Hence there exists an inverse quantity $r'^{\lambda\mu}$:

$$(5)\qquad R'\, r'^{(\mu\lambda)} = \frac{\partial R'}{\partial R'_{\lambda\mu}} \quad ; \quad r'^{\lambda\mu} R'_{\mu\nu} = r'^{\mu\lambda} R'_{\nu\mu} = A^\lambda_\nu .$$

When $F'_{\mu\lambda}$ and $G'_{\mu\lambda}$ are the antisymmetrical and the symmetrical part of $R'_{\mu\lambda}$:

$$(6)\qquad F'_{\mu\lambda} = R'_{[\mu\lambda]} \quad ; \quad G'_{\mu\lambda} = R'_{(\mu\lambda)}\ ^1)$$

and when the word function $\mathfrak{H} = H\sqrt{-R}$ (scalar density) is a still unknown function of $G'_{\mu\lambda}$ and $F'_{\mu\lambda}$, we then have the variation equation:

$$(7)\qquad \bar{d}\int \mathfrak{H}\, d\tau = \int v'^{\lambda\mu}\, \bar{d} R'_{\mu\lambda}\, d\tau = 0\ ^2),$$

in which

$$(8a)\qquad v'^{\lambda\mu} = v'^{(\lambda\mu)}\sqrt{-R'} = (g'^{\lambda\mu} + f'^{(\lambda\mu)})\sqrt{-R'}$$

$$(8b)\qquad g'^{\lambda\mu}\sqrt{-R'} = \frac{\partial \mathfrak{H}}{\partial G'_{\mu\lambda}} \quad ; \quad f'^{\lambda\mu}\sqrt{-R'} = \frac{\partial \mathfrak{H}}{\partial F'_{\mu\lambda}}.$$

When we substitute into (7) the value of (4), we get for $n = 4$

$$(9)\quad 0 = \int \mathfrak{v}'^{\lambda\mu}\, d\tau \left\{ \bar{d} R^*_{\mu\lambda} - {}^1/_2 \bar{d}\left(\frac{\partial S_\lambda}{\partial x^\mu} - \frac{\partial S_\mu}{\partial x^\lambda}\right) + 2\bar{d}\left(\frac{\partial S_\lambda}{\partial x^\mu} - \Gamma^\nu_{\lambda\mu} S\right) - {}^3/_4 \bar{d}(S_\lambda S_\mu)\right\},$$

an equation that, $R^*_{\mu\nu}$ being independent of S_ν, is equivalent with the two equations

$$(10)\qquad d A^\alpha_{\lambda\mu}\{-A^\mu_\alpha(\nabla^*_\beta v'^{\lambda\beta} - P_\beta v'^{\lambda\beta}) + \nabla_\alpha v'^{\lambda\mu} - P_\alpha v'^{\lambda\mu} - {}^3/_2 S_\alpha v'^{\lambda\mu}\} = 0$$

$$(11)\quad d S_\lambda\{\nabla^*_\mu f'^{\lambda\mu} - P_\mu f'^{\lambda\mu} - {}^3/_2(\nabla^*_\mu v'^{\lambda\mu} - P_\mu v'^{\lambda\mu}) - {}^3/_2 S_\mu g'^{\lambda\mu}\} = 0,$$

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[1]) In this paper $v_{(\mu} w_{\nu)}$ means ${}^1/_2(v_\mu w_\nu + v_\nu w_\mu)$.

[2]) We use the variation symbol $\bar{d}$ in place of δ to prevent confusion with the symbol δ of the covariant differentiation.

in which P_λ is a vector depending on $R'_{\mu\nu}$ and $r'^{\lambda\mu}$ in the following way:

$$P_\mu = {}^1/_2\, R'_{\lambda\nu}\nabla^*_\mu r'^{\lambda\nu} = -\frac{\partial\, log\sqrt{-R'}}{\partial x^\mu} + A^\alpha_{\alpha\mu}. \tag{12}$$

Since $A^\nu_{\lambda\mu}$ is symmetrical in $\lambda\mu$, we get from (10):

$$\boxed{\begin{gathered} -A^{(\mu}_\alpha\nabla^*_\beta g'^{\lambda)\beta} + A^{(\mu}_\alpha P_\beta g'^{\lambda)\beta} - A^{(\mu}_\alpha\nabla^*_\beta f'^{\lambda)\beta} + A^{(\mu}_\alpha P_\beta f'^{\lambda)\beta} + \\ + \nabla^*_\alpha g'^{\lambda\mu} - P_\alpha g'^{\lambda\mu} - {}^3/_2\, S_\alpha g'^{\lambda\mu} = 0 \end{gathered}} \tag{I}$$

and from (11):

$$\boxed{\nabla^*_\mu f'^{\lambda\mu} - P_\mu f'^{\lambda\mu} - {}^3/_2(\nabla^*_\mu r'^{\lambda\mu} - P_\mu r'^{\lambda\mu}) - {}^3/_2\, S_\mu g'^{\lambda\mu} = 0.} \tag{II}$$

For $\nabla^*_\mu f'^{\lambda\mu} - P_\mu f'^{\lambda\mu}$ we introduce the notation i'^λ. It is easily shown that

$$i'^\lambda = \nabla^*_\mu f'^{\lambda\mu} - P_\mu f'^{\lambda\mu} = \frac{1}{\sqrt{-R'}}\,\frac{\partial f'^{\lambda\mu}\sqrt{-R'}}{\partial x^\mu}. \tag{13}$$

From (*I*) follows by contracting, $\alpha = \mu$:

$$\nabla^*_\mu g'^{\lambda\mu} - P_\mu g'^{\lambda\mu} = -i'^\lambda - S_\mu g'^{\lambda\mu}. \tag{14}$$

When this value is substituted into (*I*), we get

$$\nabla^*_\alpha g'^{\lambda\mu} - P_\alpha g'^{\lambda\mu} = -{}^2/_3\, A^{(\mu}_\alpha i'^{\lambda)} - A^{(\mu}_\alpha g'^{\lambda)\beta} S_\beta + {}^3/_2\, S_\alpha g'^{\lambda\mu}. \tag{15}$$

In the supposition that also the determinant $|g'^{\lambda\mu}|$ does not vanish this equation can be simplified by the introduction of the tensor

$$g^{\lambda\mu} = \frac{\sqrt{-R'}}{\sqrt{-g}}\, g'^{\lambda\mu} \quad ; \quad g = |g^{\lambda\mu}|^{-1} \tag{16}$$

as *fundamental tensor* and the vector

$$i_\nu = \frac{\sqrt{-R'}}{\sqrt{-g}}\, i'^\nu. \tag{17}$$

Then, because of

$$P_\mu - {}^1/_2\, g_{\lambda\nu}\nabla^*_\mu g^{\lambda\nu} = -\frac{\partial}{\partial x^\mu}\, log\,\frac{\sqrt{-R'}}{\sqrt{-g}}, \tag{18}$$

the equation (15) passes into:

$$(19)\quad \nabla^*_\alpha g^{\lambda\nu} - {}^1/_2\, g^{\lambda\nu}\, g_{\beta\gamma}\, \nabla_\alpha g^{\beta\gamma} = -{}^2/_3\, A^{(\mu}_\alpha\, i^{\lambda)} - A^{(\mu}_\alpha\, S^{\lambda)} + {}^2/_3\, S_\alpha\, g^{\lambda\nu}.$$

Transvection of this equation with $g_{\lambda\mu}$ gives:

$$(20)\quad -g_{\beta\gamma}\, \nabla_\alpha g^{\beta\gamma} = -{}^2/_3\, i_\alpha + 5\, S_\alpha,$$

so that we get the resulting equation:

$$(21)\quad \nabla^*_\alpha g^{\lambda\nu} = -{}^2/_3\, A^{(\nu}_\alpha\, i^{\lambda)} + {}^1/_3\, i_\alpha\, g^{\lambda\nu} - A^{(\nu}_\alpha\, S^{\lambda)} - S_\alpha\, g^{\lambda\nu}$$

and

$$(III)\quad \boxed{\nabla'_\alpha g^{\lambda\mu} = -{}^2/_3\, A^{(\mu}_\alpha\, i^{\nu)} + {}^1/_3\, i_\alpha\, g^{\lambda\mu} - 2\, S_\alpha\, g^{\lambda\nu},}$$

in which ∇' is the differential operator belonging to $\Gamma'^{\nu}_{\lambda\mu}$.

From (21) we deduce:

$$(22)\quad A^{\nu}_{\lambda\mu} = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} - {}^1/_2\, g_{\lambda\mu}\, i^\nu + {}^1/_6\, A^\nu_\lambda\, i_\mu + {}^1/_6\, A^\nu_\mu\, i_\lambda - {}^1/_2\, A^\nu_\lambda\, S_\mu - {}^1/_2\, A^\nu_\mu\, S_\lambda.$$

so that, with regard to (1):

$$(23)\quad \Gamma'^{\nu}_{\lambda\mu} = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} - {}^1/_2\, g_{\lambda\mu}\, i^\nu + {}^1/_6\, A^\nu_\lambda\, i_\mu + {}^1/_6\, A^\nu_\mu\, i_\lambda - A^\nu_\lambda\, S_\mu.$$

Substituting (22) into (3), we obtain:

$$(24)\quad R^*_{\mu\lambda} = K_{\mu\lambda} + {}^1/_6\, (\nabla^*_\mu i_\lambda - \nabla^*_\lambda i_\mu) + {}^1/_6\, i_\mu\, i_\lambda - {}^1/_2 (\nabla^*_\mu S_\lambda - \nabla^*_\lambda S_\mu) - \\ - {}^3/_2\, \nabla_\mu S_\lambda + {}^9/_4\, S_\mu\, S_\lambda,$$

in which $K_{\lambda\mu}$ is the contracted curvature quantity $K^{\cdots\nu}_{\omega\mu\lambda}$ belonging to the fundamental tensor $g_{\mu\nu}$. By substituting (24) into (4) we obtain the field equations:

$$(IV)\quad \boxed{\begin{aligned} R'_{\mu\lambda} &= K_{\mu\lambda} + {}^1/_6\, (\nabla^*_\mu\, i_\lambda - \nabla^*_\lambda\, i_\mu) + {}^1/_6\, i_\mu\, i_\lambda - (\nabla^*_\mu\, S_\lambda - \nabla^*_\lambda\, S_\mu) \\ &= K_{\mu\lambda} + {}^1/_6 \left(\frac{\partial i_\lambda}{\partial x^\mu} - \frac{\partial i_\mu}{\partial x^\lambda} \right) + {}^1/_6\, i_\mu\, i_\lambda - \left(\frac{\partial S_\lambda}{\partial x^\mu} - \frac{\partial S_\mu}{\partial x^\lambda} \right) \end{aligned}}$$

From (IV) follows for the bivector $F'_{\mu\lambda}$ of the electromagnetic field:

$$F'_{\mu\lambda} = R'_{[\mu\lambda]} = {}^1/_6 \left(\frac{\partial i_\lambda}{\partial x^\mu} - \frac{\partial i_\mu}{\partial x^\lambda} \right) - \left(\frac{\partial S_\lambda}{\partial x^\mu} - \frac{\partial S_\mu}{\partial x^\lambda} \right). \tag{25}$$

We now return to the equation (II) obtained from the variation principle. With regard to (13), (14) and (17) this equation leads to

$$i^\nu = 0. \tag{26}$$

Since i^ν has the character of a current vector, it is not allowed to consider variations of the *alternating* part of $\Gamma'^{\nu}_{\lambda\mu}$, when we wish to keep the current vector in the equations. In regions where only an electromagnetic field exists and no current, the variation principle remains valid without any restriction.

The expressions (IV) and (25) only differ from those of EINSTEIN by the terms in S_λ, hence an electromagnetic field is also possible in places with vanishing current vector i^ν. There the vector S_λ behaves as a potential vector.

We can further make the following important remarks:

1. In the field equations (IV) S_λ does not contribute to the *symmetrical* part of $R'_{\mu\lambda}$.

2. When there is no current the displacement is by (III) *conformal*, the fundamental tensor being diminished with $2\, dx^\alpha\, S_\alpha\, g^{\lambda\mu}$ when the pseudoparallel displacement is dx^ν.

3. When there is no current and no potential (23) passes into the ordinary equation of the gravitational field, in the same way as EINSTEIN's equation.

3. *The potentialvector* S_λ. It is remarkable that here the potential vector S_λ occurs as unambiguously determinated, not as a vector to which an arbitrary gradient vector may be added. This difficulty disappears when we make the supposition that the parameters which define the displacement are not the same for covariant and for contravariant vectors[1]) and thus no longer adopt the invariance of transvection. It is namely possible to alter covariant parameters independently of the transformation of the original variables by changing the *measure*[2]) of the *covariant* vectors. This change of measure

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[1]) For these displacements cf. the above mentioned paper in Math. Zeitschrift **13**.

[2]) This change of measure has nothing to do with an introduction of a ds.

(27) $$\tau\, {}'w_\lambda = w_\lambda$$

in which τ is an arbitrary function, leaves the parameters of the contravariant displacement unaltered, while the covariant parameters, which we will also further denote with $\Gamma'^{\nu}_{\lambda\mu}$, will be transformed in the following way:

(28) $$'\Gamma'^{\nu}_{\lambda\mu} = \Gamma'^{\nu}_{\lambda\mu} - \frac{\partial\, lg\, \tau}{\partial x^\mu} A^\nu_\lambda .$$

Such a change of measure cannot be applied in the same easy way to contravariant vectors, the new components $\tau^{-1}\, dx^\nu$ being in general no more exact differentials. In this case we would be obliged to consider space-time as a system of non-exact differentials, and it would no more be possible to represent a point by four finite coordinates. This case has doubtlessly but little attraction so long as there are other possibilities.

When we wish to "loose" the vector S_ν in the above mentioned sense, we have only to consider the $\Gamma'^{\nu}_{\lambda\mu}$ as the parameters of the *covariant* displacement and to define the $\Gamma^{\nu}_{\lambda\mu}$, the parameters of the contravariant displacement, in the following way:

(29) $$\Gamma_{\lambda\mu} = \Gamma'^{\nu}_{\lambda\mu} + S_\mu A^\nu_\lambda = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} - {}^1/_2\, g_{\lambda\mu}\, i^\nu + {}^1/_6\, A^\nu_\lambda\, i_\mu + {}^1/_6\, A^\nu_\mu\, i_\lambda ,$$

We then have obtained that $\Gamma^{\nu}_{\lambda\mu}$ is independent of S_λ and that, when covariant measure is changed, S_λ is transformed in the following way:

(30) $$'S_\mu = S_\mu + \frac{\partial\, lg\, \tau}{\partial x_\mu} .$$

It is very remarkable that by (23) $\Gamma'^{\nu}_{\lambda\mu}$ has just a form that leads to the desired transformation of the potential vector. If f.i. $\Gamma^{\nu}_{\lambda\mu}$ contained a term with $S_\lambda A^\nu_\mu$, it would not be possible to obtain an equation of the form (30).

Representing the covariant differential operator determined by $\Gamma^{\nu}_{\lambda\mu}$ and $\Gamma^{\nu}_{\lambda\mu}$ by ∇, (III) is changed into:

(*III'*) $$\nabla_\alpha\, g^{\lambda\mu} = - {}^3/_8\, A^{(\mu}_\alpha\, i^{\lambda)} + {}^1/_8\, i_\alpha\, g^{\lambda\mu}$$
$$\nabla_\alpha\, g_{\lambda\mu} = - {}^1/_8\, g_{\alpha\mu}\, i_\lambda - {}^1/_8\, g_{\alpha\lambda}\, i_\mu + {}^1/_8\, i_\alpha\, g_{\lambda\mu} - 2\, S_\alpha\, g_{\lambda\mu} .$$

The tensor $g_{\lambda\mu}$ is a quantity variable with transformation of covariant measure, for its components do *not* change, while the

components of a genuine quantity of second order obtain the factor τ^{-2}. When the current vanishes, this quantity has the same character as the variable fundamental tensor of WEYL's theory, and $-2\,S_\alpha$ behaves as the vector which WEYL calls φ_α.

4. *On the law of conservation of energy and momentum.* The law of conservation of energy and momentum in gravitation theory is a consequence of the identity of BIANCHI. The form of this identity is known for non-symmetrical displacements and for displacements with non-invariant transvection [1]). Hence it must be possible to deduce, starting with this identity, an equation that can be considered as an analogon of the equation that expresses the law of energy and momentum. This possibility exists already before any supposition is made relating to the special form of Hamilton's function.

[1]) Cf. Math. Zeitschrift 1923, 17, p. 111—115; R. WEITZENBÖCK, Invariantentheorie (NOORDHOFF, Groningen 1923), p. 357.

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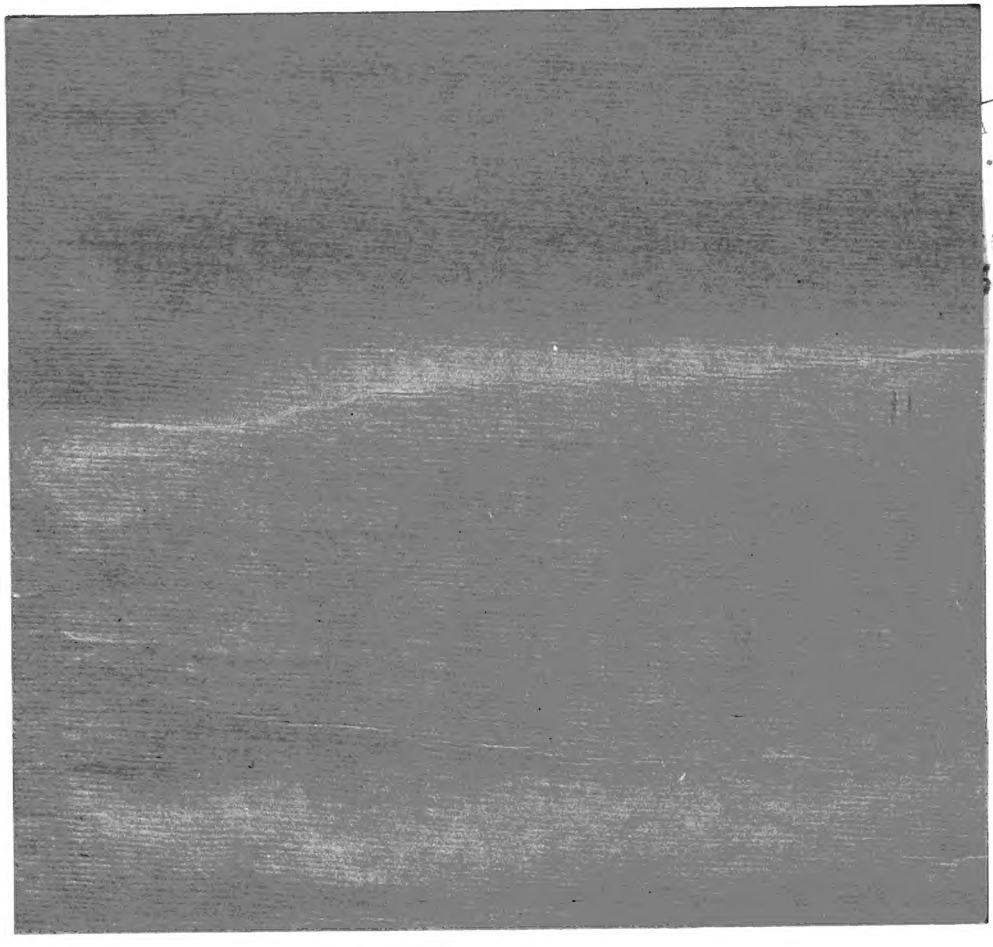
PROCEEDINGS OF THE SECTION OF SCIENCES

VOLUME XXVI
— (Nos. 1—10) —

PUBLISHED BY
"KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN", AMSTERDAM
1923

(Translated from: „Verslag van de Gewone Vergaderingen der Wis- en Natuurkundige Afdeeling" Vols. XXXI and XXXII).

DRUKKERIJ HOLLAND
AMSTERDAM



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